Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 185948, 19 pages
doi:10.1155/2012/185948

Research Article

On the Sets of Convergence for Sequences of the q-Bernstein Polynomials with $q > 1$

Sofiya Ostrovska and Ahmet Yaşar Özban

Department of Mathematics, Atilim University, Incek, 06836 Ankara, Turkey

Correspondence should be addressed to Sofiya Ostrovska, ostrovsk@atilim.edu.tr

Received 27 March 2012; Accepted 19 June 2012

Academic Editor: Ngai-Ching Wong

The aim of this paper is to present new results related to the convergence of the sequence of the q-Bernstein polynomials $\{B_{n,q}(f;x)\}$ in the case $q > 1$, where f is a continuous function on $[0,1]$. It is shown that the polynomials converge to f uniformly on the time scale $\mathbb{J}_q = \{q^{-j}\}_{j=0}^{\infty} \cup \{0\}$, and that this result is sharp in the sense that the sequence $\{B_{n,q}(f;x)\}_{n=1}^{\infty}$ may be divergent for all $x \in R \setminus \mathbb{J}_q$. Further, the impossibility of the uniform approximation for the Weierstrass-type functions is established. Throughout the paper, the results are illustrated by numerical examples.

1. Introduction

Let $f : [0,1] \to \mathbb{C}$, $q > 0$, and $n \in \mathbb{N}$. Then, the *q-Bernstein polynomial* of f is defined by

$$B_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q;x), \tag{1.1}$$

where

$$p_{nk}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}, \quad k = 0,1,\ldots n, \tag{1.2}$$

with $\begin{bmatrix} n \\ k \end{bmatrix}_q$ being the *q-binomial* coefficients given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \tag{1.3}$$

and $(x;q)_m$ being the q-Pochhammer symbol:

$$(x;q)_0 = 1, \qquad (x;q)_m = \prod_{s=0}^{m-1}(1-xq^s), \qquad (x;q)_\infty = \prod_{s=0}^{\infty}(1-xq^s). \tag{1.4}$$

Here, for any nonnegative integer k,

$$[k]_q! = [1]_q[2]_q \cdots [k]_q \quad (k = 1,2,\ldots), \qquad [0]_q! \ := 1 \tag{1.5}$$

are the *q-factorials* with $[k]_q$ being the *q-integer* given by

$$[k]_q = 1 + q + \cdots + q^{k-1} \quad (k = 1,2,\ldots), \qquad [0]_q := 0. \tag{1.6}$$

We use the notation from [[1], Ch. 10].

The polynomials $p_{n0}(q;x), p_{n1}(q;x), \ldots, p_{nn}(q;x)$, called the *q-Bernstein basic polynomials*, form the *q-Bernstein basis* in the linear space of polynomials of degree at most n.

Although, for $q = 1$, the q-Bernstein polynomial $B_{n,q}(f;x)$ turns into the classical Bernstein polynomial $B_n(f;x)$:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \tag{1.7}$$

conventionally, the name "q-Bernstein polynomials" is reserved for the case $q \neq 1$.

Based on the q-Bernstein polynomials, the *q-Bernstein operator* on $C[0,1]$ is given by

$$B_{n,q} : f \longmapsto B_{n,q}(f;\cdot). \tag{1.8}$$

A detailed review of the results on the q-Bernstein polynomials along with an extensive bibliography has been provided in [2]. In this field, new results concerning the properties of the q-Bernstein polynomials and/or their various generalizations are still coming out (see, e.g, papers [3–8], all of which have appeared after [2]).

The popularity of the q-Bernstein polynomials is attributed to the fact that they are closely related to the q-binomial and the q-deformed Poisson probability distributions (cf. [9]). The q-binomial distribution plays an important role in the q-boson theory, providing a q-deformation for the quantum harmonic formalism. More specifically, it has been used to construct the binomial state for the q-boson. Meanwhile, the q-deformed Poisson distribution, which is the limit form of q-binomial one, defines the energy distribution in a q-analogue of the coherent state [10]. Another motivation for this study is that various estimates related to the natural sequences of functions and operators in functional spaces, convergence theorems, and estimates for the rates of convergence are of decisive nature in the modern functional analysis and its applications (see, e.g., [4, 11, 12]).

The q-Bernstein polynomials retain some of the properties of the classical Bernstein polynomials. For example, they possess the end-point interpolation property:

$$B_{n,q}(f;0) = f(0), \quad B_{n,q}(f;1) = f(1), \quad n = 1,2,\ldots, \ q > 0, \tag{1.9}$$

and leave the linear functions invariant:

$$B_{n,q}(at+b;x) = ax+b, \quad n = 1,2,\ldots, \ q > 0. \tag{1.10}$$

In addition, the q-Bernstein basic polynomials (1.2) satisfy the identity

$$\sum_{k=0}^{n} p_{nk}(q;x) = 1 \quad \forall n = 1,2,\ldots, \ \forall q > 0. \tag{1.11}$$

Furthermore, the q-Bernstein polynomials admit a representation via the divided differences given by (3.3), as well as demonstrate the saturation phenomenon (see [2, 7, 13]).

Despite the similarities such as those indicated above, the convergence properties of the q-Bernstein polynomials for $q \neq 1$ are essentially *different* from those of the classical ones. What is more, the cases $0 < q < 1$ and $q > 1$ in terms of convergence are not similar to each other, as shown in [14, 15]. This absence of similarity is brought about by the fact that, for $0 < q < 1$, $B_{n,q}$ are *positive* linear operators on $C[0,1]$, whereas for $q > 1$, no positivity occurs. In addition, the case $q > 1$ is aggravated by the rather irregular behavior of basic polynomials (1.2), which, in this case, combine the fast increase in magnitude with the sign oscillations. For a detailed examination of this situation, see [16], where, in particular, it has been shown that the norm $\|B_{n,q}\|$ increases rather rapidly in both n and q. Namely,

$$\|B_{n,q}\| \sim \frac{2}{e} \cdot \frac{q^{n(n-1)/2}}{n} \quad \text{as } n \longrightarrow \infty, q \longrightarrow +\infty. \tag{1.12}$$

This puts serious obstacles in the analysis of the convergence for $q > 1$. The challenge has inspired some papers by a number of authors dealing with the convergence of q-Bernstein polynomials in the case $q > 1$ (see, e.g., [7, 17]). However, there are still many open problems related to the behavior of the q-Bernstein polynomials with $q > 1$ (see the list of open problems in [2]).

In this paper, it is shown that the time scale

$$\mathbb{J}_q = \left\{q^{-j}\right\}_{j=0}^{\infty} \cup \{0\} \tag{1.13}$$

is the "minimal" set of convergence for the q-Bernstein polynomials of continuous functions with $q > 1$, in the sense that every sequence $\{B_{n,q}(f;x)\}$ converges uniformly on $\mathbb{J}_q$. Moreover, it is proved that $\mathbb{J}_q$ is the only set of convergence for some continuous functions.

The paper is organized as follows. In Section 2, we present results concerning the convergence of the q-Bernstein polynomials on the time scale $\mathbb{J}_q$. Section 3 is devoted to the q-Bernstein polynomials of the Weierstrass-type functions. Some of the results throughout the paper are also illustrated using numerical examples.

2. The Convergence of the q-Bernstein Polynomials on $\mathbb{J}_q$

In this paper, $q > 1$ is considered fixed. It has been shown in [15], that, if a function f is analytic in $D_\varepsilon = \{z : |z| < 1 + \varepsilon\}$, then it is uniformly approximated by its q-Bernstein polynomials on any compact set in D_ε, and, in particular, on $[0, 1]$.

In this study, attention is focused on the q-Bernstein polynomials of "bad" functions, that is, functions which do not have an analytic continuation from $[0, 1]$ to the unit disc. In general, such functions are not approximated by their q-Bernstein polynomials on $[0, 1]$. Moreover, their q-Bernstein polynomials may tend to infinity at some points of $[0, 1]$ (a simple example has been provided in [15]). Here, it is proved that the divergence of $\{B_{n,q}(f; x)\}$ may occur everywhere outside of $\mathbb{J}_q$, which is a "minimal" set of convergence.

However, in spite of this negative information, it will be shown that, for any $f \in C[0, 1]$, the sequence of its q-Bernstein polynomials converges uniformly on the time scale $\mathbb{J}_q$.

The next statement generalizing Lemma 1 of [15] can be regarded as a discrete analogue of the Popoviciu Theorem.

Theorem 2.1. *Let $f \in C[0, 1]$. Then*

$$\left|B_{n,q}\left(f; q^{-j}\right) - f\left(q^{-j}\right)\right| \leq 2\omega_f\left(\sqrt{\frac{q^{-j}(1 - q^{-j})}{[n]_q}}\right), \quad j \in \mathbb{Z}_+, \tag{2.1}$$

where ω_f is the modulus of continuity of f on $[0, 1]$.

Corollary 2.2. *If $j \in \mathbb{Z}_+$, then*

$$\left|B_{n,q}\left(f; q^{-j}\right) - f\left(q^{-j}\right)\right| \leq 2\omega_f\left(\frac{1}{2\sqrt{[n]_q}}\right), \tag{2.2}$$

that is, $B_{n,q}(f; x)$ converges uniformly to $f(x)$ on the time scale $\mathbb{J}_q$.

Proof. The proof is rather straightforward. First, notice that $p_{nk}(q; q^{-j}) \geq 0$ for all n, k, j, while $\sum_{k=0}^{n} p_{nk}(q; q^{-j}) = 1$ by virtue of (1.11). Then

$$\begin{aligned}
\left|B_{n,q}\left(f; q^{-j}\right) - f\left(q^{-j}\right)\right| &\leq \sum_{k=0}^{n}\left|f\left(\frac{[k]_q}{[n]_q}\right) - f\left(q^{-j}\right)\right| p_{nk}\left(q; q^{-j}\right) \\
&\leq \sum_{k=0}^{n} \omega_f\left(\left|\frac{[k]_q}{[n]_q} - q^{-j}\right|\right) p_{nk}\left(q; q^{-j}\right) \\
&\leq \omega_f(\delta) \sum_{k=0}^{n}\left\{1 + \frac{1}{\delta^2}\left(\frac{[k]_q}{[n]_q} - q^{-j}\right)^2\right\} p_{nk}\left(q; q^{-j}\right)
\end{aligned} \tag{2.3}$$

for any $\delta > 0$. Plain calculations (see, e.g., [13], formula (2.7)) show that

$$B_{n,q}\left((t-x)^2;x\right) = \frac{x(1-x)}{[n]_q}, \tag{2.4}$$

which implies that

$$\left|B_{n,q}\left(f;q^{-j}\right) - f\left(q^{-j}\right)\right| \le \omega_f(\delta)\cdot\left\{1+\frac{1}{\delta^2}\cdot\frac{q^{-j}\left(1-q^{-j}\right)}{[n]_q}\right\}. \tag{2.5}$$

Then, one can immediately derive the result by choosing $\delta = \sqrt{q^{-j}(1-q^{-j})/[n]_q}$. □

Remark 2.3. In [7], Wu has shown that if $f \in C^1[0,1]$, then for any $j \in \mathbb{Z}_+$, one has:

$$\left|B_{n,q}\left(f;q^{-j}\right) - f\left(q^{-j}\right)\right| \le C_j\left(q^{-n}\right), \quad n \longrightarrow \infty, \quad \text{where } C_j \longrightarrow \infty \quad \text{as } j \to \infty. \tag{2.6}$$

The condition $f \in C^1[0,1]$ cannot be left out completely, as the following example shows.

Example 2.4. Consider a function $f \in C[0,1]$ satisfying

$$f(x) = \begin{cases} 0 & \text{if } x \in \left[0,q^{-2}\right] \cup \left(q^{-1},1\right], \\ \left(q^{-1}-x\right)^{\alpha} & \text{if } x \in \left[\dfrac{\left(q^{-2}+q^{-1}\right)}{2}, q^{-1}\right], \end{cases} \tag{2.7}$$

where $0 < \alpha < 1$. Then, for n large enough, we have

$$\begin{aligned} B_{n,q}\left(f;q^{-1}\right) - f\left(q^{-1}\right) &= f\left(\frac{[n-1]_q}{[n]_q}\right)p_{n,n-1}\left(q;q^{-1}\right) \\ &= \left(\frac{q-1}{q}\right)^{\alpha}\cdot\frac{\left(q^n-1\right)^{1-\alpha}}{q^n} \ge C\,q^{-n\alpha}, \end{aligned} \tag{2.8}$$

where C is a positive constant independent from n.

As it has been already mentioned, the behavior of the q-Bernstein polynomials in the case $q > 1$ outside of the time scale $\mathbb{J}_q$ may be rather unpredictable. The next theorem shows that the sequence $\{B_{n,q}(f;x)\}$ may be divergent for all $x \in \mathbb{R} \setminus \mathbb{J}_q$.

Theorem 2.5. *Let $f(x) = x^{\alpha}$, $0 < \alpha \le 1/2$. If $q \ge 2$, then*

$$B_{n,q}(f;x) \to \infty \quad \textit{as } n \to \infty \quad \forall x \in \mathbb{R} \setminus \mathbb{J}_q. \tag{2.9}$$

Proof. The q-Bernstein polynomial of f is

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \left(\frac{[k]_q}{[n]_q} \right)^{\alpha} p_{nk}(q;x) = \frac{1}{[n]_q^{\alpha}} \sum_{k=1}^{n} [k]_q^{\alpha} p_{nk}(q;x). \tag{2.10}$$

Since for $k = 1, 2, \ldots, n-1$ one has

$$\begin{aligned} p_{nk}(q;x) &= \frac{(q^n - 1)\cdots(q^{n-k+1}-1)}{(q^k-1)\cdots(q-1)} x^k (x;q)_{n-k} \\ &= \frac{q^{(2n-k+1)k/2}(q^{-n};q)_k}{(q^k-1)\cdots(q-1)} (-1)^{n-k} q^{(n-k)(n-k-1)/2} x^n \left(\frac{1}{x};\frac{1}{q}\right)_{n-k} \\ &= (-1)^n q^{n(n-1)/2} x^n \cdot \frac{(-1)^k q^k (q^{-n};q)_k}{(q^k-1)\cdots(q-1)} \cdot \left(\frac{1}{x};\frac{1}{q}\right)_{n-k}, \end{aligned} \tag{2.11}$$

it follows that

$$B_{n,q}(f;x) = \frac{(-1)^n q^{n(n-1)/2} x^n}{[n]_q^{\alpha}} \cdot T(n,q,x), \tag{2.12}$$

where

$$T(n;q;x) := \frac{(-1)^n [n]_q^{\alpha}}{q^{n(n-1)/2}} + \sum_{k=0}^{n-1} \frac{(-1)^k [k]_q^{\alpha} q^k (q^{-n};q)_k}{(q^k-1)\cdots(q-1)} \cdot \left(\frac{1}{x};\frac{1}{q}\right)_{n-k}. \tag{2.13}$$

Obviously,

$$\lim_{n\to\infty} \frac{q^{n(n-1)/2} x^n}{[n]_q^{\alpha}} = \infty \quad \text{for any } x \neq 0. \tag{2.14}$$

As such, the theorem will be proved if it is shown that

$$\lim_{n\to\infty} T(n,q,x) \neq 0 \quad \text{for } x \notin \mathbb{J}_q. \tag{2.15}$$

As $\lim_{n\to\infty}(q^{-n(n-1)/2}(-1)^n [n]_q^{\alpha}) = 0$, it suffices to prove that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} c_{kn} \neq 0 \quad \text{when } x \notin \mathbb{J}_q, \tag{2.16}$$

where

$$c_{kn} := \begin{cases} \dfrac{(-1)^k [k]_q^\alpha q^k (q^{-n};q)_k}{(q^k-1)\cdots(q-1)} \cdot \left(\dfrac{1}{x};\dfrac{1}{q}\right)_{n-k} & \text{if } k \leq n-1, \\ 0 & \text{if } k \geq n. \end{cases} \tag{2.17}$$

The fact that $(q^{-n};q)_k \leq 1$ and the inequality

$$\left|\left(\frac{1}{x};\frac{1}{q}\right)_{n-k}\right| \leq \left(-\frac{1}{|x|};\frac{1}{q}\right)_{n-k} \leq \left(-\frac{1}{|x|};\frac{1}{q}\right)_\infty \tag{2.18}$$

lead to

$$|c_{kn}| \leq \frac{q^k [k]_q^\alpha}{(q^k-1)\cdots(q-1)}\left(-\frac{1}{|x|};\frac{1}{q}\right)_\infty =: d_k. \tag{2.19}$$

Now, since

$$\lim_{n\to\infty} c_{kn} = \frac{(-1)^k q^k [k]_q^\alpha}{(q^k-1)\cdots(q-1)}\left(\frac{1}{x};\frac{1}{q}\right)_\infty, \tag{2.20}$$

and the series $\sum_{k=0}^{\infty} d_k$ is convergent, the Lebesgues dominated convergence theorem implies

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} c_{kn} = \sum_{k=0}^{\infty}\lim_{n\to\infty} c_{kn} = \frac{1}{(q-1)^\alpha}\left(\frac{1}{x};\frac{1}{q}\right)_\infty \cdot \sum_{k=0}^{\infty}(-1)^k a_k, \tag{2.21}$$

where $a_k = q^k (q^k-1)^\alpha/(q^k-1)\cdots(q-1)$, $k = 1,2,\ldots$. Moreover,

$$\frac{1}{(q-1)^\alpha}\left(\frac{1}{x};\frac{1}{q}\right)_\infty \neq 0 \quad \text{whenever } x \notin \mathbb{J}_q. \tag{2.22}$$

How about the sum of the series in (2.21)? Consider the following two cases.

Case 1. $0 < \alpha < 1/3$.

Let us show that $a_{k+1} < a_k$, $k = 1,2,\ldots$ for $q \geq 2$. Since

$$\frac{a_{k+1}}{a_k} = \frac{q}{(q^{k+1}-1)^{1-\alpha}(q^k-1)^\alpha}, \tag{2.23}$$

for $k \geq 2$ it follows that

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &\leq \frac{q}{\left(q^3-1\right)^{1-\alpha}\left(q^2-1\right)^{\alpha}} \\
&\leq \frac{q}{\left(q-1\right)\left(q^2+q+1\right)^{1-\alpha}\left(q+1\right)^{\alpha}} \\
&\leq \frac{q}{\left(q-1\right)\left(q+1\right)} < 1.
\end{aligned} \tag{2.24}$$

Notice that (2.24) holds for any $\alpha \in (0,1)$. In addition, if $k = 1$, then

$$\frac{a_2}{a_1} = \frac{q}{\left(q^2-1\right)^{1-\alpha}\left(q-1\right)^{\alpha}} = \frac{q}{\left(q-1\right)\left(q+1\right)^{1-\alpha}}. \tag{2.25}$$

The function in the r.h.s. is monotone decreasing in q, so

$$\frac{a_2}{a_1} \leq \frac{2}{1 \cdot 3^{1-\alpha}} \leq \frac{2}{\sqrt[3]{9}} < 1. \tag{2.26}$$

Thus, $\{a_k\}_{k=1}^{\infty}$ is a strictly decreasing sequence. Since all $(a_{2k-1} - a_{2k})$ are strictly positive, it follows that

$$\sum_{k=0}^{\infty}(-1)^k a_k = -\left[(a_1-a_2)+(a_3-a_4)+\cdots+(a_{2k-1}-a_{2k})+\cdots\right] < 0. \tag{2.27}$$

Case 2. $1/3 \leq \alpha \leq 1/2$.

Estimate (2.24) implies that $\sum_{k=5}^{\infty}(-1)^k a_k < 0$. To prove the theorem, it suffices to show that $a_1 - a_2 + a_3 - a_4 > 0$ when $q \geq 2$. Denoting $a_i = (q(q-1)^{\alpha}/(q-1))g_i(q)$, $i = 1,2,3,4$, we write the following:

$$a_1 - a_2 + a_3 - a_4 = \frac{q(q-1)^{\alpha}}{q-1}\left[g_1(q) - g_2(q) + g_3(q) - g_4(q)\right] =: \frac{q(q-1)^{\alpha}}{q-1}K(q). \tag{2.28}$$

We are left to show that $K(q)$ is strictly positive for the specified values of q and α. First of all, notice that $g_1(q) = 1$, while $g_2(q)$, $g_3(q)$, and $g_4(q)$ are strictly decreasing in q on $(0,+\infty)$. Hence, for $q \in [2,5/2]$,

$$K(q) \geq 1 - g_1(2) + g_2\left(\frac{5}{2}\right) - g_3(2) = 1 - \frac{2}{3}\cdot 3^{\alpha} + \frac{200}{2457}\left(\frac{39}{4}\right)^{\alpha} - \frac{8}{315}\cdot 15^{\alpha} =: L(\alpha). \tag{2.29}$$

The function $L(\alpha)$ is strictly decreasing on $[1/3,1/2]$. Indeed,

$$L'(\alpha) = -\frac{2}{3}\cdot 3^{\alpha}\ln 3 + \frac{200}{2457}\left(\frac{39}{4}\right)^{\alpha}\ln\left(\frac{39}{4}\right) - \frac{8}{315}\cdot 15^{\alpha}\ln 15 \tag{2.30}$$

and, for $\alpha \in [1/3, 1/2]$,

$$L'(\alpha) \leq -\frac{2}{3} \cdot 3^{1/3} \ln 3 + \frac{200}{2457} \left(\frac{39}{4}\right)^{1/2} \ln\left(\frac{39}{4}\right) - \frac{8}{315} \cdot 15^{1/3} \ln 15 \leq -0.4332 < 0, \tag{2.31}$$

whence $L(\alpha) \geq L(1/2) \geq 1.096 \times 10^{-3} > 0$ for $\alpha \in [1/3, 1/2]$.

Similarly, for $q \in [5/2, 3]$,

$$\begin{aligned} K(q) &\geq 1 - g_1\left(\frac{5}{2}\right) + g_2(3) - g_3\left(\frac{5}{2}\right) \\ &= 1 - \frac{10}{21} \cdot \left(\frac{7}{2}\right)^{\alpha} + \frac{9}{208} \cdot 13^{\alpha} - \frac{8000}{1496313} \cdot \left(\frac{203}{8}\right)^{\alpha} =: M(\alpha). \end{aligned} \tag{2.32}$$

Applying the same reasoning as done for $L(\alpha)$, it can be shown that $M(\alpha)$ is strictly decreasing on $[1/3, 1/2]$. Since $M(1/2) \geq 0.238 > 0$, it follows that $M(\alpha) > 0$ for all $\alpha \in [1/3, 1/2]$.

Finally, for $q \in [3, +\infty)$, we obtain

$$K(q) \geq 1 - g_1(3) - g_3(3) = 1 - \frac{3}{8} \cdot 4^{\alpha} - \frac{27}{16640} \cdot 40^{\alpha} =: N(\alpha). \tag{2.33}$$

Obviously, $N(\alpha)$ is a strictly decreasing function for all $\alpha \in \mathbb{R}$, whence, for $\alpha \in [1/3, 1/2]$,

$$N(\alpha) \geq N\left(\frac{1}{2}\right) \geq 0.239 > 0, \tag{2.34}$$

which completes the proof. □

Remark 2.6. It can be seen from the proof that, the statement of the theorem is true for any $\alpha \in (0, 1)$ and $q \geq q_0(\alpha)$.

An illustrative example is supplied below.

Example 2.7. Let $f(x) = \sqrt[3]{x}$. The graphs of $y = f(x)$ and $y = B_{n,q}(f; x)$ for $q = 2$ and $n = 4, 5$ are exhibited in Figure 1. Similarly, Figure 2 represents the graphs of $y = f(x)$ and $y = B_{n,q}(f; x)$ for $q = 2$ and $n = 6, 7$ over the subintervals $[0, 0.5]$ and $[0.5, 1]$, respectively. In addition, Table 1 presents the values of the *error function* $E(n, q, x) := B_{n,q}(f; x) - f(x)$ with $q = 2$ at some points $x \in [0, 1]$. The points are taken both in $\mathbb{J}_q$ and in $[0, 1] \setminus \mathbb{J}_q$. It can be observed from Table 1 that, while at the points $x \in \mathbb{J}_q$, the values of the error function are close to 0, at the points $x \notin \mathbb{J}_q$, the values of the error function may be very large in magnitude.

Remark 2.8. Table 1 also shows that while the error function changes its sign for different values of x, for $x = q^{-j} \in \mathbb{J}_q$, its values are negative, that is, $B_{n,q}(t^{1/3}; q^{-j}) < f(q^{-j})$ for $q^{-j} \in \mathbb{J}_q$. This is a particular case of the following statement.

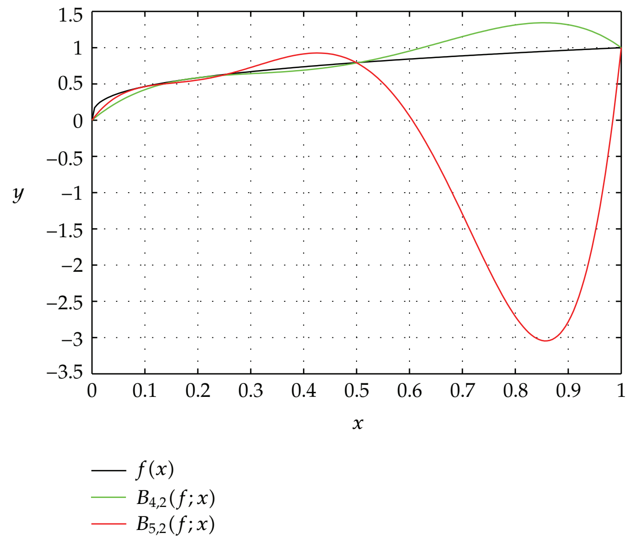

Figure 1: Graphs of $y = f(x)$ and $y = B_{n,2}(f;x)$, $n = 4, 5$.

Theorem 2.9. *Let* $q > 1$*. If* $f(x)$ *is convex (concave) on* $[0,1]$*, then*

$$B_{n,q}\left(f;q^{-j}\right) \geq f\left(q^{-j}\right) \quad \left(\text{correspondingly } B_{n,q}\left(f;q^{-j}\right) \leq f\left(q^{-j}\right)\right), \tag{2.35}$$

for all $q^{-j} \in \mathbb{J}_q$*.*

Proof. It can be readily seen from (1.10) and (1.11) that

$$\sum_{k=0}^{n} p_{nk}\left(q;q^{-j}\right) = 1, \qquad \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} p_{nk}\left(q;q^{-j}\right) = q^{-j}, \tag{2.36}$$

while $p_{nk}(q;q^{-j}) \geq 0$. By virtue of Jensen's inequality, if f is convex on $[0,1]$, then whenever $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in [a,b]$, there holds the following:

$$\sum_{k=0}^{n} \lambda_k f(x_k) \geq f\left(\sum_{k=0}^{n} \lambda_k x_k\right). \tag{2.37}$$

for all $\lambda_0, \lambda_1, \ldots, \lambda_n \geq 0$ satisfying $\sum_{k=0}^{n} \lambda_k = 1$. Setting

$$x_k = \frac{[k]_q}{[n]_q}, \quad \lambda_k = p_{nk}\left(q;q^{-j}\right), \quad k = 0, 1, \ldots, n, \tag{2.38}$$

and observing that

$$\sum_{k=0}^{n} \lambda_k f(x_k) = B_{n,q}\left(f;q^{-j}\right), \tag{2.39}$$

the required result is derived. □

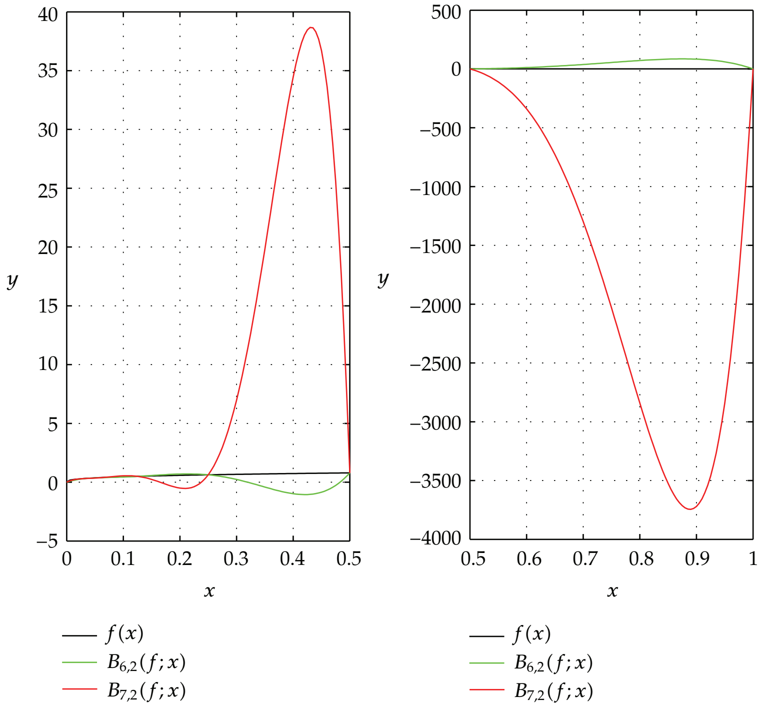

Figure 2: Graphs of $y = f(x)$ and $y = B_{n,2}(f;x)$, $n = 6, 7$.

Example 2.10. Let

$$f(x) = \begin{cases} q^2 x & \text{if } 0 \le x \le q^{-2}, \\ q^2 x - \dfrac{(q^2 x - 1)^2}{q^2 - 1} & \text{if } q^{-2} < x \le 1. \end{cases} \tag{2.40}$$

The function is concave on $[0,1]$ and, hence, according to the previous results, $B_{n,q}(f;q^{-j}) \to f(q^{-j})$ as $n \to \infty$ from below for all $j \in \mathbb{Z}_+$. To examine the behavior of polynomials $B_{n,q}(f;x)$ for $x \notin \mathbb{J}_q$, consider the auxiliary function:

$$g(x) = f(x) - q^2 x = \begin{cases} 0 & \text{if } 0 \le x \le q^{-2}, \\ -\dfrac{(q^2 x - 1)^2}{q^2 - 1} & \text{if } q^{-2} < x \le 1. \end{cases} \tag{2.41}$$

Since $[n-k]_q/[n]_q \le q^{-k}$ for $k = 0, 1, \ldots, n$, and $[n-1]_q/[n]_q \ge q^{-2}$ whenever $q^n \ge q+1$, it follows that, for sufficiently large n,

$$B_{n,q}(g;x) = g\left(\frac{[n-1]_q}{[n]_q}\right) p_{n,n-1}(q;x) + g(1) p_{nn}(x). \tag{2.42}$$

Table 1: The values of $E(n,q,x) = B_{n,q}(f;x) - f(x)$ at some points $x \in [0,1]$.

x	$E(3,2,x)$	$E(4,2,x)$	$E(10,2,x)$	$E(15,2,x)$	$E(20,2,x)$	$E(25,2,x)$	$E(30,2,x)$
$(q+1)/2q$	-8.76×10^{-2}	0.336	7.07×10^{9}	-6.08×10^{26}	1.76×10^{51}	-1.71×10^{83}	5.55×10^{122}
$1/q$	-9.00×10^{-3}	-4.02×10^{-3}	-5.70×10^{-5}	-1.78×10^{-6}	-5.56×10^{-8}	-1.74×10^{-9}	-5.43×10^{-11}
$(q+1)/2q^2$	9.88×10^{-3}	-4.63×10^{-2}	-1.15×10^{7}	3.09×10^{22}	-2.79×10^{45}	8.47×10^{75}	-8.62×10^{113}
$1/q^2$	-2.39×10^{-2}	-9.99×10^{-3}	-1.36×10^{-4}	-4.23×10^{-6}	-1.32×10^{-7}	-4.13×10^{-9}	-1.29×10^{-10}
$(q+1)/2q^3$	-6.66×10^{-2}	1.39×10^{-3}	4.90×10^{4}	-4.09×10^{18}	1.16×10^{40}	-1.09×10^{69}	3.48×10^{105}
$1/q^3$	-0.126	-2.05×10^{-2}	-2.52×10^{-4}	-7.84×10^{-6}	-2.45×10^{-7}	-7.66×10^{-9}	-2.39×10^{-10}
$(q+1)/2q^4$	-0.159	-5.06×10^{-2}	$-465.$	1.21×10^{15}	-1.06×10^{35}	3.15×10^{62}	-3.13×10^{97}
$1/q^4$	-0.190	-9.47×10^{-2}	-4.29×10^{-4}	-1.33×10^{-5}	-4.17×10^{-7}	-1.30×10^{-8}	-4.07×10^{-10}
$(q+1)/2q^5$	-0.201	-0.12	9.37	-7.49×10^{11}	2.06×10^{30}	-1.91×10^{56}	5.93×10^{89}
$1/q^5$	-0.206	-0.145	-7.07×10^{-4}	-2.19×10^{-5}	-6.84×10^{-7}	-2.14×10^{-8}	-6.68×10^{-10}
$(q+1)/2q^6$	-0.204	-0.155	-0.393	9.54×10^{8}	-8.21×10^{25}	2.37×10^{50}	-2.30×10^{82}
$1/q^6$	-0.194	-0.16	-1.15×10^{-3}	-3.53×10^{-5}	-1.10×10^{-6}	-3.45×10^{-8}	-1.08×10^{-9}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1/q^{19}$	-1.24×10^{-2}	-1.24×10^{-2}	-1.22×10^{-2}	-1.05×10^{-2}	-5.43×10^{-4}	-1.43×10^{-5}	-4.45×10^{-7}
$(q+1)/2q^{20}$	-1.13×10^{-2}	-1.13×10^{-2}	-1.11×10^{-2}	-9.82×10^{-3}	-1.21×10^{-3}	1.27	-2.23×10^{10}
$1/q^{20}$	-9.84×10^{-3}	-9.84×10^{-3}	-9.75×10^{-3}	-8.87×10^{-3}	-2.24×10^{-3}	-2.28×10^{-5}	-7.06×10^{-7}
$(q+1)/2q^{21}$	-8.94×10^{-3}	-8.94×10^{-3}	-8.87×10^{-3}	-8.22×10^{-3}	-2.86×10^{-3}	-0.102	2.91×10^{7}
$1/q^{21}$	-7.81×10^{-3}	-7.81×10^{-3}	-7.76×10^{-3}	-7.33×10^{-3}	-3.49×10^{-3}	-5.51×10^{-2}	-1.12×10^{-6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1/q^{49}$	-1.21×10^{-5}	-1.21×10^{-5}	-1.21×10^{-5}	-1.21×10^{-5}	-1.21×10^{-5}	-1.21×10^{-5}	-1.21×10^{-5}
$(q+1)/2q^{50}$	-1.10×10^{-5}	-1.10×10^{-5}	-1.10×10^{-5}	-1.10×10^{-5}	-1.10×10^{-5}	-1.10×10^{-5}	-1.10×10^{-5}
$1/q^{50}$	-9.61×10^{-6}	-9.61×10^{-6}	-9.61×10^{-6}	-9.61×10^{-6}	-9.61×10^{-6}	-9.61×10^{-6}	-9.61×10^{-6}

Plain computations reveal

$$g\left(\frac{[n-1]_q}{[n]_q}\right)=-\frac{(q-1)\left(q^n-q-1\right)^2}{\left(q^n-1\right)^2(q+1)}, \tag{2.43}$$

yielding

$$B_{n,q}(g;x)=-\frac{\left(q^n-q-1\right)^2}{\left(q^n-1\right)(q+1)}x^{n-1}(1-x)-\left(q^2-1\right)x^n. \tag{2.44}$$

Consequently, for $x \notin \mathbb{J}_q$, one obtains

$$\lim_{n\to\infty} B_{n,q}(g;x)=\begin{cases} 0 & \text{if } |x|<q^{-1}, \\ \infty & \text{if } |x|>q^{-1}. \end{cases} \tag{2.45}$$

Since, by (1.10), $B_{n,q}(f;x)=q^2x+B_{n,q}(g;x)$, it follows that:

$$\lim_{n\to\infty} B_{n,q}(f;x)=\begin{cases} q^2x & \text{if } |x|<q^{-1}, \\ \infty & \text{if } |x|>q^{-1},\ x\neq 1, \\ \dfrac{q^2+1}{q+1} & \text{if } x=q^{-1}, \\ 1 & \text{if } x=1. \end{cases} \tag{2.46}$$

For $x=-q^{-1}$, the limit does not exist. Additionally, it is not difficult to see that $B_{n,q}(f;x)\to f(x)$ as $n\to\infty$ uniformly on any compact set inside $(-1/q^2,1/q^2)$, while on any interval outside of $(-1/q^2,1/q^2)$, the function $f(x)$ is not approximated by its q-Bernstein polynomials. This agrees with the result from [17], Theorem 2.3. The graphs of $f(x)$ and $B_{n,q}(f;x)$ for $q=2$, $n=5$ and 8 on $[0,1]$ are given in Figure 3. The values of the error function at some points $x\in\mathbb{J}_q$ and at some exemplary points $x\notin\mathbb{J}_q$ are given in Table 2.

Remark 2.11. Following Charalambides [9], consider a sequence of random variables $\{X_n^{(j)}\}_{n=1}^{\infty}$ possessing the distributions $P_n^{(j)}$ given by

$$\mathbf{P}\left\{X_n^{(j)}=\frac{[n-k]_q}{[n]_q}\right\}=p_{n,n-k}\left(q;q^{-j}\right),\quad k=0,1,\dots,n. \tag{2.47}$$

Let $I(q^{-j})$ denote a random variable with the δ-distribution concentrated at q^{-j}. Theorem 2.1 implies that $X_n^{(j)}\to I(q^{-j})$ in distribution.

Generally speaking, Theorem 2.1 shows that the q-Bernstein polynomials with $q>1$ possess an "interpolation-type" property on $\mathbb{J}_q$. Information on interpolation of functions with nodes on a geometric progression can be found in, for example, [18] by Schoenberg.

Table 2: The values of $E(n,q,x) = B_{n,q}(f;x) - f(x)$ at some points $x \in [0,1]$.

x	$E(3,2,x)$	$E(4,2,x)$	$E(10,2,x)$	$E(15,2,x)$	$E(20,2,x)$	$E(25,2,x)$	$E(30,2,x)$
$(5q+1)/6q$	-2.38×10^{-2}	1.11×10^{-2}	-11.8	$-268.$	-5.57×10^{3}	-1.15×10^{5}	-2.39×10^{6}
$(11q+7)/18q$	-7.23×10^{-2}	1.07×10^{-3}	-8.13	$-101.$	-1.12×10^{3}	-1.21×10^{4}	-1.32×10^{5}
$(q+1)/2q$	-9.97×10^{-2}	-1.20×10^{-2}	-5.21	-47.4	$-368.$	-2.80×10^{3}	-2.13×10^{4}
$(5q+13)/18q$	-0.151	-4.69×10^{-2}	-1.4	-6.64	-24.6	-85.6	$-293.$
$(q+5)/6q$	-0.172	-6.54×10^{-2}	-0.528	-1.81	-4.6	-10.6	-23.7
$1/q$	-0.19	-8.89×10^{-2}	-1.30×10^{-3}	-4.07×10^{-5}	-1.27×10^{-6}	-3.97×10^{-8}	-1.24×10^{-9}
$(q+1)/2q^2$	-0.180	-9.98×10^{-2}	0.052	7.59×10^{-2}	8.16×10^{-2}	8.29×10^{-2}	8.32×10^{-2}
$1/q^2$	-0.103	-5.57×10^{-2}	-9.75×10^{-4}	-3.05×10^{-5}	-9.54×10^{-7}	-2.98×10^{-8}	-9.31×10^{-10}
$(q+1)/2q^3$	-5.38×10^{-2}	-2.38×10^{-2}	-7.92×10^{-5}	-5.89×10^{-7}	-4.37×10^{-9}	-3.24×10^{-11}	-2.40×10^{-13}
$1/q^3$	-2.21×10^{-2}	-7.15×10^{-3}	-2.22×10^{-6}	-2.17×10^{-9}	-2.12×10^{-12}	-2.07×10^{-15}	-2.02×10^{-18}
$(q+1)/2q^4$	-1.20×10^{-2}	-3.04×10^{-3}	-1.72×10^{-7}	-4.01×10^{-11}	-9.29×10^{-15}	-2.15×10^{-18}	-4.99×10^{-22}
$1/q^4$	-5.09×10^{-3}	-9.05×10^{-4}	-4.64×10^{-9}	-1.42×10^{-13}	-4.34×10^{-18}	-1.32×10^{-22}	-4.04×10^{-27}
$(q+1)/2q^5$	-2.8×10^{-3}	-3.83×10^{-4}	-3.54×10^{-10}	-2.57×10^{-15}	-1.86×10^{-20}	-1.35×10^{-25}	-9.78×10^{-31}
$1/q^5$	-1.22×10^{-3}	-1.14×10^{-4}	-9.35×10^{-12}	-8.96×10^{-18}	-8.55×10^{-24}	-8.15×10^{-30}	-7.77×10^{-36}
$(q+1)/2q^6$	-6.77×10^{-4}	-4.81×10^{-5}	-7.08×10^{-13}	-1.61×10^{-19}	-3.64×10^{-26}	-8.25×10^{-33}	-1.87×10^{-39}
$1/q^6$	-2.98×10^{-4}	-1.43×10^{-5}	-1.86×10^{-14}	-5.56×10^{-22}	-1.66×10^{-29}	-4.94×10^{-37}	-1.47×10^{-44}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1/q^{19}$	-4.33×10^{-12}	-2.61×10^{-17}	-1.13×10^{-49}	-9.21×10^{-77}	-7.44×10^{-104}	-6.01×10^{-131}	-4.86×10^{-158}
$(q+1)/2q^{20}$	-2.44×10^{-12}	-1.10×10^{-17}	-8.52×10^{-51}	-1.64×10^{-78}	-3.15×10^{-106}	-6.03×10^{-134}	-1.16×10^{-161}
$1/q^{20}$	-1.08×10^{-12}	-3.26×10^{-18}	-2.22×10^{-52}	-5.62×10^{-81}	-1.42×10^{-109}	-3.58×10^{-138}	-9.04×10^{-167}
$(q+1)/2q^{21}$	-6.09×10^{-13}	-1.37×10^{-18}	-1.66×10^{-53}	-1.00×10^{-82}	-6.0×10^{-112}	-3.60×10^{-141}	-2.15×10^{-170}
$1/q^{21}$	-2.71×10^{-13}	-4.07×10^{-19}	-4.33×10^{-55}	-3.43×10^{-85}	-2.71×10^{-115}	-2.14×10^{-145}	-1.68×10^{-175}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1/q^{49}$	-3.76×10^{-30}	-2.11×10^{-44}	-5.98×10^{-131}	-3.4×10^{-203}	-1.93×10^{-275}	-1.09×10^{-347}	-6.17×10^{-420}
$(q+1)/2q^{50}$	-2.11×10^{-30}	-8.88×10^{-45}	-4.49×10^{-132}	-6.06×10^{-205}	-8.14×10^{-278}	-1.09×10^{-350}	-1.47×10^{-423}
$1/q^{50}$	-9.39×10^{-31}	-2.63×10^{-45}	-1.17×10^{-133}	-2.08×10^{-207}	-3.67×10^{-281}	-6.50×10^{-355}	-1.15×10^{-428}

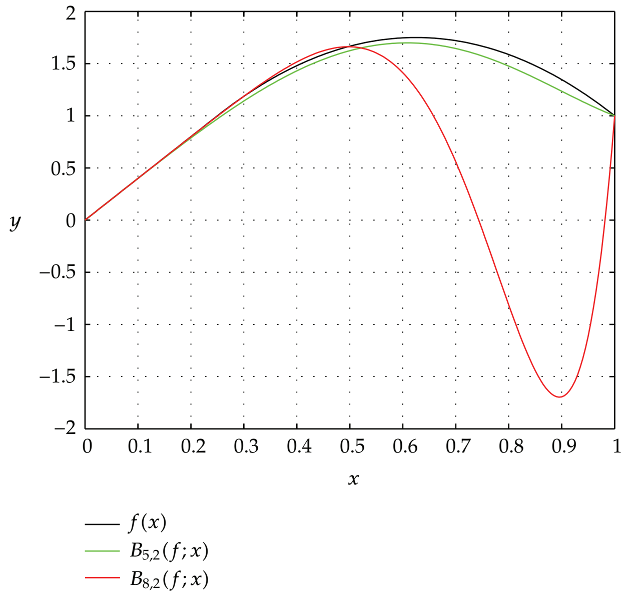

Figure 3: Graphs of $y = f(x)$ and$y = B_{n,2}(f; x)$, $n = 5, 8$.

3. On the q-Bernstein Polynomials of the Weierstrass-Type Functions

In this section, the q-Bernstein polynomials of the functions with "bad" smoothness are considered. Let $\varphi(x) \in C[-1, 1]$ satisfy the condition:

$$\varphi(0) > \varphi(x) \quad \text{for } x \in [-1, 1] \setminus \{0\}. \tag{3.1}$$

The letter φ will also denote a 2-periodic continuation of $\varphi(x)$ on $(-\infty, \infty)$.

Definition 3.1. Let $a, b \in \mathbb{R}$ satisfy $0 < a < 1 < ab$. A function $f(x)$ is said to be *Weierstrass-type* if

$$f(x) = \sum_{k=0}^{\infty} a^k \varphi\left(b^k x\right). \tag{3.2}$$

Notice that $f(x)$ is continuous if and only if $\varphi(-1) = \varphi(1)$. For $\varphi(x) = \cos \pi x$ and a special choice of a and b (see, e.g., [19, Section 4]), the classical Weierstrass continuous nowhere differentiable function is obtained. In [19], one can also find an exhaustive bibliography on this function and similar ones. For $\varphi(x) = 1 - |x|$, a function analogous to the Van der Waerden continuous nowhere differentiable function appears.

The aim of this section is to prove the following statement.

Theorem 3.2. *If $f(x)$ is a Weierstrass-type function, then the sequence $B_{n,q}(f; x)$ of its q-Bernstein polynomials is not uniformly bounded on any interval $[0, c]$.*

Proof. To prove the theorem, the following representation of q-Bernstein polynomials (see [15], formulae (6) and (7)) is used:

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \lambda_{kn} f\left[0; \frac{1}{[n]_q}; \dots; \frac{[k]_q}{[n]_q}\right] x^k, \tag{3.3}$$

where

$$\lambda_{0n} = \lambda_{1n} = 1, \qquad \lambda_{kn} = \left(1 - \frac{1}{[n]_q}\right) \cdots \left(1 - \frac{[k-1]_q}{[n]_q}\right), \quad k = 2, \dots, n, \tag{3.4}$$

and $f[x_0; x_1; \dots; x_k]$ denote the divided differences of f, that is,

$$\begin{gathered} f[x_0] = f(x_0), \qquad f[x_0; x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \dots, \\ f[x_0; x_1; \dots; x_k] = \frac{f[x_1; \dots; x_k] - f[x_0; \dots; x_{k-1}]}{x_k - x_0}. \end{gathered} \tag{3.5}$$

When $q = 1$, the well-known representation for the classical Bernstein polynomials is recovered and the numbers λ_{kn} are the eigenvalues of the Bernstein operator, see [20], Chapter 4, Section 4.1 and [21]. The latter result has been extended to the case $q \neq 1$ in [15].

Clearly, it suffices to consider the case $0 < c < 1$. From (3.3), it follows that

$$B'_{n,q}(f;0) = \lambda_{1n} f\left[0; \frac{1}{[n]_q}\right] = [n]_q \left\{ f\left(\frac{1}{[n]_q}\right) - f(0) \right\}, \tag{3.6}$$

and, hence,

$$\left|B'_{n,q}(f;0)\right| = [n]_q \left\{ f(0) - f\left(\frac{1}{[n]_q}\right) \right\} = [n]_q \sum_{k=0}^{\infty} a^k \left\{ \varphi(0) - \varphi\left(\frac{b^k}{[n]_q}\right) \right\}. \tag{3.7}$$

What remains is to find a lower bound for $|B'_{n,q}(f;0)|$. Due to (3.1), all terms of the series are nonnegative and, therefore,

$$\left|B'_{n,q}(f;0)\right| \geq [n]_q a^j \left\{ \varphi(0) - \varphi\left(\frac{b^j}{[n]_q}\right) \right\} \qquad \text{for any } j = 0, 1, \dots \tag{3.8}$$

Let $j = j_n$ be chosen in such a way that

$$\frac{1}{b} < \frac{b^{j_n}}{[n]_q} \leq 1. \tag{3.9}$$

For $n > b$, such a choice is possible because, in this case, inequality (3.9) implies that

$$0 < \frac{\ln [n]_q}{\ln b} - 1 < j_n \leq \frac{\ln [n]_q}{\ln b}. \tag{3.10}$$

Since the length of the interval $(\ln [n]_q / \ln b - 1, \ln [n]_q / \ln b]$ is 1, there is a positive integer, say, j_n, such that $j_n \in (\ln [n]_q / \ln b - 1, \ln [n]_q / \ln b]$. The obvious inequality $[n]_q > q^{n-1}$ implies the following:

$$j_n \geq (n-1)\frac{\ln q}{\ln b} - 1 =: n\frac{\ln q}{\ln b} - A, \tag{3.11}$$

with $A = (\ln q / \ln b) + 1$ being a positive constant. Then, for $n > b$, it follows that

$$\left|B'_{n,q}(f;0)\right| \geq [n]_q \ a^{j_n} \min_{t\in[1/b,1]} \{\varphi(0) - \varphi(t)\} := \tau [n]_q a^{j_n}, \tag{3.12}$$

where $\tau > 0$ due to (3.1). Consequently,

$$\left|B'_{n,q}(f;0)\right| \geq \tau \frac{[n]_q}{b^{j_n}} (ab)^{j_n} \geq \tau (ab)^{j_n} \geq \tau (ab)^{n(\ln q / \ln b) - A}, \tag{3.13}$$

which leads to

$$\left|B'_{n,q}(f;0)\right| \geq C\rho^n, \tag{3.14}$$

where $C = \tau (ab)^{-A}$ is a positive constant and $\rho = (ab)^{(\ln q / \ln b)} > 1$. Now, assume that $\{B_{n,q}(f;x)\}$ is uniformly bounded on $[0,c]$, that is, $|B_{n,q}(f;x)| \leq M$ for all $x \in [0,c]$. By Markov's Inequality (cf., e.g., [22], Chapter 4, Section 1, pp. 97-98) it follows that

$$\left|B'_{n,q}(f;0)\right| \leq \frac{2M}{c} n^2 \ \forall n = 1,2,\ldots, \tag{3.15}$$

This proves the theorem because the latter estimate contradicts (3.14). □

To present an illustrative example, let us denote the Nth partial sum of the series in (3.2) by h_N, that is:

$$h_N(x) = \sum_{k=0}^{N} a^k \varphi\left(b^k x\right). \tag{3.16}$$

Clearly, the function h_N is an approximation of (3.2) satisfying the error estimate

$$E_N(x) = \left|f(x) - h_N(x)\right| \leq \max_{t\in[-1,1]} \left|\varphi(t)\right| \frac{a^{N+1}}{1-a}, \quad \forall x \in [0,1]. \tag{3.17}$$

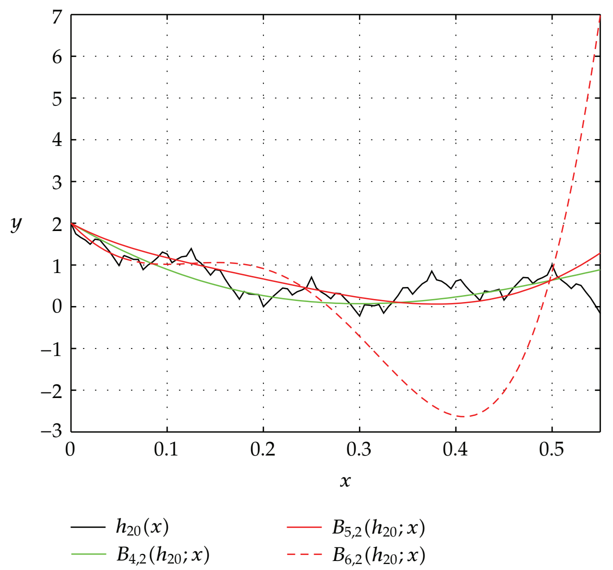

Figure 4: Graphs of $y = h_{20}(x)$ and $y = B_{n,2}(f;x)$, $n = 4, 5, 6$.

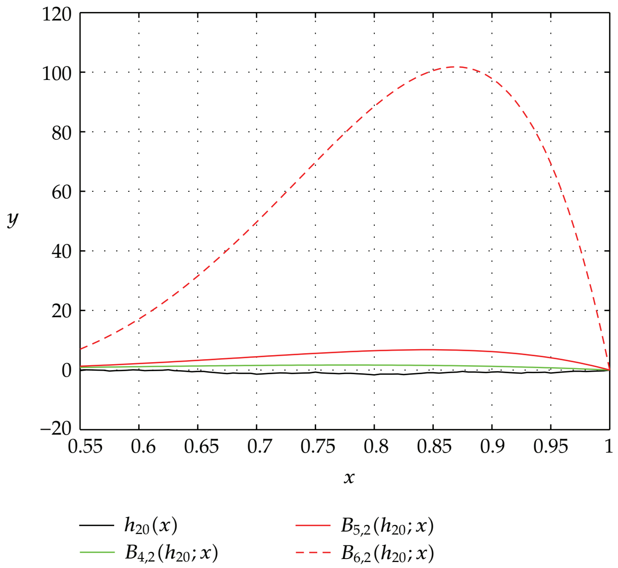

Figure 5: Graphs of $y = h_{20}(x)$ and $y = B_{n,2}(f;x)$, $n = 4, 5, 6$.

Example 3.3. Let $\varphi(x) = (\cos \pi x)$, $a = 1/2$, and $b = 4$. For $N = 20$, one has $E_{20}(x) \le 10^{-6}$. The graphs of $h_{20}(x)$ and the associated q-Bernstein polynomials $B_{n,q}(h_{20};x)$ for $q = 2$, $n = 4, 5$, and 6 on the subintervals $[0, 0.55]$ and $[0.55, 1]$ are presented in Figures 4 and 5, respectively.

Acknowledgment

The authors would like to express their sincere gratitude to Mr. P. Danesh from Atilim University Academic Writing and Advisory Centre for his help in the preparation of the paper.

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, vol. 71, Cambridge University Press, Cambridge, UK, 1999.

[2] S. Ostrovska, "The first decade of the q-Bernstein polynomials: results and perspectives," *Journal of Mathematical Analysis and Approximation Theory*, vol. 2, no. 1, pp. 35–51, 2007.

[3] N. Mahmudov, "The moments of q-Bernstein operators in the case $0 < q < 1$," *Numerical Algorithms*, vol. 53, no. 4, pp. 439–450, 2010.

[4] M. Popov, "Narrow operators (a survey)," in *Function Spaces IX*, vol. 92, pp. 299–326, Banach Center Publications, 2011.

[5] P. Sabancıgil, "Higher order generalization of q-Bernstein operators," *Journal of Computational Analysis and Applications*, vol. 12, no. 4, pp. 821–827, 2010.

[6] H. Wang, "Properties of convergence for ω, q-Bernstein polynomials," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 2, pp. 1096–1108, 2008.

[7] Z. Wu, "The saturation of convergence on the interval $[0, 1]$ for the q-Bernstein polynomials in the case $q > 1$," *Journal of Mathematical Analysis and Applications*, vol. 357, no. 1, pp. 137–141, 2009.

[8] X.-M. Zeng, D. Lin, and L. Li, "A note on approximation properties of q-Durrmeyer operators," *Applied Mathematics and Computation*, vol. 216, no. 3, pp. 819–821, 2010.

[9] C. A. Charalambides, "The q-Bernstein basis as a q-binomial distribution," *Journal of Statistical Planning and Inference*, vol. 140, no. 8, pp. 2184–2190, 2010.

[10] S. C. Jing, "The q-deformed binomial distribution and its asymptotic behaviour," *Journal of Physics A*, vol. 27, no. 2, pp. 493–499, 1994.

[11] I. Ya. Novikov and S. B. Stechkin, "Fundamentals of wavelet theory," *Russian Mathematical Surveys*, vol. 53, no. 6, pp. 53–128, 1998.

[12] M. I. Ostrovskiĭ, "Regularizability of inverse linear operators in Banach spaces with a basis," *Siberian Mathematical Journal*, vol. 33, no. 3, pp. 470–476, 1992.

[13] V. S. Videnskii, "On some classes of q-parametric positive linear operators," in *Selected Topics in Complex Analysis*, vol. 158, pp. 213–222, Birkhäuser, Basel, Switzerland, 2005.

[14] A. Il'inskii and S. Ostrovska, "Convergence of generalized Bernstein polynomials," *Journal of Approximation Theory*, vol. 116, no. 1, pp. 100–112, 2002.

[15] S. Ostrovska, "q-Bernstein polynomials and their iterates," *Journal of Approximation Theory*, vol. 123, no. 2, pp. 232–255, 2003.

[16] S. Ostrovska and A. Y. Özban, "The norm estimates of the q-Bernstein operators for varying $q > 1$," *Computers & Mathematics with Applications*, vol. 62, no. 12, pp. 4758–4771, 2011.

[17] S. Ostrovska, "On the approximation of analytic functions by the q-Bernstein polynomials in the case $q > 1$," *Electronic Transactions on Numerical Analysis*, vol. 37, pp. 105–112, 2010.

[18] I. J. Schoenberg, "On polynomial interpolation at the points of a geometric progression," *Proceedings of the Royal Society of Edinburgh*, vol. 90, no. 3-4, pp. 195–207, 1981.

[19] A. Pinkus, "Weierstrass and approximation theory," *Journal of Approximation Theory*, vol. 107, no. 1, pp. 1–66, 2000.

[20] G. G. Lorentz, *Bernstein Polynomials*, Chelsea, New York, NY, USA, 2nd edition, 1986.

[21] S. Cooper and S. Waldron, "The eigenstructure of the Bernstein operator," *Journal of Approximation Theory*, vol. 105, no. 1, pp. 133–165, 2000.

[22] R. A. DeVore and G. G. Lorentz, *Constructive Aapproximation*, vol. 303, Springer, Berlin, Germany, 1993.