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Research Article

Threshold Effects for the Generalized Friedrichs Model with the Perturbation of Rank One

Saidakhmat Lakaev,[1] Arsmah Ibrahim,[2] and Shaxzod Kurbanov[1]

[1] *Department of Mathematics, Samarkand State University, University Boulevard 15, Samarkand 140104, Uzbekistan*
[2] *Department of Mathematics, Faculty of Computer science and Mathematics, MARA University Technology, 40450 Shah Alam, Selangor Darul Ehsan, Malaysia*

Correspondence should be addressed to Saidakhmat Lakaev, slakaev@mail.ru

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A family $H_\mu(p)$, $\mu > 0$, $p \in \mathbb{T}^2$ of the Friedrichs models with the perturbation of rank one, associated to a system of two particles, moving on the two-dimensional lattice $\mathbb{Z}^2$ is considered. The existence or absence of the unique eigenvalue of the operator $H_\mu(p)$ lying below threshold depending on the values of $\mu > 0$ and $p \in U_\delta(0) \subset \mathbb{T}^2$ is proved. The analyticity of corresponding eigenfunction is shown.

1. Introduction

In celebrated work [1] of Simon and Klaus it has been considered a family of the Schrödinger operators $H = -\Delta + \mu V$ and, a situation where as μ tends to μ_0 some eigenvalue $e_i(\mu)$ tends to 0, that is, as $\mu \downarrow \mu_0$ an eigenvalue is absorbed into continuous spectrum, and conversely, for any $\mu : \mu > \mu_0$ continuous spectrum *gives birth* to a new eigenvalue. This phenomenon in [1] is called *coupling constant threshold.*

In [2] the Hamiltonian of a system of two identical quantum mechanical particles (bosons) moving on the d-dimensional lattice $\mathbb{Z}^d$, $d \geq 3$ and interacting via zero-range repulsive pair potentials has been considered. For the associated two-particle Schrödinger operator $H_\mu(k)$, $k \in \mathbb{T}^d = (-\pi, \pi]^d$ the existence of *coupling constant threshold* $\mu_0 = \mu_0(k) > 0$ has been proved: the operator has none eigenvalue for any $0 < \mu \leq \mu_0$, but for each $\mu > \mu_0$ it has a unique eigenvalue $z(\mu, k)$ above the upper threshold of the spectrum.

Note that in [1] the existence of a coupling constant threshold has been assumed, at the same time in [2] the *coupling constant threshold* has been definitely found by the given data of the Hamiltonian.

We remark that for the Hamiltonians of a system of two arbitrary particles moving on $\mathbb{R}^d$ or $\mathbb{Z}^d$, $d \geq 1$ the *coupling constant threshold* vanishes, if $d = 1, 2$ and the *coupling constant threshold* is positive, if $d \geq 3$.

Notice also that for the Hamiltonians of a system of two identical particles moving on $\mathbb{R}^d$ or $\mathbb{Z}^d$, $d = 1, 2$ the *coupling constant threshold* vanishes, if particles are bosons and the *coupling constant threshold* is positive, if particles are fermions.

In [3] for a wide class of the two-particle Schrödinger operators $H_\mu(k)$ on the d-dimensional lattice $\mathbb{Z}^d$, $d \geq 3$, k being the two-particle quasimomentum, it has been proved that if the following two assumptions (i) and (ii) are satisfied, then for all $k \neq 0$, the discrete spectrum of $H_\mu(k)$ below its threshold is nonempty. The assumptions are (i) the two-particle Schrödinger operator $H_\mu(0)$, corresponding to the zero value of the quasimomentum k, has *a coupling constant threshold* $\mu_0(0) > 0$ and (ii) the one-particle free Hamiltonians in the coordinate representation generate positivity preserving semigroups.

In [4] a family of the Friedrichs models $H_\mu(p)$, $\mu > 0$, $p \in (-\pi, \pi]^3$ with perturbation of rank one associated to a system of two particles on the three-dimensional lattice $\mathbb{Z}^3$ has been considered. In some special case of the multiplication operator and under the assumption that the operator $H_\mu(0)$, $0 \in \mathbb{T}^3$ has a *coupling constant threshold* $\mu_0(0) > 0$, the existence of a unique eigenvalue, below the threshold of the spectrum of $H_{\mu_0(0)}(p)$, $p \in (-\pi, \pi]^3$ for all nontrivial values of $p \in \mathbb{T}^3$, has been proved.

In the present paper, a family of the Friedrichs models $H_\mu(p)$, $\mu > 0$, $p \in U_\delta(0) \subset \mathbb{T}^2$, where $U_\delta(0)$ is a δ-neighborhood of the point $p = 0 \in \mathbb{T}^2$ with perturbation of rank one associated to a system of two particles on the two-dimensional lattice $\mathbb{Z}^2$ interacting via pair local potentials, is considered and the following results have been obtained.

(i) If the parameters of the Friedrichs model satisfy some conditions (see Theorem 2.3), then there exists *a coupling constant threshold* $\mu_0 = \mu_0(p) > 0$: for any $0 < \mu \leq \mu_0(p)$ the operator has none eigenvalue; at the same time for any $\mu > \mu_0(p)$ it has a unique eigenvalue $z(\mu, p)$, lying below its threshold of the spectrum. Moreover an explicit expression for the corresponding eigenfunction is found and its analyticity is proven.

(ii) If the parameters of the Friedrichs model do not satisfy conditions mentioned in (i), then the operator has none positive *coupling constant threshold,* that is, for any $\mu > 0$ the operator $H_\mu(p)$ has a unique eigenvalue $z(\mu, p)$, lying below its threshold of the spectrum.

(iii) A criterion for being the threshold $m(p)$, $p \in U_\delta(0)$ of the spectrum of $H_\mu(p)$ a virtual level of the operator $H_\mu(p)$ is proven.

Note that the generalized Friedrichs models appear in the problems of quantum mechanics [5], solid state physics [6], and quantum field theory [7, 8] and the existence of its eigenvalues and resonances have been studied in [9, 10].

In [11] a special family of generalized Friedrichs models has been considered and the existence of eigenvalues for some values of quasimomentum $p \in \mathbb{T}^d$ of the system, lying in a neighborhood of some $p_0 \in \mathbb{T}^d$, has been proved.

2. Notions and Assumptions: The Main Results

Let $\mathbb{Z}$ be the one-dimensional hypercubic lattice and $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2 = (-\pi, \pi]^2$ be the two-dimensional torus, the dual group of $\mathbb{Z}^2$ (Brillion zone). Note that operations addition and multiplication by number of the elements of torus $\mathbb{T}^2 \equiv (-\pi, \pi]^2 \subset \mathbb{R}^2$ is defined as operations in $\mathbb{R}^2$ by the module $(2\pi\mathbb{Z})^2$.

Let $L_2(\mathbb{T}^2)$ be the Hilbert space of square-integrable functions defined on the torus $\mathbb{T}^2$ and $\mathbf{C}^1$ be one-dimensional complex Hilbert space.

We consider the family of generalized Friedrichs model acting in $L_2(\mathbb{T}^2)$ as follows:

$$H_\mu(p) = H_0(p) - \mu\Phi^*\Phi, \quad \mu > 0. \tag{2.1}$$

Here

$$\begin{aligned} \Phi &: L_2\left(\mathbb{T}^2\right) \longrightarrow \mathbf{C}^1, \qquad \Phi f = (f, \varphi)_{L_2(\mathbb{T}^2)}, \\ \Phi^* &: \mathbf{C}^1 \longrightarrow L_2\left(\mathbb{T}^2\right), \qquad \Phi^* f_0 = \varphi(q) f_0, \end{aligned} \tag{2.2}$$

where $(\cdot, \cdot)_{L_2(\mathbb{T}^2)}$ is inner product in $L_2(\mathbb{T}^2)$ and $H_0(p)$, $p \in \mathbb{T}^2$ is the multiplication operator by function $w_p(\cdot) := w(p, \cdot)$, that is,

$$(H_0(p) f)(q) = w_p(q) f(q), \quad f \in L_2\left(\mathbb{T}^2\right). \tag{2.3}$$

Note that for any $f \in L_2(\mathbb{T}^2)$ and $g_0 \in \mathbf{C}^1$ the equality

$$(\Phi f, g_0)_{\mathbf{C}^1} = (f, \Phi^* g_0)_{L_2(\mathbb{T}^2)}, \tag{2.4}$$

holds.

The following assumption will be needed throughout the paper.

Assumption 2.1. The following conditions are satisfied:

(i) the function $\varphi(\cdot)$ is nontrivial real-analytic on $\mathbb{T}^2$;

(ii) the function $w(\cdot, \cdot)$ is real-analytic on $(\mathbb{T}^2)^2 = \mathbb{T}^2 \times \mathbb{T}^2$ and has a unique nondegenerated minimum at $(0, 0) \in (\mathbb{T}^2)^2$.

The perturbation $v = \Phi^*\Phi$ is positive operator of rank one. Consequently, by well-known Weyl's theorem [12], the essential spectrum of $H_\mu(p)$ fills the following segment on the real axis:

$$\sigma_{\text{ess}}(H_\mu(p)) = \sigma_{\text{ess}}(H_0(p)) = [m(p), M(p)], \tag{2.5}$$

where

$$m(p) = \min_{q \in \mathbb{T}^2} w_p(q), \qquad M(p) = \max_{q \in \mathbb{T}^2} w_p(q). \tag{2.6}$$

By Assumption 2.1 there exist such δ-neighborhood $U_\delta(0) \subset \mathbb{T}^2$ of the point $p = 0 \in \mathbb{T}^2$ and analytic vector function $q_0 : U_\delta(0) \to \mathbb{T}^2$ that for any $p \in U_\delta(0)$ the point $q_0(p) = (q_0^{(1)}(p), q_0^{(2)}(p)) \in \mathbb{T}^2$ is a unique nondegenerated minimum of the function $w_p(\cdot)$ (see Lemma 3.2).

Moreover, in the case $\varphi(q_0(p)) = 0$, $p \in U_\delta(0)$ the following integral

$$\int_{\mathbb{T}^2} \frac{\varphi^2(s)ds}{w_p(s) - m(p)} > 0, \tag{2.7}$$

exists (see Lemma 3.4) and we introduce a parameter $\mu(p)$ as

$$\frac{1}{\mu(p)} = \int_{\mathbb{T}^2} \frac{\varphi^2(s)ds}{w_p(s) - m(p)} > 0. \tag{2.8}$$

If $\varphi(q_0(p)) \neq 0$, $p \in U_\delta(0)$, then we define $\mu(p)$ as $\mu(p) = 0$.

Definition 2.2. The number $z = m(p)$ is called a virtual level of the operator $H_\mu(p)$, if the equation $H_\mu(p)f = m(p)f$ has a nonzero solution $f \in L_1(\mathbb{T}^2) \setminus L_2(\mathbb{T}^2)$, where $L_1(\mathbb{T}^2)$ is the Banach space of integrable functions on $\mathbb{T}^2$. The corresponding solution f is called a virtual state of the operator $H_\mu(p)$.

In the following theorem we assert that for any $\mu > \mu(p)$ there exists a unique eigenvalue $E(\mu, p)$, lying below the essential spectrum, of the operator $H_\mu(p)$, $p \in U_\delta(0)$, but for $0 < \mu \leq \mu(p)$, $p \in U_\delta(0)$ the operator $H_\mu(p)$ has none eigenvalue outside the essential spectrum. It is proved that for fixed $p \in U_\delta(0)$, the function $E(\cdot, p)$ is analytic in $(\mu(p), +\infty)$.

Moreover, this theorem provides a criterion, for being the bottom $m(p)$, $p \in U_\delta(0)$ of the essential spectrum of $H_\mu(p)$, a virtual level of the operator $H_\mu(p)$.

Theorem 2.3. *Let Assumption 2.1 holds and $p \in U_\delta(0)$. Then the following assertions are true.*

(i) *If $\mu > \mu(p)$, then the operator $H_\mu(p)$ has a unique eigenvalue $E(\mu, p)$, lying below the essential spectrum of $H_\mu(p)$. The function $E(\cdot, p)$ is monotonously decreasing real-analytic function in the interval $(\mu(p), +\infty)$ and the function $E(\mu, \cdot)$ is real-analytic in $U_\delta(0)$. The corresponding eigenfunction*

$$\Psi(\mu; p, \cdot, E(\mu, p)) = \frac{C\mu\varphi(\cdot)}{w_p(\cdot) - E(\mu, p)}, \tag{2.9}$$

is analytic on $\mathbb{T}^2$, where $C \neq 0$ is a normalizing constant. Moreover, the mappings

$$\begin{aligned} &\Psi_\mu : U_\delta(0) \longrightarrow L_2\left(\mathbb{T}^2\right), \qquad p \longmapsto \Psi(\mu; p, \cdot, E(\mu, p)) \in L_2\left(\mathbb{T}^2\right), \\ &\Psi_p : (\mu(p), +\infty) \longrightarrow L_2\left(\mathbb{T}^2\right), \qquad \mu \longmapsto \Psi(\mu; p, \cdot, E(\mu, p)) \in L_2\left(\mathbb{T}^2\right), \end{aligned} \tag{2.10}$$

are vector-valued analytic functions in $U_\delta(0)$ and $(\mu(p), +\infty)$, respectively.

(ii) *If* $\varphi(q_0(p)) = 0$ *and* $0 < \mu < \mu(p)$, *then the operator* $H_\mu(p)$ *has none eigenvalue in* $(-\infty, m(p)]$.

(iii) *If* $\varphi(q_0(p)) = 0, \nabla\varphi(q_0(p)) = ((\partial\varphi/\partial q_1)(q_0(p)), (\partial\varphi/\partial q_2)(q_0(p))) \neq 0$ *and* $\mu = \mu(p)$, *then the number* $z = m(p)$ *is a virtual level of the operator* $H_\mu(p)$ *and the corresponding virtual state is of the form:*

$$f_p(\cdot) = \frac{C\mu(p)\varphi(\cdot)}{w_p(\cdot) - m(p)} \in L_1\left(\mathbb{T}^2\right) \setminus L_2\left(\mathbb{T}^2\right), \tag{2.11}$$

where $C \neq 0$ *is a normalizing constant.*

(iv) *If* $\varphi(q_0(p)) = 0, \nabla\varphi(q_0(p)) = ((\partial\varphi/\partial q_1)(q_0(p)), (\partial\varphi/\partial q_2)(q_0(p))) = 0$ *and* $\mu = \mu(p)$, *then the number* $m(p) = w_p(q_0(p))$ *is an eigenvalue of the operator* $H_\mu(p)$ *and the corresponding eigenfunction is of the form*

$$f_p(q) = \frac{C\mu(p)\varphi(\cdot)}{w_p(\cdot) - m(p)} \in L_2\left(\mathbb{T}^2\right), \tag{2.12}$$

where $C \neq 0$ *is a normalizing constant.*

Remark 2.4. Notice that if $\varphi(q_0(p)) \neq 0$, then $\mu(p) = 0$. So, in this case the number $z = m(p)$ is neither a virtual level nor an eigenvalue for the operator $H_\mu(p)$.

Remark 2.5. From the positivity of $\Phi^*\Phi$ it follows that the operator $H_\mu(p)$ has none eigenvalue lying above $M(p)$.

3. Proof of the Results

We postpone the proof of the theorem after several lemmas and remarks.

For any $\mu > 0$ and $p \in \mathbb{T}^2$ we define in $\mathbb{C} \setminus [m(p); M(p)]$ an analytic function $\Delta(\mu, p; \cdot)$ (the Fredholm determinant $\Delta(\mu, p; \cdot)$, associated to the operator $H_\mu(p)$) as

$$\Delta(\mu, p; \cdot) = 1 - \mu\Omega(p; \cdot), \tag{3.1}$$

where

$$\Omega(p; z) = \int_{\mathbb{T}^2} \frac{\varphi^2(s)ds}{w_p(s) - z}, \quad p \in \mathbb{T}^2,\ z \in \mathbb{C} \setminus \left[m(p); M(p)\right]. \tag{3.2}$$

Lemma 3.1. *For any* $\mu \in (\mu(p), +\infty)$ *and* $p \in U_\delta(0)$ *the number* $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_\mu(p))$, $p \in \mathbb{T}^2$ *is an eigenvalue of the operator* $H_\mu(p)$ *if and only if, when*

$$\Delta(\mu, p; z) = 0. \tag{3.3}$$

The corresponding eigenfunction

$$f_{\mu,p}(\cdot) = \frac{C\mu\varphi(\cdot)}{w_p(\cdot) - z}, \tag{3.4}$$

is analytic on $\mathbb{T}^2$*, where* $C \neq 0$ *is a normalizing constant [4].*

Lemma 3.2. *Let Assumption 2.1 holds. Then there exist such a* δ*-neighborhood* $U_\delta(0) \subset \mathbb{T}^2$ *of the point* $p = 0$ *and analytic function* $q_0 : U_\delta(0) \to \mathbb{T}^2$ *that for any* $p \in U_\delta(0)$ *the point* $q_0(p)$ *is a unique non degenerated minimum of* $w_p(\cdot)$*.*

Proof. By Assumption 2.1, the square matrix

$$A(0) = \left(\frac{\partial^2 w_0}{\partial q_i \partial q_j}(0)\right)_{i,j=1}^{2} > 0, \tag{3.5}$$

is positively defined and $\nabla w_0(0) = 0$. Then by the implicit function theorem in analytic case there exist a δ-neighborhood $U_\delta(0) \subset \mathbb{T}^2$ of $p = 0 \in \mathbb{T}^2$ and a unique analytic vector function $q_0(\cdot) : U_\delta(0) \to \mathbb{T}^2$ such that $\nabla w_p(q_0(p)) = 0$ and

$$A(p) = \left(\frac{\partial^2 w_p}{\partial q_i \partial q_j}(q_0(p))\right)_{i,j=1}^{2} > 0, \quad p \in U_\delta(0). \tag{3.6}$$

Hence for any $p \in U_\delta(0)$ the point $q_0(p)$ is a unique non degenerated minimum of the function $w_p(\cdot)$. □

Remark 3.3. We note that by the parametrical Morse lemma for any $p \in U_\delta(0)$ there exists a map $s = \psi(y,p)$ of the sphere $W_\gamma(0) \subset \mathbb{R}^2$ with radius $\gamma > 0$ and center at $y = 0$ to a neighborhood $U(q_0(p))$ of the point $q_0(p)$ that in $U(q_0(p))$ the function $w_p(\psi(y,p))$ can be represented as

$$w_p(\psi(y,p)) = m(p) + y^2. \tag{3.7}$$

Here the function $\psi(y,\cdot)$ (resp., $\psi(\cdot,p)$) is analytic in $U_\delta(0)$ (resp., $W_\gamma(0)$) and $\psi(0,p) = q_0(p)$. Moreover, the Jacobian $J(\psi(y,p))$ of the mapping $s = \psi(y,p)$ is analytic in $W_\gamma(0)$ and positive, that is $J(\psi(y,p)) > 0$ for all $p \in U_\delta(0)$ and for all $y \in W_\gamma(0)$.

Lemma 3.4. *Let Assumption 2.1 holds. Then the integral*

$$\xi(p) = \int_{\mathbb{T}^2} \frac{\varphi^2(s) - \varphi^2(q_0(p))}{w_p(s) - m(p)} ds, \tag{3.8}$$

exists and defines an analytic function in $U_\delta(0)$*.*

Proof. We represent the function

$$\xi(p,z) = \int_{\mathbb{T}^2} \frac{\varphi^2(s) - \varphi^2(q_0(p))}{w_p(s) - z} ds, \tag{3.9}$$

as

$$\xi(p,z) = \xi_1(p,z) + \xi_2(p,z), \tag{3.10}$$

where

$$\begin{aligned} \xi_1(p,z) &= \int_{U(q_0(p))} \frac{\varphi^2(s) - \varphi^2(q_0(p))}{w_p(s) - z} ds, \\ \xi_2(p,z) &= \int_{\mathbb{T}^2 \setminus U(q_0(p))} \frac{\varphi^2(s) - \varphi^2(q_0(p))}{w_p(s) - z} ds, \end{aligned} \tag{3.11}$$

and $U(q_0(p))$ is a neighborhood of $q_0(p)$.

Observe that by Assumption 2.1 for any $p \in U_\delta(0)$ the function $\xi_2(p,z)$ is analytic at the point $z = m(p)$.

According to Remark 3.3 in the integral for $\xi_1(p,z)$ a change of variables $s = \psi(y,p)$ implies

$$\xi_1(p,z) = \int_{W_\gamma(0)} \frac{\varphi^2(\psi(y,p)) - \varphi^2(q_0(p))}{y^2 + m(p) - z} J(\psi(y,p)) dy, \tag{3.12}$$

where $J(\psi(y,p))$ is the Jacobian of the mapping $\psi(y,p)$.

Passing to spherical coordinates as $y = rv$, we obtain

$$\xi_1(p,z) = \int_0^\gamma \frac{r}{r^2 + m(p) - z} \left\{ \int_{\Omega_2} \left[\varphi^2(\psi(rv,p)) - \varphi^2(q_0(p)) \right] J(\psi(rv,p)) dv \right\} dr, \tag{3.13}$$

where Ω_2 is a unit sphere in $\mathbb{R}^2$ and dv its element. Inner integral can be represented as

$$\int_{\Omega_2} \left[\varphi^2(\psi(rv,p)) - \varphi^2(q_0(p)) \right] J(\psi(rv,p)) dv = \sum_{n=1}^{\infty} \tau_n(p) r^{2n}, \tag{3.14}$$

where the Pizetti coefficients $\tau_n(p)$, $n = 1, 2, \ldots$ are analytic in $U_\delta(0)$ [13].

Thus we have that

$$\xi_1(p,z) = \sum_{n=1}^{\infty} \tau_n(p) \int_0^\gamma \frac{r^{2n+1} dr}{r^2 + m(p) - z}. \tag{3.15}$$

From (3.15) it follows that the following limit exists

$$\xi_1(p)=\lim_{z\to m(p)-0}\xi_1(p,z)=\lim_{z\to m(p)-0}\sum_{n=1}^{\infty}\tau_n(p)\int_0^{\gamma}\frac{r^{2n+1}dr}{r^2+m(p)-z}=\sum_{n=1}^{\infty}\frac{\gamma^{2n}}{2n}\tau_n(p), \tag{3.16}$$

and, consequently,

$$\xi(p)=\lim_{z\to m(p)-0}\xi(p,z)=\xi_1(p)+\xi_2(p), \tag{3.17}$$

where $\xi_2(p)=\xi_2(p,m(p))$. Observe that the functions in the right hand side of (3.17) are analytic in $p\in U_\delta(0)$. So, the function $\xi(p)$ is analytic in $p\in U_\delta(0)$. □

Proposition 3.5. *For $\zeta<0$ the following equalities hold:*

$$I_n(\zeta)=\int_0^{\delta}\frac{r^{2n+1}dr}{r^2-\zeta}=-\frac{1}{2}\zeta^n\ln(-\zeta)+\widehat{I}_n(\zeta),\quad n=0,1,2,\ldots, \tag{3.18}$$

where $\ln(-\zeta)$ is real for $\zeta<0$ and $\widehat{I}_n(\zeta)$ is an analytic function in a neighborhood of the origin [14].

In the following lemma we establish an expansion of $\Delta(\mu,p;z)$ in a half neighborhood $(m(p)-\varepsilon,m(p))$ of the point $z=m(p)$.

Lemma 3.6. *Assume Assumption 2.1. Then for any $\mu>0, p\in U_\delta(0)$ and sufficiently small $m(p)-z>0$ the function $\Delta(\mu,p;\cdot)$ can be represented as the following convergent series:*

$$\begin{gathered}\Delta(\mu,p;z)=1-\mu\alpha_0(p)\ \ln(m(p)-z)+\frac{\mu}{2}\ln(m(p)-z)\sum_{n=1}^{\infty}\alpha_n(p)(m(p)-z)^n-\mu F(p,z),\\ \alpha_0(p)=-\frac{1}{2}\varphi^2(q_0(p))J(q_0(p)),\\ F(p,z)=\sum_{n=0}^{\infty}c_n(p)(m(p)-z)^n,\end{gathered} \tag{3.19}$$

where $\alpha_n(p),c_n(p)$, $n=0,1,2,\ldots$ are real numbers.

Proof. The function $\Omega(p;\cdot)$ can be written as a sum of the following functions:

$$\Omega(p;\cdot)=I_1(p,\cdot)+I_2(p,\cdot)+I_3(p,\cdot), \tag{3.20}$$

where

$$I_1(p,z) = \varphi^2(q_0(p)) \int_{U(q_0(p))} \frac{ds}{w_p(s) - z}, \qquad I_2(p,z) = \varphi^2(q_0(p)) \int_{\mathbb{T}^2 \setminus U(q_0(p))} \frac{ds}{w_p(s) - z},$$
$$I_3(p,z) = \int_{\mathbb{T}^2} \frac{(\varphi^2(s) - \varphi^2(q_0(p)))ds}{w_p(s) - z}, \tag{3.21}$$

and $U(q_0(p))$ is a neighborhood of the point $q_0(p)$, $p \in U_\delta(0)$.

Since $\min_{q \in \mathbb{T}^2} w_p(q) = w_p(q_0(p))$ for any $p \in U_\delta(0)$, the function $I_2(p,z)$ is analytic at $z = m(p)$. According to Lemma 3.4 the function $I_3(p, m(p))$ is analytic in $U_\delta(0)$.

A change of variables $s = \psi(y,p)$ in the integral $I_1(p,z)$ yields

$$I_1(p,z) = \varphi^2(q_0(p)) \int_{W_\gamma(0)} \frac{J(\psi(y,p))dy}{m(p) + y^2 - z}. \tag{3.22}$$

Passing to spherical coordinates by $y = rv$ we obtain

$$I_1(p,z) = \varphi^2(q_0(p)) \int_0^\gamma \int_{\Omega_2} \frac{J(\psi(rv,p))r\,dv\,dr}{m(p) + r^2 - z}, \tag{3.23}$$

and hence

$$I_1(p,z) = \varphi^2(q_0(p)) \int_0^\gamma \left(\int_{\Omega_2} J(\psi(rv,p))dv \right) \frac{r\,dr}{m(p) + r^2 - z}, \tag{3.24}$$

where Ω_2 is unit sphere in $\mathbb{R}^2$. Since

$$\int_{\Omega_2} J(\psi(rv,p))dv = \sum_{n=0}^{\infty} \tilde{\alpha}_n(p) r^{2n}, \tag{3.25}$$

where $\tilde{\alpha}_n(p), n = 0, 1, \ldots$ are the Pizetti coefficients, we get

$$I_1(p,z) = \varphi^2(q_0(p)) \sum_{n=0}^{\infty} \tilde{\alpha}_n(p) \int_0^\gamma \frac{r^{2n+1}dr}{m(p) + r^2 - z}, \tag{3.26}$$

where $\tilde{\alpha}_0(p) = J(q_0(p))$. Using Proposition 3.5 we have

$$\sum_{n=0}^{\infty} \tilde{\alpha}_n(p) \int_0^\gamma \frac{r^{2n+1}dr}{m(p) + r^2 - z} = -\frac{1}{2} \ln(m(p) - z) \sum_{n=0}^{\infty} \alpha_n(p)(m(p) - z)^n + \Phi(p,z), \tag{3.27}$$

where $\Phi(p,z) = \sum_{n=0}^{\infty} \beta_n(p)(m(p) - z)^n$ and $\widehat{\alpha}_n(p) = (-1)^n \tilde{\alpha}_n(p)$. Using relations (3.27) and (3.21) and putting (3.26) in (3.20) we get required relation (3.19). □

Lemma 3.7. *Let Assumption 2.1 hold. Then for any $p \in U_\delta(0)$ consider*

(i) *if $\varphi(q_0(p)) = \nabla\varphi(q_0(p)) = 0$, then*

$$f_p(q) = \frac{\varphi(\cdot)}{w_p(\cdot) - m(p)} \in L_2\left(\mathbb{T}^2\right); \tag{3.28}$$

(ii) *if $\varphi(q_0(p)) = 0$, $\nabla\varphi(q_0(p)) \neq 0$, then $f_p \in L_1(\mathbb{T}^2) \setminus L_2(\mathbb{T}^2)$.*

Proof. We consider the following integral:

$$I(p) = \int_{\mathbb{T}^2} \frac{F(s)ds}{\left(w_p(s) - m(p)\right)^k}, \tag{3.29}$$

where $F(\cdot)$ is a continuous function on $\mathbb{T}^2$ and $k \in N$. By Lemma 3.2 for any $p \in U_\delta(0)$ the function $w_p(\cdot)$ has a unique non degenerated minimum at $q = q_0(p)$. Then there exist a neighborhood $U(q_0(p)) \subset \mathbb{T}^2$ of the point $q = q_0(p)$ and positive number $c_p > 0$ that

$$c_p \leq w_p(q) - m(p), \quad q \in \mathbb{T}^2 \setminus U(q_0(p)). \tag{3.30}$$

We represent the function $I(\cdot)$ as a sum of two functions:

$$I(\cdot) = I_1(\cdot) + I_2(\cdot), \tag{3.31}$$

where

$$I_1(p) = \int_{U(q_0(p))} \frac{F(s)ds}{\left(w_p(s) - m(p)\right)^k}, \qquad I_2(p) = \int_{\mathbb{T}^2 \setminus U(q_0(p))} \frac{F(s)ds}{\left(w_p(s) - m(p)\right)^k}. \tag{3.32}$$

From (3.30) we get that $I_2(p) < \infty$. In the integral for $I_1(p)$ making a change of variables $s := \psi(y,p)$ one obtains

$$I_1(p) = \int_{W_\gamma(0)} \frac{F(\psi(y,p))J(\psi(y,p))dy}{y^{2k}}, \tag{3.33}$$

where $J(\psi(y,p))$ is the Jacobian of the mapping $s = \psi(y,p)$.

(i) Let $F(s) = \varphi^2(s), k = 2$. Then from (3.33) we get

$$I_1(p) = \int_{W_\gamma(0)} \frac{\varphi^2(\psi(y,p))J(\psi(y,p))dy}{y^4}. \tag{3.34}$$

Passing to spherical coordinates by $y = rv$ we get

$$I_1(p) = \int_0^\gamma \left(\int_{\Omega_2} \varphi^2(\psi(rv,p))J(\psi(rv,p))dv\right) r^{-3}dr. \tag{3.35}$$

Expanding the function $\varphi(\psi(rv,p))$ to the Taylor series at $r=0$ we obtain

$$\varphi(\psi(rv,p)) = \varphi(q_0(p)) + \sum_{i=1}^{2} \frac{\partial\varphi}{\partial\psi^{(i)}}(q_0(p)) \left(\sum_{j=1}^{2} \frac{\partial\psi^{(i)}}{\partial y_j}(0,p) v_j \right) r + g(r,v) r^2, \quad y_j = r v_j, \tag{3.36}$$

where $g(\cdot,v)$ is continuous in $W_\gamma(0)$ and $v_1^2+v_2^2=1$. By condition of part (i) of this lemma and from equality (3.36) it follows that (3.35) has the following form:

$$I_1(p) = \int_0^\gamma G(p,r)\,dr, \qquad G(p,r) = r\int_{\Omega_2} g^2(r,v) J(\psi(rv,p))\,dv. \tag{3.37}$$

Since the function $G(p,\cdot)$ is continuous in $[0,\gamma]$, we have $I_1(p) < \infty$. Taking into account $\|f\|^2_{L_2(\mathbb{T}^2)} = I(p)$, from (3.31) we get that $f \in L_2(\mathbb{T}^2)$.

Now we show that if the conditions of part (i) of Lemma 3.7 are not satisfied, that is, $\varphi(q_0(p)) \neq 0$ or $\nabla\varphi(q_0(p)) \neq 0$, then the function defined by (3.28) does not belong to $L_2(\mathbb{T}^2)$.

Let $\varphi(q_0(p)) = 0$ and $\nabla\varphi(q_0(p)) \neq 0$. We will show that

$$C(v) = \sum_{i=1}^{2} \frac{\partial\varphi}{\partial\psi^{(i)}}(q_0(p)) \left(\sum_{j=1}^{2} \frac{\partial\psi^{(i)}}{\partial y_j}(0,p) v_j \right) \neq 0, \quad v \in \Omega_2. \tag{3.38}$$

Assume the converse, let

$$\sum_{i=1}^{2} c_i \sum_{j=1}^{2} u_{ij} v_j = \sum_{j=1}^{2}\sum_{i=1}^{2} c_i u_{ij} \cdot v_j = 0, \tag{3.39}$$

where $c_i = (\partial\varphi/\partial\psi^{(i)})(q_0(p))$ and $u_{ij} = (\partial\psi^{(i)}/\partial y_j)(0,p)$, $i,j=1,2$. Since the function v_j, $j = 1,2$ are linearly independent, the last equality holds if and only if, when

$$\sum_{i=1}^{2} c_i u_{ij} = 0, \quad j = 1,2. \tag{3.40}$$

Observe that $\det(u_{ij})_{i,j=1}^2 = J(q_0(p)) \neq 0$. Consequently, the equalities (3.40) hold if and only if, when $c_1 = c_2 = 0$. This contradicts the fact that $\nabla\varphi(q_0(p)) \neq 0$. Thus, $C(v) \neq 0$. Hence the equality (3.35) has the form

$$I_1(p) = \int_0^\gamma \widetilde{G}(p,r)\,dr, \qquad \widetilde{G}(p,r) = r^{-1}\int_{\Omega_2} \widetilde{g}^2(r,v) J(\psi(rv,p))\,dv,\ \widetilde{g}(r,v) = C(v) + g(r,v) r. \tag{3.41}$$

Since

$$\int_0^{\gamma} r^{-1} dr = \infty, \qquad \lim_{r\to 0} \frac{\widetilde{G}(p,r)}{r^{-1}} = J(q_0(p)) \int_{\Omega_2} C^2(v)dv > 0, \tag{3.42}$$

by the theorem on comparison of improper integrals, we get that $I_1(p) = \infty$ and therefore $f \notin L_2(\mathbb{T}^2)$.

In case of $\varphi(q_0(p)) \neq 0$ the relation $f \notin L_2(\mathbb{T}^2)$ can be proven analogously.

(ii) Let $F(s) = |\varphi(s)|$, $k = 1$. Then from (3.33) we get

$$I_1(p) = \int_{W_\gamma(0)} \frac{|\varphi(\psi(y,p))| J(\psi(y,p))dy}{y^2}. \tag{3.43}$$

Passing to spherical coordinates by $y = rv$ we obtain

$$I_1(p) = \int_0^{\gamma} \left[\int_{\Omega_2} |\varphi(\psi(rv,p))| J(\psi(rv,p))dv \right] r^{-1} dr. \tag{3.44}$$

By the condition of part (ii) of Lemma 3.7 and from (3.36) we get

$$I_1(p) = \int_0^{\gamma} \left[\int_{\Omega_2} |C(v) + g(r,v)r| J(\psi(rv,p))dv \right] dr. \tag{3.45}$$

Since the function under the integral sign is continuous in $[0,\gamma]$, it follows that $I_1(p) < \infty$. Thus $I(p) < \infty$. Taking into account $\|f\|_{L_1(\mathbb{T}^2)} = I(p)$ we obtain $f \in L_1(\mathbb{T}^2)$. Consequently, from part (i) of Lemma 3.7 it follows that $f \in L_1(\mathbb{T}^2) \setminus L_2(\mathbb{T}^2)$. □

Lemma 3.8. *Let the point $s = q_0(p)$, $p \in U_\delta(0)$ a unique non degenerated minimum of the function $w_p(s)$, and $\varphi(q_0(p)) = 0$. Then for any $\mu > 0$ the equation*

$$H_\mu(p)f = m(p)f. \tag{3.46}$$

has a nonzero solution if and only if

$$\Delta(\mu,p;m(p)) = 1 - \mu \int_{\mathbb{T}^2} \frac{\varphi^2(q)dq}{w_p(q) - m(p)} = 0. \tag{3.47}$$

In this case the nonzero function

$$f_{\mu,p}(\cdot) = \frac{C\mu\varphi(\cdot)}{w_p(\cdot) - m(p)} \in L_1\left(\mathbb{T}^2\right), \tag{3.48}$$

is a solution of (3.46), *where* $C \neq 0$ *a normalizing constant.*

This lemma can be proved as Lemma 3.1 taking into account part (ii) of Lemma 3.7.

Now we prove the main results.

Proof of Theorem 2.3. (i) Observe that $\Delta(\mu, p; \cdot)$ is continuous and monotonously decreasing in $(-\infty, m(p))$. Moreover,

$$\lim_{z \to -\infty} \Delta(\mu, p; z) = 1. \tag{3.49}$$

By definition, if $\varphi(q_0(p)) \neq 0$, then $\mu(p) = 0$. So, for $\mu > \mu(p) = 0$, Lemma 3.6 gives that

$$\lim_{z \to m(p)-0} \Delta(\mu, p; z) = -\infty. \tag{3.50}$$

Analogously, if $\varphi(q_0(p)) = 0$, then for $\mu > \mu(p) > 0$ the inequality

$$\lim_{z \to m(p)-0} \Delta(\mu, p; z) = 1 - \frac{\mu}{\mu(p)} < 0, \tag{3.51}$$

holds.

The continuity of function $\Delta(\mu, p; \cdot)$ in $(-\infty, m(p))$ yields that the equation $\Delta(\mu, p; z) = 0$ has a unique solution $z = E(\mu, p) < m(p)$ and hence Lemma 3.1 yields that the operator $H_\mu(p), p \in U_\delta(0)$ has a unique eigenvalue $E(\mu, p)$.

Since for any $p \in U_\delta(0)$ and $\mu \in (\mu(p), +\infty)$ the number $z = E(\mu, p)$ is a solution of the equation $\Delta(\mu, p; z) = 0$ and the function $\Delta(\mu, \cdot; z)$ (resp., $\Delta(\cdot, p; z)$) is real analytic in $U_\delta(0)$ (resp., $(\mu(p), +\infty)$), the implicit function theorem implies that $E(\mu, \cdot)$ (resp., $E(\cdot, p)$) is real analytic in $U_\delta(0)$ (resp., $(\mu(p), +\infty)$).

Note that the function $\Delta(\cdot, p; z)$ monotonously decreases in $(\mu(p), \infty)$ and hence for any $\mu_1 > \mu_2 > \mu(p)$ the eigenvalues $E(\mu_1, p)$ and $E(\mu_2, p)$ satisfy the relations:

$$0 = \Delta(\mu_1, p; E(\mu_1, p)) = \Delta(\mu_2, p; E(\mu_2, p)) > \Delta(\mu_1, p; E(\mu_2, p)). \tag{3.52}$$

Using the monotonicity of the function $\Delta(\mu_1, p; \cdot)$ in $(-\infty, m(p))$ we obtain that $E(\mu_1, p) < E(\mu_2, p)$, that is, $E(\cdot, p)$ is monotonously decreases in $(\mu(p), \infty)$.

Lemma 3.1 implies that if for any $\mu \in (\mu(p), +\infty)$ and $p \in U_\delta(0)$ the number $E(\mu, p)$ is an eigenvalue of the operator $H_\mu(p), p \in U_\delta(0)$, then the function

$$\Psi(\mu; p, \cdot, E(\mu, p)) = \frac{C\mu\varphi(\cdot)}{w_p(\cdot) - E(\mu, p)} \in L_2(\mathbb{T}^2), \tag{3.53}$$

is a solution of the equation

$$H_\mu(p)\Psi(\mu; p, \cdot, E(\mu, p)) = E(\mu, p)\Psi(\mu; p, \cdot, E(\mu, p)), \tag{3.54}$$

where $C \neq 0$ is a a normalizing constant.

Analyticity of the function $\Psi(\mu; p, \cdot, E(\mu, p))$ follows from the analyticity of the functions $\varphi(\cdot)$ and $[w_p(\cdot) - E(\mu, p)]^{-1}$ in $\mathbb{T}^2$ and the representation (3.53). The functions $E(\mu, \cdot)$

and $w(\cdot,q)$ are analytic in $U_\delta(0)$ and for any $q \in \mathbb{T}^d$ the inequality $w_p(q) - E(\mu,p) > 0$ holds, therefore representation (3.53) yields that the mapping $p \mapsto \Psi(\mu;p,\cdot,E(\mu,p))$ is analytic in $U_\delta(0)$. Analogously the analyticity of the function $E(\cdot,p))$ implies that the mapping $\mu \mapsto \Psi(\mu;p,\cdot,E(\mu,p))$ is analytic in $(\mu(p),+\infty)$.

(ii) Let $\varphi(q_0(p)) = 0$ and $0 < \mu < \mu(p)$. Since

$$\lim_{z\to m(p)-0} \Delta(\mu,p;z) = \Delta(\mu,p;m(p)) = 1 - \frac{\mu}{\mu(p)} > 0, \tag{3.55}$$

we have $\Delta(\mu,p;z) > 0$, $z \in (-\infty, m(p)]$ and Lemma 3.1 yields that the operator $H_\mu(p), p \in U_\delta(0)$ does not have any eigenvalue in $(-\infty, m(p)]$.

The statements (iii) and (iv) of Theorem 2.3 follows from Lemmas 3.7 and 3.8. $\square$

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