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Research Article

Analytic Solutions of Some Self-Adjoint Equations by Using Variable Change Method and Its Applications

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Many applications of various self-adjoint differential equations, whose solutions are complex, are produced (Arfken, 1985; Gandarias, 2011; and Delkhosh, 2011). In this work we propose a method for the solving some self-adjoint equations with variable change in problem, and then we obtain a analytical solutions. Because this solution, an exact analytical solution can be provided to us, we benefited from the solution of numerical Self-adjoint equations (Mohynl-Din, 2009; Allame and Azal, 2011; Borhanifar et al. 2011; Sweilam and Nagy, 2011; Gülsu et al. 2011; Mohyud-Din et al. 2010; and Li et al. 1996).

1. Introduction

Many applications of science to solve many differential equations, we find that these equations are self-adjoint equations and solve relatively complex because they are forced to use numerical methods, which are contained several errors [1–6].

There are several methods for solving equations, there one of which can be seen in the literature [7–11], where the change of variables is very complicated to use.

In this paper, for solving analytical some self-adjoint equations, we get a method with variable change in problem, and then we obtain a analytical solutions.

Before going to the main point, we start to introduce three following equations.

1.1. Self-Adjoint Equation

A second-order linear homogeneous differential equation is called self-adjoint if and only if it has the following form [10–13]:

$$\frac{d}{dx}\left(h(x)y'\right) + \psi(x)y = 0 \quad a < x < b, \tag{1.1}$$

where $h(x) > 0$ on (a, b) and $\psi(x)$, $h'(x)$ are continuous functions on $[a, b]$.

1.2. Self-Adjointization Factor

By multiplying both sides, a second order linear homogeneous equation in a function $\mu(x)$ can be changed into a self-adjoint equation. Namely, we consider the following linear homogeneous equation:

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1.2}$$

where $P(x)$ is a non-zero function on $[a, b]$.

By multiplying both sides in $\mu(x)$, we have

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0. \tag{1.3}$$

If we check the self-adjoint condition, we have:

$$\begin{gathered} \frac{d}{dx}\left(\mu(x)P(x)\right) = \mu(x)Q(x) \\ \Longrightarrow \mu' P + \mu P' = \mu Q \Longrightarrow \frac{d\mu}{\mu} = \frac{Q - P'}{P}dx. \end{gathered} \tag{1.4}$$

Thus

$$\mu(x) = \frac{A}{P(x)}\operatorname{Exp}\left(\int \frac{Q}{P}dx\right), \tag{1.5}$$

where A is a real number that will be specified exactly during the process.

If we multiply both sides of (1.2) and (1.5) by each other, then we have the following form of self-adjoint equation:

$$\frac{d}{dx}\left(\mu(x)P(x)y'\right) + \mu(x)R(x)y = 0. \tag{1.6}$$

From now on, we will focus on the self-adjoint equations shown in (1.1).

1.3. Wronskian

The Wronskian of two functions f and g is [12, 14]

$$W(x) = W(f, g) = f'g - fg'. \tag{1.7}$$

More generally, for n real- or complex-valued functions$f_1, f_2, \ldots, f_n$, which are $n-1$ times differentiable on an interval I, the Wronskian $W(x) = W(f_1, \ldots, f_n)$ as a function on I is defined by

$$W(x) = \begin{vmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}. \tag{1.8}$$

That is, it is the determinant of the matrix constructed by placing the functions in the first row, the first derivative of each function in the second row, and so on through the $(n-1)$st derivative, thus forming a square matrix sometimes called a fundamental matrix.

When the functions f_i are solutions of a linear differential equation, the Wronskian can be found explicitly using Abel's identity, even if the functions f_i are not known explicitly.

Theorem 1.1. *If $P(x)y'' + Q(x)y' + R(x)y = 0$, then*

$$W(x) = e^{-\int (Q/P)dx}. \tag{1.9}$$

Proof. Let two solution of equation by y_1 and y_2, then, since these solutions satisfy the equation, we have

$$\begin{gathered} Py_1'' + Qy_1' + Ry_1 = 0, \\ Py_2'' + Qy_2' + Ry_2 = 0. \end{gathered} \tag{1.10}$$

Multiplying the first equation by y_2, the second by y_1, and subtracting, we find

$$P \cdot \left(y_1 y_2'' - y_2 y_1''\right) + Q \cdot \left(y_1 y_2' - y_2 y_1'\right) = 0. \tag{1.11}$$

Since Wronskian is given by $W = y_1 y_2' - y_2 y'$, thus

$$P \cdot \frac{dW}{dx} + Q \cdot W = 0. \tag{1.12}$$

Solving, we obtain an important relation known as Abel's identity, given by

$$W(x) = e^{-\int (Q/P)dx}. \tag{1.13}$$

□

2. The Solving Some Self-Adjoint Equation

Now, we show that self-adjoint equation (1.1) is changeable to two linear differential equations:

$$\begin{gathered}\frac{d}{dx}(h(x)y') + \psi(x)y = 0 \\ \Longrightarrow h(x)y'' + h'(x)y' + \psi(x)y = 0 \\ \Longrightarrow y'' + \frac{h'(x)}{h(x)}y' + \frac{\psi(x)}{h(x)}y = 0.\end{gathered} \tag{2.1}$$

By replacing of change variable $y = u(x) \cdot v(x)$, where $u(x)$ and $v(x)$ are continuous and differentiable functions, we obtain

$$(u'' \cdot v + 2u' \cdot v' + u \cdot v'') + \frac{h'(x)}{h(x)}(u' \cdot v + u \cdot v') + \frac{\psi(x)}{h(x)}u \cdot v = 0, \tag{2.2}$$

or

$$u'' + \left(2\frac{v'}{v} + \frac{h'}{h}\right)u' + \left(\frac{v'' + (h'/hv') + ((\psi/h) \cdot v)}{v}\right)u = 0. \tag{2.3}$$

Now, $u(x)$ and $v(x)$ values are calculated with the following assumptions:

$$2\frac{v'}{v} + \frac{h'}{h} = 0, \tag{2.4}$$

$$v'' + \frac{h'}{h}v' + \frac{\psi}{h}v = 0. \tag{2.5}$$

Now, corresponding to equation (2.4), we have

$$v(x) = e^{-1/2\int (h'/h)dx} = (h(x))^{-1/2} = \frac{1}{\sqrt{h(x)}} = \sqrt{W(x)}, \tag{2.6}$$

where $W(x)$ is Wronskian.
Also, corresponding to (2.4), (2.5) and (2.6), we have

$$\left(-\frac{h''}{2h} + \frac{3h'^2}{4h^2}\right)v + \frac{h'}{h}\left(-\frac{h'}{2h}\right)v + \frac{\psi}{h}v = 0, \tag{2.7}$$

or

$$\varphi(x) = \frac{h''}{2} - \frac{h'^2}{4h}. \tag{2.8}$$

Thus, if in (1.1) the following relations are established

$$\varphi(x) = \frac{h''}{2} - \frac{h'^2}{4h}, \tag{2.9}$$

then, we have from (2.3), (2.4), (2.5), and (2.6):

$$\begin{aligned} v(x) &= \frac{1}{\sqrt{h(x)}}, \\ u(x) &= C_1 x + C_2, \end{aligned} \tag{2.10}$$

and, the answer to self-adjoint equation (1.1) will be

$$y(x) = v(x) \cdot u(x) = \frac{1}{\sqrt{h(x)}}(C_1 x + C_2). \tag{2.11}$$

3. Applications and Examples

Example 3.1. Solve the equation:

$$\frac{d}{dx}\left(\alpha(1+\beta x)^{\gamma} y'\right) + \frac{\alpha\beta^2\gamma(\gamma-2)}{4}(1+\beta x)^{\gamma-2} y = 0, \tag{3.1}$$

where α, β, γ are constants and $\alpha \neq 0$ [7–9].

Solution 1. By virtue of (1.1), we have

$$h(x) = \alpha(1+\beta x)^{\gamma}, \qquad \varphi(x) = \frac{\alpha\beta^2\gamma(\gamma-2)}{4}(1+\beta x)^{\gamma-2}. \tag{3.2}$$

Obviously, that (2.9) is established, so we have:

$$y(x) = v(x) \cdot u(x) = \frac{1}{\sqrt{\alpha(1+\beta x)^{\gamma}}}(C_1 x + C_2). \tag{3.3}$$

Example 3.2. Solve the equation:

$$\frac{d}{dx}\left(\alpha e^{\gamma x} y'\right) + \frac{\alpha\gamma^2}{4} e^{\gamma x} y = 0, \tag{3.4}$$

where α, γ are constants and $\alpha \neq 0$ [7–9].

Solution 2. By virtue of (1.1), we have

$$h(x) = \alpha \cdot e^{\gamma x}, \qquad \varphi(x) = \frac{\alpha\gamma^2}{4} e^{\gamma x}. \tag{3.5}$$

Obviously, that (2.9) is established, so we have:

$$y(x) = v(x) \cdot u(x) = \frac{1}{\sqrt{\alpha e^{\gamma x}}}(C_1 x + C_2). \tag{3.6}$$

Example 3.3. Solve the equation:

$$\frac{d}{dx}\left(\alpha \cdot x^n y'\right) + \frac{\alpha \cdot n \cdot (n-2)}{4} x^{n-2} y = 0, \tag{3.7}$$

where α, n are constants and $\alpha \neq 0$ [7–9].

Solution 3. By virtue of (1.1), (2.9), and (2.11), we have

$$y(x) = v(x) \cdot u(x) = \frac{1}{\sqrt{\alpha \cdot x^n}}(C_1 x + C_2). \tag{3.8}$$

4. Conclusion

The governing equation for stability analysis of a variable cross-section bar subject to variably distributed axial loads, dynamic analysis of multi-storey building, tall building, and other systems is written in the form of a unified self-adjoint equation of the second order. These are reduced to Bessel's equation in this paper.

The key step in transforming the unified equation to self-adjoint equation is the selection of $h(x)$ and $\varphi(x)$ in (1.1).

Many difficult problems in the field of static and dynamic mechanics are solved by the unified equation proposed in this paper.

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