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Research Article

Existence of Solutions for Fractional Integro-Differential Equation with Multipoint Boundary Value Problem in Banach Spaces

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By means of the fixed-point theorem in the cone of strict-set-contraction operators, we consider the existence of a nonlinear multi-point boundary value problem of fractional integro-differential equation in a Banach space. In addition, an example to illustrate the main results is given.

1. Introduction

The purpose of this paper is to establish the existence results of positive solution to nonlinear fractional boundary value problem

$$\begin{gathered}
D_{0^+}^{q}u(t)+f\left(t,u,u',\dots,u^{(n-2)},Tu,Su\right)=\theta,\quad 0<t<1,\ n-1<q\le n,\\
u(0)=u'(0)=\cdots=u^{(n-2)}(0)=\theta,\qquad u^{(n-2)}(1)=\sum_{i=1}^{m-2}a_i u^{(n-2)}(\eta_i)
\end{gathered}\tag{1.1}$$

in a Banach space E, where θ is the zero element of E, and $n\ge 2,\ 0<\eta_1<\cdots<\eta_{m-2}<1,\ a_i>0\ (i=1,2,\dots,m-2)$, $D_{0^+}^{\alpha}$ is Riemann-Liouville fractional derivative, and

$$Tu(t)=\int_0^t K(t,s)u(s)ds,\qquad Su(t)=\int_0^1 H(t,s)u(s)ds,\tag{1.2}$$

where $K \in C[B, R_+]$, $B = \{(t,s) \in I \times I : t \geq s\}$, $H \in C[I \times I, R_+]$, $I = [0,1]$, and R_+ denotes the set of all nonnegative numbers.

Fractional differential equations have gained importance due to their numerous applications in many fields of science and engineering including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, and probability. For details see [1–3] and the references therein. In recent years, there are some papers dealing with the existence of the solutions of initial value problems or linear boundary value problems for fractional differential equations by means of techniques of nonlinear analysis (fixed-point theorems, Leray-Schauder theory, lower and upper solutions method, and so forth), see for example, [4–23].

In [8], by means of the fixed-point theorem for the mixed monotone operator, the authors considers unique existence of positive to singular boundary value problems for fractional differential equation

$$\begin{gathered} D_{0^+}^{q} u(t) + a(t) f\left(t, u, u', \ldots, u^{(n-2)}\right) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \end{gathered} \tag{1.3}$$

where $D_{0^+}^{q}$ is Riemann-Liouville fractional derivative of order $n-1 < q \leq n$, $n \geq 2$.

In [11], El-Shahed and Nieto study the existence of nontrivial solutions for a multi-point boundary value problem for fractional differential equations

$$\begin{gathered} D_{0^+}^{q} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\ n-1 < q \leq n,\ n \in N, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \end{gathered} \tag{1.4}$$

where $n \geq 2$, $\eta_i \in (0,1)$, $a_i > 0$ $(i = 1, 2, \ldots, m-2)$, and $D_{0^+}^{q}$ is Riemann-Liouville fractional derivative. Under certain growth conditions on the nonlinearity, several sufficient conditions for the existence of nontrivial solution are obtained by using Leray-Schauder nonlinear alternative. And then, Goodrich [24] was concerned with a partial extension of the problem (1.3)

$$\begin{gathered} D_{0^+}^{q} u(t) = f(t, u(t)), \quad 0 < t < 1,\ n - n < q \leq n-2, \\ u^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \qquad D_{0^+}^{p} u(1) = 0, \quad 1 \leq p \leq n-2, \end{gathered} \tag{1.5}$$

and the authors derived the Green function for the problem (1.5) and showed that it satisfies certain properties.

By the contraction mapping principle and the Krasnoselskii's fixed-point theorem, Zhou and Chu [13] discussed the existence and uniqueness results for the following fractional differential equation with multi-point boundary conditions:

$$\begin{gathered} {}^{C}D_{0^+}^{q} u(t) + f(t, u, Ku, Su) = 0, \quad 0 < t < 1,\ 1 < q < 2, \\ a_1 u(0) - b_1 u'(0) = d_1 u(\eta_1), \qquad a_2 u(1) - b_2 u'(1) = d_2 u(\eta_2), \end{gathered} \tag{1.6}$$

where ${}^{C}D_{0^+}^{q}$ is the Caputo's fractional derivative, a_1, a_2, b_1, b_2, d_1, and d_2 are real numbers, $0 < \eta_1$, and $\eta_2 < 1$.

In [20], Staněk has discussed the existence of positive solutions for the singular fractional boundary value problem

$$\begin{gathered} D^{q}u(t) + f(t, u, u', D^{p}u) = 0, \quad 2 < q < 3,\ 0 < p < 1, \\ u(0) = 0, \qquad u'(0) = u'(1) = 0. \end{gathered} \tag{1.7}$$

However, to the best of the author's knowledge, a few papers can be found in the literature dealing with the existence of solutions to boundary value problems of fractional differential equations in Banach spaces. In [25], Salem investigated the existence of Pseudo solutions for the following nonlinear m-point boundary value problem of fractional type

$$\begin{gathered} D_{0^+}^{q}u(t) + a(t)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \zeta_i u(\eta_i) \end{gathered} \tag{1.8}$$

in a reflexive Banach space E, where $D_{0^+}^{q}$ is the Pseudo fractional differential operator of order $n - 1 < q \leq n$, $n \geq 2$.

In [26], by the monotone iterative technique and mönch fixed-point theorem, Lv et al. investigated the existence of solution to the following Cauchy problems for differential equation with fractional order in a real Banach space E

$${}^{C}D^{q}u(t) = f(t, u(t)), \qquad u(0) = u_0, \tag{1.9}$$

where ${}^{C}D^{q}u(t)$ is the Caputo's derivative order, $0 < q < 1$.

By means of Darbo's fixed-point theorem, Su [27] has established the existence result of solutions to the following boundary value problem of fractional differential equation on unbounded domain $[0, +\infty)$

$$\begin{gathered} D_{0^+}^{q}u(t) = f(t, u(t)), \quad t \in [0, +\infty),\ 1 < q \leq 2, \\ u(0) = \theta, \qquad D_{0^+}^{q-1}u(\infty) = u_\infty \end{gathered} \tag{1.10}$$

in a Banach space E. $D_{0^+}^{q}$ is the Riemann-Liouville fractional derivative.

Motivated by the above mentioned papers [8, 13, 24, 25, 27, 28] but taking a quite different method from that in [26–29]. By using fixed-point theorem for strict-set-contraction operators and introducing a new cone Ω, we obtain the existence of at least two positive solutions for the BVP (1.1) under certain conditions on the nonlinear term in Banach spaces. Our results are different from those of [8, 13, 24, 25, 28, 30]. Note that the nonlinear term f depends on u and its derivatives $u', u'', \ldots, u^{(n-2)}$.

2. Preliminaries and Lemmas

Let the real Banach space E with norm $\|\cdot\|$ be partially ordered by a cone P of E; that is, $u \leq v$ if and only if $v - u \in P$; and P is said to be normal if there exists a positive constant N such that $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\|$, where the smallest N is called the normal constant of P. For details on cone theory, see [31].

The basic space used in this paper is $C[I, E]$. For any $u \in C[I, E]$, evidently, $(C[I, E], \|\cdot\|_C)$ is a Banach space with norm $\|u\|_C = \sup_{t\in I}|u(t)|$, and $P = \{u \in C[I, E] : u(t) \geq \theta \text{ for } t \in I\}$ is a cone of the Banach space $C[I, E]$.

Definition 2.1 (see [31]). Let V be a bounded set in a real Banach space E, and $\alpha(V) = \inf\{\delta > 0 : V = \cup_{i=1}^{m} V_i$, all the diameters of $V_i \leq \delta\}$. Clearly, $0 \leq \alpha(V) < \infty$. $\alpha(V)$ is called the Kuratovski measure of noncompactness.

We use α, α_C to denote the Kuratowski noncompactness measure of bounded sets in the spaces E, $C(I, E)$, respectively.

Definition 2.2 (see [31]). Let E_1, E_2 be real Banach spaces, $S \subset E_1$. $T : S \to E_2$ is a continuous and bounded operator. If there exists a constant k, such that $\alpha(T(S)) \leq k\alpha(S)$, then T is called a k-set contraction operator. When $k < 1$, T is called a strict-set-contraction operator.

Lemma 2.3 (see [31]). *If $D \subset C[I, E]$ is bounded and equicontinuous, then $\alpha(D(t))$ is continuous on I and*

$$\alpha_C(D) = \max_{t\in I} \alpha(D(t)), \quad \alpha\left(\left\{\int_I u(t)dt : u \in D\right\}\right) \leq \int_I \alpha(D(t))dt, \tag{2.1}$$

where $D(t) = \{u(t) : u \in D, t \in I\}$.

Definition 2.4 (see [2, 3]). The left-sided Riemann-Liouville fractional integral of order $q > 0$ of a function $y : R_+^0 \to R$ is given by

$$I_{0+}^{q} y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)ds. \tag{2.2}$$

Definition 2.5 (see [2, 3]). The fractional derivative of order $q > 0$ of a function $y : R_+^0 \to R$ is given by

$$D_{0+}^{q} y(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} y(s)ds, \tag{2.3}$$

where $n = [q] + 1, [q]$ denotes the integer part of number q, provided that the right side is pointwise defined on R_+^0.

Lemma 2.6 (see [2, 3]). *Let $q > 0$. Then the fractional differential equation*

$$D_{0+}^{q} y(t) = 0 \tag{2.4}$$

has a unique solution $y(t) = c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n}$, $c_i \in R$, $i = 1, 2, \ldots, n$; here $n - 1 < q \leq n$.

Lemma 2.7 (see [2, 3]). *Let $q > 0$. Then the following equality holds for $y \in L(0,1)$, $D_{0+}^{q} y \in L(0,1)$,*

$$I_{0+}^{q} D_{0+}^{q} y(t) = y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_N t^{q-N} \tag{2.5}$$

for some $c_i \in R$, $i = 1, 2, \ldots, N$; here N is the smallest integer greater than or equal to q.

Lemma 2.8 (see [31]). *Let K be a cone in a Banach space E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a strict-set-contraction operator such that either:*

(i) $\|Tx\| \leq \|x\|$, $x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \geq \|x\|$, $x \in K \cap \partial\Omega_2$, *or*

(ii) $\|Tx\| \geq \|x\|$, $x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \leq \|x\|$, $x \in K \cap \partial\Omega_2$,

then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

For convenience, we list some following assumptions.

(H1) There exist $a \in C[I, R_+]$ and $h \in C[R_+^{n+1}, R_+]$ such that

$$\left\| f(t, u_1, \ldots, u_{n+1}) \right\| \leq a(t) h(\|u_1\|, \ldots, \|u_{n+1}\|), \quad \forall t \in I,\ u_k \in P,\ k = 1, \ldots, n+1. \tag{3.1}$$

(H2) $f : I \times P_r^{n+1} \to P$, for any $r > 0$, f is uniformly continuous on $I \times P_r^{n+1}$ and there exist nonnegative constants L_k, $k = 1, \ldots, n+1$, with

$$\frac{2}{\Gamma(q-n+2)\eta^*} \left(\sum_{k=1}^{n-2} \frac{L_k}{(n-2-k)!} + L_{n-1} + \frac{a^*}{(n-3)!} L_n + \frac{b^*}{(n-3)!} L_{n+1} \right) < 1 \tag{3.2}$$

such that

$$\alpha\left(f(t, D_1, D_2, \ldots, D_{n+1})\right) \leq \sum_{k=1}^{n+1} L_k \alpha(D_k), \quad \forall t \in I,\ \text{bounded sets } D_k \in P_r, \tag{3.3}$$

where $a^* = \max\{K(t,s) : (t,s) \in B\}$, $b^* = \max\{H(t,s) : (t,s) \in I \times I\}$, $P_r = \{u \in P : \|u\| \leq r\}$, $\eta^* = 1 - \sum_{i=1}^{m-2} a_i \eta_i^{q-n+1}$.

Lemma 3.1. *Given $y \in C[I, E]$ and $1 - \sum_{i=1}^{m-2} a_i \eta_i^{q-n+1} \neq 0$ hold. Then the unique solution of*

$$\begin{gathered} D_{0^+}^{q-n+2} x(t) + y(t) = 0, \quad 0 < t < 1,\ n-1 < q \leq n,\ n \geq 2, \\ x(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\eta_i), \end{gathered} \tag{3.4}$$

is

$$x(t)=\int_0^1 G(t,s)y(s)ds, \tag{3.5}$$

where

$$G(t,s)=g(t,s)+\frac{\sum_{i=1}^{m-2}a_i g(\eta_i,s)}{\eta^*}t^{q-n+1}, \tag{3.6}$$

$$g(t,s)=\begin{cases}\dfrac{(t(1-s))^{q-n+1}-(t-s)^{q-n+1}}{\Gamma(q-n+2)}, & s\le t,\\ \dfrac{(t(1-s))^{q-n+1}}{\Gamma(q-n+2)}, & t\le s.\end{cases} \tag{3.7}$$

Proof. Deduced from Lemma 2.7, we have

$$x(t)=-I_{0+}^{q-n+2}y(t)+c_1t^{q-n+1}+c_2t^{q-n} \tag{3.8}$$

for some $c_1,c_2\in R$. Consequently, the general solution of (3.4) is

$$x(t)=-\int_0^t\frac{(t-s)^{q-n+1}}{\Gamma(q-n+2)}y(s)ds+c_1t^{q-n+1}+c_2t^{q-n}. \tag{3.9}$$

By boundary value conditions $x(0)=0,\ x(1)=\sum_{i=1}^{m-2}a_ix(\eta_i)$, there is $c_2=0$, and

$$c_1=\frac{1}{1-\sum_{i=1}^{m-2}a_i\eta_i^{q-n+1}}\int_0^1\frac{(1-s)^{q-n+1}}{\Gamma(q-n+2)}y(s)ds-\frac{\sum_{i=1}^{m-2}a_i}{1-\sum_{i=1}^{m-2}a_i\eta_i^{q-n+1}}\int_0^{\eta_i}\frac{(\eta_i-s)^{q-n+1}}{\Gamma(q-n+2)}y(s)ds. \tag{3.10}$$

Therefore, the solution of problem (3.4) is

$$\begin{aligned}x(t)=&-\int_0^t\frac{(t-s)^{q-n+1}}{\Gamma(q-n+2)}y(s)ds+\frac{t^{q-n+1}}{1-\sum_{i=1}^{m-2}a_i{\eta_i}^{q-n+1}}\int_0^1\frac{(1-s)^{q-n+1}}{\Gamma(q-n+2)}y(s)ds\\&-\frac{\sum_{i=1}^{m-2}a_it^{q-n+1}}{1-\sum_{i=1}^{m-2}a_i\eta_i^{q-n+1}}\int_0^{\eta_i}\frac{(\eta_i-s)^{q-n+1}}{\Gamma(q-n+2)}y(s)ds\\=&\int_0^1G(t,s)y(s)ds.\end{aligned} \tag{3.11}$$

The proof is complete. □

Moreover, there is one paper [8] in which the following statement has been shown.

Lemma 3.2. *The function $g(t,s)$ defined in (3.7) satisfying the following properties:*

(1) *$g(t,s) \geq 0$ is continuous on $[0,1]\times[0,1]$, and $g(t,s) \leq t^{q-n+1}/\Gamma(q-n+2)$, $g(t,s) \leq g(s,s)$ for all $0 \leq t,s \leq 1$;*

(2) *there exists a positive function $\rho_0 \in C(0,1)$ such that $\min_{\gamma\leq t\leq\delta} g(t,s) \geq \rho_0(s)g(t,s), s \in (0,1)$, where $0<\gamma<\delta<1$ and*

$$\rho_0(s) = \begin{cases} \dfrac{(\delta(1-s))^{q-n+1}-(\delta-s)^{q-n+1}}{(s(1-s))^{q-n+1}}, & s \in (0,\xi], \\ \left(\dfrac{\gamma}{s}\right)^{q-n+1}, & s \in [\xi,1), \end{cases} \tag{3.12}$$

where $\gamma<\xi<\delta$ is the solution of

$$(\delta(1-\xi))^{q-n+1} - (\delta-\xi)^{q-n+1} = \left(\gamma(1-\xi)\right)^{q-n+1}. \tag{3.13}$$

For our purpose, one assumes that

(H3) *$\eta^* = 1-\sum_{i=1}^{m-2} a_i\eta_i^{q-n+1} > 0$ and $0<\gamma\leq\min\{2\delta-1,\delta/2\}$, $2/3\leq\delta<1$, where γ, δ are the constants in (2) of Lemma 3.2.*

Remark 3.3. We note that if (H3) holds, then the function $G(t,s)$ defined in (3.4) is satisfying the following properties:

(i) $G(t,s)\geq 0$ is continuous on $[0,1]\times[0,1]$, and $G(t,s)\leq \Delta t^{q-n+1}$, for all $0\leq t$, $s\leq 1$, where $\Delta^{-1} = \eta^*\Gamma(q-n+2)$;

(ii) $G(t,s)\leq G(s)$ for all $0\leq t$, $s\leq 1$, where

$$G(s) = g(s,s) + \frac{\sum_{i=1}^{m-2} a_i g(\eta_i,s)}{\eta^*}. \tag{3.14}$$

Indeed, it is obvious from (1) of Lemma 3.2 and (3.6) that

$$\begin{aligned} G(t,s) &\leq g(s,s) + \frac{1}{\eta^*}\sum_{i=1}^{m-2} a_i g(\eta_i,s) = G(s) \\ &\leq \frac{t^{q-n+1}}{\Gamma(q-n+2)} + \frac{\sum_{i=1}^{m-2} a_i\eta_i^{q-n+1}}{\eta^*\Gamma(q-n+2)} t^{q-n+1} \leq \Delta t^{q-n+1}. \end{aligned} \tag{3.15}$$

Lemma 3.4. *Let $u(t) = I_{0^+}^{n-2}x(t)$, $x\in C[I,E]$. Then the problem (1.1) can be transformed into the following modified problem:*

$$\begin{gathered} D_{0^+}^{q-n+2}x(t) + f\left(s, I_{0^+}^{n-2}x(s),\dots,I_{0^+}^{1}x(s), x(s), T\left(I_{0^+}^{n-2}x(s)\right), S\left(I_{0^+}^{n-2}x(s)\right)\right) = \theta, \\ x(0)=\theta, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\eta_i), \end{gathered} \tag{3.16}$$

where $0 < t < 1$, $n-1 < q \le n$, $n \ge 2$. *Moreover, if* $x \in C[I,E]$ *is a solutions of problem* (3.16), *then the function* $u(t) = I_{0^+}^{n-2}x(t)$ *is a solution of* (1.1).

The proof follows by routine calculations.

To obtain a positive solution, we construct a cone Ω by

$$\Omega = \left\{x(t) \in P : x(t) \ge \frac{\lambda}{3}x(s),\ t \in I^*,\ s \in I\right\}, \tag{3.17}$$

where $P = \{x \in C[I,E], x(t) \ge \theta, t \in I\}$, $\lambda = \min\{\min_{\gamma\le t\le\delta}\rho(t), \gamma^{q-n+1}\}$, $I^* = [\gamma,\delta]$.

Let

$$(Ax)(t) = \int_0^1 G(t,s)f\left(s, I_{0^+}^{n-2}x(s),\dots,x(s),T\left(I_{0^+}^{n-2}x(s)\right),S\left(I_{0^+}^{n-2}x(s)\right)\right)ds, \quad 0 \le t \le 1. \tag{3.18}$$

Lemma 3.5. *Assume that (H1)–(H3) hold. Then* $A : \Omega \to \Omega$ *is a strict-set-contraction operator.*

Proof. Let $x \in \Omega$. Then, it follows from Remark 3.3. that

$$\begin{aligned}
(Ax)(t) &\le \int_0^1 G(s)f\left(s, I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)ds \\
&= \left(\int_0^\gamma + \int_\gamma^\delta + \int_\delta^1\right)G(s)f\left(s, I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)ds \\
&\le 3\int_\gamma^\delta G(s)f\left(s, I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)ds,
\end{aligned} \tag{3.19}$$

here, by (H3), we know that $\gamma < \delta - \gamma$ and $\delta - \gamma > 1 - \delta$.

From (3.6) and (3.18), we obtain

$$\begin{aligned}
\min_{t\in[\gamma,\delta]}(Ax)(t) &= \min_{t\in[\gamma,\delta]}\int_0^1 G(t,s)f\left(s, I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)ds \\
&\ge \int_\gamma^\delta\left(g(t,s) + \frac{t^{q-n+1}}{\eta^*}\sum_{i=1}^{m-2}a_i g(\eta_i,s)\right) \\
&\quad \times f\left(s, I_{0^+}^{n-2}x_j(s),\dots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)ds
\end{aligned}$$

$$\begin{aligned}
&\geq \int_{\gamma}^{\delta}\left(\rho_0(s)g(s,s)+\frac{\gamma^{q-n+1}}{\eta^*}\sum_{i=1}^{m-2}a_i g(\eta_i,s)\right)\\
&\quad\times f\left(s,I_{0^+}^{n-2}x_j(s),\ldots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)ds\\
&\geq \lambda\int_{\gamma}^{\delta}G(s)f\left(s,I_{0^+}^{n-2}x_j(s),\ldots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)ds\\
&\geq \frac{\lambda}{3}(Ax)(t'),\quad t'\in I,
\end{aligned}\tag{3.20}$$

which implies that $(Ax)(t)\in\Omega$; that is, $A(\Omega)\subset\Omega$.

Next, we prove that A is continuous on Ω. Let $\{x_j\}$, $\{x\}\subset\Omega$, and $\|x_j-x\|_\Omega\to 0$ $(j\to\infty)$. Hence $\{x_j\}$ is a bounded subset of Ω. Thus, there exists $r>0$ such that $r=\sup_j\|x_j\|_\Omega<\infty$ and $\|x\|_\Omega\leq r$. It is clear that

$$\begin{aligned}
\left\|(Ax_j)(t)-(Ax)(t)\right\| &= \int_0^1 G(t,s)\left\|f\left(s,I_{0^+}^{n-2}x_j(s),\ldots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)\right.\\
&\qquad\left.-f\left(s,I_{0^+}^{n-2}x(s),\ldots,x(s),T\left(I_{0^+}^{n-2}x(s)\right),S\left(I_{0^+}^{n-2}x(s)\right)\right)\right\|ds\\
&\leq \Delta t^{q-n+1}\left\|f\left(s,I_{0^+}^{n-2}x_j(s),\ldots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)\right.\\
&\qquad\left.-f\left(s,I_{0^+}^{n-2}x(s),\ldots,x(s),T\left(I_{0^+}^{n-2}x(s)\right),S\left(I_{0^+}^{n-2}x(s)\right)\right)\right\|ds.
\end{aligned}\tag{3.21}$$

According to the properties of f, for all $\varepsilon>0$, there exists $J>0$ such that

$$\begin{aligned}
&\left\|f\left(s,I_{0^+}^{n-2}x_j(s),\ldots,x_j(s),T\left(I_{0^+}^{n-2}x_j(s)\right),S\left(I_{0^+}^{n-2}x_j(s)\right)\right)\right.\\
&\quad\left.-f\left(s,I_{0^+}^{n-2}x(s),\ldots,x(s),T\left(I_{0^+}^{n-2}x(s)\right),S\left(I_{0^+}^{n-2}x(s)\right)\right)\right\|<\frac{\varepsilon}{\Delta},
\end{aligned}\tag{3.22}$$

for $j\geq J$, for all $t\in I$.

Therefore, for all $\varepsilon>0$, for any $t\in I$ and $j\geq J$, we get

$$\left\|(Ax_j)(t)-(Ax)(t)\right\|<t^{q-n+1}\varepsilon\leq\varepsilon.\tag{3.23}$$

This implies that A is continuous on Ω.

By the properties of continuous of $G(t,s)$, it is easy to see that A is equicontinuous on I.

Finally, we are going to show that A is a strict-set-contraction operator. Let $D \subset \Omega$ be bounded. Then by condition (H1), Lemma 3.1 implies that $\alpha_C(AD) = \max_{t\in I} \alpha((AD)(t))$. It follows from (3.18) that

$$\begin{aligned}
\alpha((AD)(t)) &\leq \alpha\Big(\overline{co}\Big\{G(t,s)f\Big(s, I_{0^+}^{n-2}x(s),\dots,x(s),T\Big(I_{0^+}^{n-2}x(s)\Big),S\Big(I_{0^+}^{n-2}x(s)\Big)\Big) \\
&\qquad : s\in[0,t],\ t\in I,\ x\in D\Big\}\Big) \\
&\leq \Delta\cdot\alpha\Big(\Big\{f\Big(s, I_{0^+}^{n-2}x(s),\dots,x(s),T\Big(I_{0^+}^{n-2}x(s)\Big),S\Big(I_{0^+}^{n-2}x(s)\Big)\Big) \\
&\qquad : s\in[0,t],\ t\in I,\ x\in D\Big\}\Big) \\
&\leq \Delta\cdot\alpha\Big(f\Big(I\times\Big(I_{0^+}^{n-2}D\Big)(I)\times\cdots\times D(I)\times T\Big(I_{0^+}^{n-2}D\Big)(I)\times S\Big(I_{0^+}^{n-2}D\Big)(I)\Big)\Big) \\
&\leq \Delta\cdot\Bigg\{\sum_{k=1}^{n-2}L_k\alpha\Big(\Big(I_{0^+}^{n-1-k}D\Big)(I)\Big)+L_{n-1}\alpha(D(I))+a^*L_n\alpha\Big(T\Big(I_{0^+}^{n-2}D\Big)(I)\Big) \\
&\qquad +b^*L_{n+1}\alpha\Big(S\Big(I_{0^+}^{n-2}D\Big)(I)\Big)\Bigg\},
\end{aligned}\tag{3.24}$$

which implies

$$\begin{aligned}
\alpha_C(AD) \leq \Delta\cdot\Bigg\{&\sum_{k=1}^{n-2}L_k\alpha\Big(\Big(I_{0^+}^{n-1-k}D\Big)(I)\Big)+L_{n-1}\alpha(D(I)) \\
&+a^*L_n\alpha\Big(T\Big(I_{0^+}^{n-2}D\Big)(I)\Big)+b^*L_{n+1}\alpha\Big(S\Big(I_{0^+}^{n-2}D\Big)(I)\Big)\Bigg\}.
\end{aligned}\tag{3.25}$$

Obviously,

$$\begin{aligned}
\alpha\Big(I_{0^+}^{n-1-k}D\Big)(I) &= \alpha\left(\left\{\int_0^s\frac{(s-\tau)^{n-2-k}}{(n-2-k)!}x(\tau)d\tau : \tau\in[0,s],\ s\in I,\ k=1,\dots,n-2\right\}\right) \\
&\leq \frac{1}{(n-2-k)!}\alpha(D(I)),
\end{aligned}\tag{3.26}$$

$$\begin{aligned}
\alpha\Big(T\Big(I_{0^+}^{n-2}D\Big)\Big)(I) &= \alpha\left(\left\{\int_0^t K(t,s)\left(\int_0^s\frac{(s-\tau)^{n-2}}{(n-3)!}u(\tau)d\tau\right)ds : u\in D,\ t\in I\right\}\right) \\
&\leq \frac{a^*}{(n-3)!}\alpha(\{u(t):t\in I,\ u\in D\}) \leq \frac{a^*}{(n-3)!}\alpha(D(I)),
\end{aligned}\tag{3.27}$$

$$\begin{aligned}
\alpha\Big(S\Big(I_{0^+}^{n-2}D\Big)\Big)(I) &= \alpha\left(\left\{\int_0^1 H(t,s)\left(\int_0^s\frac{(s-\tau)^{n-2}}{(n-3)!}u(\tau)d\tau\right)ds : u\in D, t\in I\right\}\right) \\
&\leq \frac{b^*}{(n-3)!}\alpha(\{u(t):t\in I,\ u\in D\}) \ \leq \frac{b^*}{(n-3)!}\alpha(D(I)).
\end{aligned}\tag{3.28}$$

Using a similar method as in the proof of Theorem 2.1.1 in [31], we have

$$\alpha(D(I)) \leq 2\alpha_C(D). \tag{3.29}$$

Therefore, it follows from (3.26)–(3.29) that

$$\alpha_C(AD) \leq 2\Delta \cdot \left(\sum_{k=1}^{n-2} \frac{L_k}{(n-2-k)!} + L_{n-1} + \frac{a^* L_n}{(n-3)!} + \frac{b^* L_{n+1}}{(n-3)!}\right) \alpha_C(D). \tag{3.30}$$

Noticing that (3.3), we obtain that T is a strict-set-contraction operator. The proof is complete. □

Theorem 3.6. *Let cone P be normal and conditions $(H1)$~$(H3)$ hold. In addition, assume that the following conditions are satisfied.*

(H4) *There exist $u^* \in P \setminus \{\theta\}$, $c_1 \in C[I^*, R_+]$ and $h_1 \in C[P^{n+1}, R_+]$ such that*

$$\begin{gathered} f(t, u_1, \ldots, u_{n+1}) \geq c_1(t) h_1(u_1, \ldots, u_{n-1}) u^*, \quad \forall t \in I^*, \ u_k \in P, \\ \frac{h_1(u_1, \ldots, u_{n-1})}{\sum_{k=1}^{n-1} \|u_k\|} \longrightarrow \infty, \quad \text{as} \ \sum_{k=1}^{n-1} \|u_k\| \longrightarrow \infty, \ u_k \in P. \end{gathered} \tag{3.31}$$

(H5) *There exist $u_* \in P \setminus \{\theta\}$, $c_2 \in C[I^*, R_+]$, and $h_2 \in C[P^{n-1}, R_+]$ such that*

$$\begin{gathered} f(t, u_1, \ldots, u_{n+1}) \geq c_2(t) h_2(u_1, \ldots, u_{n-1}) u_*, \quad \forall t \in I^*, \ u_k \in P, \\ \frac{h_2(u_1, \ldots, u_{n-1})}{\sum_{k=1}^{n-1} \|u_i\|} \longrightarrow \infty, \quad \text{as} \ \sum_{k=1}^{n-1} \|u_k\| \longrightarrow 0, \ u_k \in P. \end{gathered} \tag{3.32}$$

(H6) *There exists a $\beta > 0$ such that*

$$N M_\beta \int_0^1 G(s) a(s) ds < \beta, \tag{3.33}$$

where $M_\beta = \max_{u_k \in P_\beta} \{h(\|u_1\|, \ldots, \|u_{n+1}\|)\}$. Then problem (1.1) has at least two positive solutions.

Proof. Consider condition (H4), there exists an $r_1 > 0$, such that

$$h_1(u_1, \ldots, u_{n-1}) \geq \frac{3N^2 \sum_{k=1}^{n-1} \|u_k\|}{\lambda^2 \int_\gamma^\delta G(s) c_1(s) ds \cdot \|u^*\|}, \quad \forall u_k \in P, \ \sum_{k=1}^{n-1} \|u_k\| \geq r_1. \tag{3.34}$$

Therefore,

$$f(t,u_1,\cdot,u_{n+1}) \geq \frac{3N^2 \sum_{k=1}^{n-1}\|u_k\|}{\lambda^2 \int_\gamma^\delta G(s)c_1(s)ds \cdot \|u^*\|} \cdot c_1(t)u^*, \quad \forall u_k \in P,\ \sum_{k=1}^{n-1}\|u_k\| \geq r_1. \tag{3.35}$$

Take

$$r_0 > \max\left\{3N\lambda^{-1}r_1,\ \beta\right\}. \tag{3.36}$$

Then for $t \in [\gamma,\delta]$, $\|x\|_\Omega = r_0$, we have, by (3.18),

$$\|x(t)\| \geq \frac{\lambda}{3N}\|x\|_\Omega \geq \frac{\lambda}{3N}r_0 > r_1. \tag{3.37}$$

Hence,

$$\begin{aligned}
(Ax)(t) &\geq \int_\gamma^\delta G(t,s)f\left(s,I_{0^+}^{n-2}x(s),\dots,x(s),T\left(I_{0^+}^{n-2}x\right)(s),S\left(I_{0^+}^{n-2}x\right)(s)\right)ds \\
&\geq \frac{3N^2}{\lambda\int_\gamma^\delta G(s)c_1(s)ds\cdot\|u^*\|}\int_\gamma^\delta G(s)\left(\sum_{k=1}^{n-2}\left\|I_{0^+}^{n-1-k}x(s)\right\| + \|x(s)\|\right)c_1(s)ds\cdot u^* \\
&\geq \frac{3N^2}{\lambda\int_\gamma^\delta G(s)c_1(s)ds\cdot\|u^*\|}\int_\gamma^\delta G(s)c_1(s)\|x(s)\|ds\cdot u^* \\
&\geq \frac{N}{\int_\gamma^\delta G(s)c_1(s)ds\cdot\|u^*\|}\|x\|_\Omega\left(\int_\gamma^\delta G(s)c_1(s)ds\right)\cdot u^* \\
&= \frac{1}{\int_\gamma^\delta G(s)c_1(s)ds\cdot\|u^*\|}\left(\int_\gamma^\delta G(s)c_1(s)ds\|u^*\|\right)\cdot\frac{N\|x\|_\Omega}{\|u^*\|}u^* \\
&\geq \frac{N\|x\|_\Omega}{\|u^*\|}\cdot u^*,
\end{aligned} \tag{3.38}$$

and consequently,

$$\|Ax\|_\Omega \geq \|x\|_\Omega, \quad \forall x \in \Omega, \|x\|_\Omega = r_0. \tag{3.39}$$

Similarly, by condition (H5), there exists $r_2 > 0$, such that

$$h_2(u_1,\dots,u_{n-1}) \geq \frac{3N^2\sum_{k=1}^{n-1}\|u_k\|}{\lambda\int_\gamma^\delta G(\xi,s)c_2(s)ds\cdot\|u_*\|}, \quad \forall u_k \in P,\ 0 < \sum_{k=1}^{n-1}\|u_k\| \leq r_2, \tag{3.40}$$

where ξ is given in (2) of Lemma 3.2. Therefore,

$$f(t,u_1,\ldots,u_{n+1}) \geq \frac{3N^2\sum_{k=1}^{n-1}\|u_k\|}{\lambda\int_\gamma^\delta G(\xi,s)c_2(s)ds\cdot\|u_*\|}\cdot c_2(t)u_*,\quad \forall u_k\in P,\ 0<\sum_{k=1}^{n-1}\|u_k\|\leq r_2. \tag{3.41}$$

Choose

$$0<r<\min\left\{\left(\sum_{k=0}^{n-2}\frac{1}{k!}\right)^{-1}r_2,\ \beta\right\}. \tag{3.42}$$

Then for $t\in[\gamma,\delta]$, $x\in\Omega$, $\|x\|_\Omega=r$, we have

$$\begin{aligned}
(Ax)(\xi) &= \int_0^1 G(\xi,s)f\left(s,I_{0^+}^{n-2}x(s),\ldots,x(s),T\left(I_{0^+}^{n-2}x\right)(s),S\left(I_{0^+}^{n-2}x\right)(s)\right)ds\\
&\geq \int_\gamma^\delta G(\xi,s)f\left(s,I_{0^+}^{n-2}x(s),\ldots,x(s),T\left(I_{0^+}^{n-2}x\right)(s),S\left(I_{0^+}^{n-2}x\right)(s)\right)ds\\
&\geq \frac{3N^2}{\lambda\int_\gamma^\delta G(\xi,s)c_2(s)ds\cdot\|u_*\|}\int_\gamma^\delta G(\xi,s)\left(\sum_{k=1}^{n-2}\left\|I_{0^+}^{n-1-k}x(s)\right\|+\|x(s)\|\right)c_2(s)ds\cdot u_*\\
&\geq \frac{3N^2}{\lambda\int_\gamma^\delta G(\xi,s)c_2(s)ds\cdot\|u_*\|}\int_\gamma^\delta G(\xi,s)\|x(s)\|c_2(s)ds\cdot u_*\\
&\geq \frac{N}{\int_\gamma^\delta G(\xi,s)c_2(s)ds\cdot\|u_*\|}\|x(s)\|_\Omega\int_\gamma^\delta G(\xi,s)c_2(s)ds\cdot u_*\\
&= \frac{1}{\int_\gamma^\delta G(\xi,s)c_2(s)ds\cdot\|u_*\|}\|x(s)\|_\Omega\left(\int_\gamma^\delta G(\xi,s)c_2(s)ds\|u_*\|\right)\cdot\frac{N\|x(s)\|_\Omega}{\|u_*\|}u_*\\
&\geq \frac{N\|x(s)\|_\Omega}{\|u_*\|}\cdot u_*,
\end{aligned} \tag{3.43}$$

which implies

$$\|(Ax)(\xi)\|_\Omega \geq \|x(s)\|_\Omega,\quad \forall x\in\Omega, \|x\|_\Omega=r, \tag{3.44}$$

that is,

$$\|Ax\|_\Omega \geq \|x(s)\|_\Omega,\quad \forall x\in\Omega, \|x\|_\Omega=r. \tag{3.45}$$

On the other hand, according to (ii) of Remark 3.3 and (3.18), we get

$$(Ax)(t) \leq \int_0^1 G(s) f\left(s, I_{0^+}^{n-2}x(s), \ldots, x(s), T\left(I_{0^+}^{n-2}x\right)(s), S\left(I_{0^+}^{n-2}x\right)(s)\right) ds. \tag{3.46}$$

By condition (H1), for $t \in I$, $x \in \Omega$, $\|x\|_\Omega = \beta$, we have

$$\left\| f(t, u_1, \ldots, u_{n+1}) \right\| \leq a(t) h(\|u_1\|, \ldots, \|u_{n+1}\|) \leq M_\beta a(t). \tag{3.47}$$

Therefore,

$$\|(Ax)(t)\|_\Omega \leq N M_\beta \cdot \int_0^1 G(s) a(s) ds < \beta = \|x\|_\Omega. \tag{3.48}$$

Applying Lemma 2.7 to (3.39), (3.45), and (3.48) yields that T has a fixed-point $x^* \in \overline{\Omega}_{r,\beta}$, $r \leq \|x^*\| \leq \beta$, and a fixed-point $x^{**} \in \overline{\Omega}_{\beta,r_0}$, $\beta \leq \|x^{**}\| \leq r_0$. Noticing (3.48), we get $\|x^*\| \neq \beta$ and $\|x^{**}\| \neq \beta$. This and Lemma 3.4 complete the proof. □

Theorem 3.7. *Let cone P be normal and conditions (H1)~(H4) hold. In addition, assume that the following condition is satisfied:*

(H7)

$$\frac{h(\|u_1\|, \ldots, \|u_{n+1}\|)}{\sum_{k=1}^{n+1} \|u_k\|} \longrightarrow 0, \quad \text{as } u_k \in P, \ \sum_{k=1}^{n+1} \|u_k\| \longrightarrow 0^+. \tag{3.49}$$

Then problem (1.1) *has at least one positive solution.*

Proof. By (H4), we can choose $r_0 > 3N\lambda^{-1} r_1$. As in the proof of Theorem 3.6, it is easy to see that (3.39) holds. On the other hand, considering (3.49), there exists $r_3 > 0$ such that

$$h(\|u_1\|, \ldots, \|u_{n+1}\|) \leq \varepsilon_0 \sum_{k=1}^{n+1} \|u_k\|, \quad \text{for } t \in I, \ u_k \in P, \ 0 < \sum_{k=1}^{n+1} \|u_k\| \leq r_3, \tag{3.50}$$

where $\varepsilon_0 > 0$ satisfies

$$\varepsilon_0 = \left(N \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} + \frac{a^* + b^*}{(n-3)!} \right\} \int_0^1 G(s) a(s) ds \right)^{-1}. \tag{3.51}$$

Choose $0 < r^* < \min\{(\sum_{k=1}^{n-1}(1/k!)+(a^*+b^*)/(n-3)!)^{-1}r_3, r_0\}$. Then for $t \in I,\ x \in \Omega,\ \|x\|_\Omega = r^*$, it follows from (3.46) that

$$\begin{aligned}
\|(Ax)(t)\| &\le N\int_0^1 G(s)\left\|f\left(s, I_{0^+}^{n-2}x(s),\ldots,x(s),T\left(I_{0^+}^{n-2}x\right)(s),S\left(I_{0^+}^{n-2}x\right)(s)\right)\right\|ds \\
&\le N\int_0^1 G(s)a(s)h\left(\left\|I_{0^+}^{n-2}x(s)\right\|,\ldots,\|x(s)\|,\left\|T\left(I_{0^+}^{n-2}x\right)(s)\right\|,\left\|S\left(I_{0^+}^{n-2}x\right)(s)\right\|\right)ds \\
&\le N\varepsilon_0\int_0^1 G(s)a(s)\left(\sum_{k=1}^{n-1}\left\|I_{0^+}^{n-1-k}x(s)\right\| + \left\|T\left(I_{0^+}^{n-2}x\right)(s)\right\| + \left\|S\left(I_{0^+}^{n-2}x\right)(s)\right\|\right)ds \\
&\le N\varepsilon_0\left(\sum_{k=1}^{n-1}\frac{1}{k!} + \frac{a^*+b^*}{(n-3)!}\right)r^*\int_0^1 G(s)a(s)ds = r^*
\end{aligned} \tag{3.52}$$

and consequently,

$$\|(Ax)(t)\|_\Omega \le \|x\|_\Omega, \quad \forall x \in \Omega,\ \|x\|_\Omega \le r^*. \tag{3.53}$$

Since $0 < r^* < r_0$, applying Lemma 2.7 to (3.39) and (3.53) yield that T has a fixed-point $x^* \in \overline{\Omega}_{r^*,r_0}, r^* \le \|x^*\| \le r_0$. This and Lemma 3.4 complete the proof. □

4. An Example

Consider the following system of scalar differential equations of fractional order

$$\begin{aligned}
-D^{5/2}u_k(t) &= \frac{(1+t)^3}{960k^3}\left\{\left[u_{2k}(t)+u'_{3k}(t)+\sum_{j=1}^{\infty}u_{2j}(t)+\sum_{j=1}^{\infty}u'_j(t)\right]^3\right. \\
&\qquad\left. + \left(3u_k(t)+3u'_{k+1}(t)+\sum_{j=1}^{\infty}u_j(t)+\sum_{j=1}^{\infty}u'_{2j}(t)\right)^{1/2}\right\} \\
&\quad + \frac{1+t^3}{36k^5}\left(\int_0^t e^{-(1+t)s}u_k(s)ds\right)^{2/3} \\
&\quad + \frac{1+t^2}{24k^4}\left(\int_0^1 e^{-s}\sin^2(t-s)\pi u_{2k}(s)ds\right), \quad t \in I, \\
u_k(0) &= u'_k(0) = 0, \quad u'_k(1) = \frac{1}{8}u'_k\left(\frac{1}{4}\right)+\frac{1}{2}u'_k\left(\frac{4}{9}\right), \quad k = 1,2,3,\ldots.
\end{aligned} \tag{4.1}$$

Conclusion. The problem (4.1) has at least two positive solutions.

Proof. Let $E = l^1 = \{u = (u_1, u_2, \ldots, u_k, \ldots) : \sum_{k=1}^{\infty} |u_k| < \infty\}$ with the norm $\|u\| = \sum_{k=1}^{\infty} |u_k|$, and $P = \{(u_1, \ldots, u_k, \ldots) : u_k \geq 0, k = 1, 2, 3, \ldots\}$. Then P is a normal cone in E with normal constant $N = 1$, and system (4.1) can be regarded as a boundary value problem of the form (1.1). In this situation, $q = 5/2$, $n = 3$, $a_1 = 1/8$, $a_2 = 1/2$, $\eta_1 = 1/4$, $\eta_2 = 4/9$, $\eta^* = 29/48$, $K(t,s) = e^{-(1+t)s}$, $H(t,s) = e^{-s}\sin^2(t-s)\pi$, $u = (u_1, \ldots, u_k, \ldots)$, $f = (f_1, f_2, \ldots, f_k, \ldots)$, in which

$$f_k(t,u,v,x,y) = \frac{(1+t)^3}{960k^3}\left\{\left(u_{2k} + v_{3k} + \sum_{j=1}^{\infty} u_{2j} + \sum_{j=1}^{\infty} v_j\right)^3 + \left(3u_k + 3v_{k+1} + \sum_{j=1}^{\infty} u_j + \sum_{j=1}^{\infty} v_{2j}\right)^{1/2}\right\} + \frac{1+t^3}{36k^5} x_k^{1/5} + \frac{1+t^2}{24k^4} y_{2k}. \tag{4.2}$$

Observing the inequality $\sum_{k=1}^{\infty}(1/k^3) < 3/2$, we get, by (4.2),

$$\begin{aligned} \|f(t,u,v,x,y)\| &= \sum_{k=1}^{\infty} |f_k(t,u,v,x,y)| \\ &\leq \frac{(1+t)^3}{2}\left(\frac{1}{40}(\|u\| + \|v\|)^3 + \frac{1}{160}(\|u\| + \|v\|)^{1/2} + \frac{1}{12}\|x\|^{1/5} + \frac{1}{8}\|y\|\right). \end{aligned} \tag{4.3}$$

Hence (H1) is satisfied for $a(t) = (1+t)^3/2$ and

$$h(u,v,x,y) = \frac{1}{40}(u+v)^3 + \frac{1}{160}(u+v)^{1/2} + \frac{1}{12}x^{1/5} + \frac{1}{8}y. \tag{4.4}$$

Now, we check condition (H2). Obviously, $f : I \times P_r^4 \to P$, for any $r > 0$, and f is uniformly continuous on $I \times P_r^4$. Let $f = f^{(1)} + f^{(2)}$, where $f^{(1)} = (f_1^{(1)}, \ldots, f_k^{(1)}, \ldots)$ and $f^{(2)} = (f_1^{(2)}, \ldots, f_k^{(2)}, \ldots)$, in which

$$\begin{aligned} f_k^{(1)}(t,u,v,x,y) &= \frac{(1+t)^3}{960k^3}\left\{\left(u_{2k} + v_{3k} + \sum_{j=1}^{\infty} u_{2j} + \sum_{j=1}^{\infty} v_j\right)^3 + \left(3u_k + 3v_{k+1} + \sum_{j=1}^{\infty} u_j + \sum_{j=1}^{\infty} v_{2j}\right)^{1/2}\right\} \\ &\quad + \frac{1+t^3}{36k^5} x_k^{1/5}, \quad (k = 1, 2, 3, \ldots), \\ f_k^{(2)}(t,u,v,x,y) &= \frac{1+t^2}{24k^4} y_{2k}, \quad (k = 1, 2, 3, \ldots). \end{aligned} \tag{4.5}$$

For any $t \in I$ and bounded subsets $D_i \subset E$, $i = 1,2,3,4$, from (4.5) and by the diagonal method, we have

$$\alpha\left(f^{(1)}(t, D_1, D_2, D_3, D_4)\right) = 0, \quad \forall t \in I, \text{ bounded sets } D_i \subset E, \ i = 1,2,3,4,$$
$$\alpha\left(f^{(2)}(I, D_1, D_2, D_3, D_4)\right) \le \frac{1}{12}\alpha(D_4), \quad \forall t \in I, D_i \subset E, \ i = 1,2,3,4. \tag{4.6}$$

It follows from (4.6) that

$$\alpha(f(I, D_1, D_2, D_3, D_4)) \le \frac{1}{12}\alpha(D_4), \quad \forall t \in I, D_i \subset E, \ i = 1,2,3,4,$$
$$\frac{2}{\Gamma(q-n+2)\eta^*}\left(\sum_{k=1}^{n-2}\frac{L_k}{(n-2-k)!} + L_{n-1} + \frac{a^*}{(n-3)!}L_n + \frac{b^*}{(n-3)!}L_{n+1}\right) \approx 0.1565 < 1. \tag{4.7}$$

that is, condition (H2) holds for $L_1 = L_2 = L_3 = 0, L_4 = 1/12$.

On the other hand, take $\gamma = 1/4$, $\delta = 3/4$. Then $1/4 = \gamma \le \min\{\delta/2, \ 2\delta - 1\} = 3/8$, $2/3 < \delta$, which implies that condition (H3) holds. By (4.2), we have

$$f_k(t, u, v, x, y) \ge \frac{(1+t)^3}{960k^3}(\|u\| + \|v\|)^3, \quad \forall t \in I^*, \ u, v, x, y \in P, \ (k = 1,2,3,\ldots),$$
$$f_k(t, u, v, x, y) \ge \frac{(1+t)^3}{960k^3}\sqrt{\|u\| + \|v\|}, \quad \forall t \in I^*, \ u, v, x, y \in P, \ (k = 1,2,3,\ldots). \tag{4.8}$$

Hence condition (H4) is satisfied for

$$c_1(t) = \frac{(1+t)^3}{960}, \qquad h_{1,k}(u, v) = (\|u\| + \|v\|)^3, \qquad u^* = \left(1, \ldots, \frac{1}{k^3}, \ldots\right), \tag{4.9}$$

in this situation,

$$h_{1,k} = \lim_{\|u\|+\|v\| \to \infty} \frac{(\|u\| + \|v\|)^3}{\|u\| + \|v\|} = \infty. \tag{4.10}$$

And condition (H5) is also satisfied for

$$c_2(t) = \frac{(1+t)^3}{960}, \qquad h_{2,k}(u, v) = \sqrt{\|u\| + \|v\|}, \qquad u_* = \left(1, \ldots, \frac{1}{k^3}, \ldots\right), \tag{4.11}$$

in this situation,

$$h_{2,k} = \lim_{\|u\|+\|v\| \to 0} \frac{\sqrt{\|u\| + \|v\|}}{\|u\| + \|v\|} = \infty. \tag{4.12}$$

Finally, choose $\beta = 1$, it is easy to check that condition (H6) is satisfied. In this case, $M_\beta \approx 0.4162$, and so

$$NM_\beta \int_0^1 G(s)a(s)ds \approx 0.7287 < \beta = 1. \tag{4.13}$$

Hence, our conclusion follows from Theorem 3.6. □

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References

[1] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.
[2] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, London, UK, 1999.
[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204, Elsevier Science, Amsterdam, The Netherlands, 2006.
[4] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis*, vol. 69, no. 8, pp. 2677–2682, 2008.
[5] V. Lakshmikantham, "Theory of fractional functional differential equations," *Nonlinear Analysis*, vol. 69, no. 10, pp. 3337–3343, 2008.
[6] V. Lakshmikantham and A. S. Vatsala, "General uniqueness and monotone iterative technique for fractional differential equations," *Applied Mathematics Letters*, vol. 21, no. 8, pp. 828–834, 2008.
[7] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.
[8] S. Zhang, "Positive solutions to singular boundary value problem for nonlinear fractional differential equation," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1300–1309, 2010.
[9] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis*, vol. 72, no. 2, pp. 916–924, 2010.
[10] C. F. Li, X. N. Luo, and Y. Zhou, "Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1363–1375, 2010.
[11] M. El-Shahed and J. J. Nieto, "Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3438–3443, 2010.
[12] Y. Zhao, H. Chen, and L. Huang, "Existence of positive solutions for nonlinear fractional functional differential equation," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3456–3467, 2012.
[13] W.-X. Zhou and Y.-D. Chu, "Existence of solutions for fractional differential equations with multi-point boundary conditions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 3, pp. 1142–1148, 2012.
[14] G. Wang, B. Ahmad, and L. Zhang, "Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1389–1397, 2011.
[15] C. Yuan, "Two positive solutions for $(n-1,1)$-type semipositone integral boundary value problems for coupled systems of nonlinear fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 2, pp. 930–942, 2012.

[16] B. Ahmad and S. Sivasundaram, "Existence of solutions for impulsive integral boundary value problems of fractional order," *Nonlinear Analysis*, vol. 4, no. 1, pp. 134–141, 2010.
[17] B. Ahmad and S. Sivasundaram, "On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order," *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 480–487, 2010.
[18] S. Hamani, M. Benchohra, and J. R. Graef, "Existence results for boundary-value problems with nonlinear fractional differential inclusions and integral conditions," *Electronic Journal of Differential Equations*, vol. 2010, pp. 1–16, 2010.
[19] A. Arikoglu and I. Ozkol, "Solution of fractional integro-differential equations by using fractional differential transform method," *Chaos, Solitons and Fractals*, vol. 40, no. 2, pp. 521–529, 2009.
[20] S. Staněk, "The existence of positive solutions of singular fractional boundary value problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1379–1388, 2011.
[21] B. Ahmad and J. J. Nieto, "Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations," *Abstract and Applied Analysis*, vol. 2009, Article ID 494720, 9 pages, 2009.
[22] M. A. Darwish, "On a perturbed functional integral equation of Urysohn type," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8800–8805, 2012.
[23] M. A. Darwish and S. K. Ntouyas, "Boundary value problems for fractional functional differential equations of mixed type," *Communications in Applied Analysis*, vol. 13, no. 1, pp. 31–38, 2009.
[24] C. S. Goodrich, "Existence of a positive solution to a class of fractional differential equations," *Applied Mathematics Letters*, vol. 23, no. 9, pp. 1050–1055, 2010.
[25] A. H. Salem, "On the fractional order m-point boundary value problem in reflexive banach spaces and weak topologies," *Journal of Computational and Applied Mathematics*, vol. 224, no. 2, pp. 565–572, 2009.
[26] Z.-W. Lv, J. Liang, and T.-J. Xiao, "Solutions to the Cauchy problem for differential equations in Banach spaces with fractional order," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1303–1311, 2011.
[27] X. Su, "Solutions to boundary value problem of fractional order on unbounded domains in a banach space," *Nonlinear Analysis*, vol. 74, no. 8, pp. 2844–2852, 2011.
[28] Y.-L. Zhao and H.-B. Chen, "Existence of multiple positive solutions for m-point boundary value problems in banach spaces," *Journal of Computational and Applied Mathematics*, vol. 215, no. 1, pp. 79–90, 2008.
[29] Y. Zhao, H. Chen, and C. Xu, "Existence of multiple solutions for three-point boundary-value problems on infinite intervals in banach spaces," *Electronic Journal of Differential Equations*, vol. 44, pp. 1–11, 2012.
[30] X. Zhang, M. Feng, and W. Ge, "Existence and nonexistence of positive solutions for a class of nth-order three-point boundary value problems in Banach spaces," *Nonlinear Analysis*, vol. 70, no. 2, pp. 584–597, 2009.
[31] D. J. Guo, V. Lakshmikantham, and X. Z. Liu, *Nonlinear Integral Equations in Abstract Spaces*, vol. 373, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.