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Research Article

Existence Results for the Distributed Order Fractional Hybrid Differential Equations

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We introduce the distributed order fractional hybrid differential equations (DOFHDEs) involving the Riemann-Liouville differential operator of order $0 < q < 1$ with respect to a nonnegative density function. Furthermore, an existence theorem for the fractional hybrid differential equations of distributed order is proved via a fixed point theorem in the Banach algebras under the mixed Lipschitz and Caratheodory conditions.

1. Introduction

The differential equations involving Riemann-Liouville differential operators of fractional order $0 < q < 1$ are very important in the modeling of several physical phenomena [1, 2]. In recent years, quadratic perturbations of nonlinear differential equations and first-order ordinary functional differential equations in Banach algebras, have attracted much attention to researchers. These type of equations have been called the hybrid differential equations [3–8]. One of the important first-order hybrid differential equations (HDE) is defined as [4, 9]

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J, \qquad x(t_0) = x_0, \tag{1.1}$$

where $J = [t_0, t_0 + a)$ is a bounded interval in $\mathbb{R}$ for some t_0 and $a \in \mathbb{R}$ with $a > 0$. Also, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R})$, such that $C(J \times \mathbb{R}, \mathbb{R})$ is the class of continuous functions and $\mathcal{C}(J \times \mathbb{R})$ is called the Caratheodory class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ which are Lebesgue integrable bounded by a Lebesgue integrable function on J. Moreover

(i) the map $t \mapsto g(t,x)$ is measurable for each $x \in \mathbb{R}$,

(ii) the map $x \mapsto g(t,x)$ is continuous for each $t \in J$.

For the above hybrid differential equation, Dhage and Lakshmikantham [9] established existence, uniqueness, and some fundamental differential inequalities. Also, they stated some theoretical approximation results for the extremal solutions between the given lower and upper solutions [10]. Later, Zhao. et al. [11] developed the following fractional hybrid differential equations involving the Riemann-Liouville differential operators of order $0<q<1$,

$$D^q\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J,$$
$$x(0) = 0, \tag{1.2}$$

where $J = [0,T]$ is bounded in $\mathbb{R}$ for some $T \in \mathbb{R}$ and $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in \mathcal{C}(J \times \mathbb{R})$.

They established the existence, uniqueness, and some fundamental fractional differential inequalities to prove existence of the extremal solutions of (1.2). Also, they considered necessary tools under the mixed Lipschitz and Caratheodory conditions to prove the comparison principle.

Now, in this article in view of the distributed order fractional derivative [12–14], we develop the distributed order fractional hybrid differential equations (DOFHDEs) with respect to a nonnegative density function.

In this regard, in Section 2 we introduce the distributed order fractional hybrid differential equation. Section 3 is about some main theorems which are used in this paper. In Section 4, we prove the existence theorem for this class of equations, and we express some special cases for the density function used in the distributed order fractional hybrid differential equation. Finally, the main conclusions are set.

2. The Fractional Hybrid Differential Equation of Distributed Order

In this section, we recall some definitions which are used throughout this paper.

Definition 2.1 (see [1, 2]). The fractional integral of order q with the lower limit t_0 for the function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > t_0, \ q > 0. \tag{2.1}$$

Definition 2.2 (see [1, 2]). The Riemann-Liouville derivative of order q with the lower limit t_0 for the function $f : [t_0, \infty) \to \mathbb{R}$ can be written as

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{q+1-n}} ds, \quad t > t_0, \ n-1 < q < n. \tag{2.2}$$

Definition 2.3. The distributed order fractional hybrid differential equation (DOFHDEs), involving the Riemann-Liouville differential operator of order $0 < q < 1$ with respect to the nonnegative density function $b(q) > 0$, is defined as

$$\int_0^1 b(q) D^q \left[\frac{x(t)}{f(t, x(t))}\right] dq = g(t, x(t)), \quad t \in J, \int_0^1 b(q) dq = 1, \\ x(0) = 0. \tag{2.3}$$

Moreover, the function $t \mapsto x/f(t, x)$ is continuous for each $x \in \mathbb{R}$, where $J = [0, T]$ is bounded in $\mathbb{R}$ for some $T \in \mathbb{R}$. Also, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R})$.

3. The Main Theorems

In this section, we state the existence theorem for the DOFHDE (2.3) on $J = [0, T]$. For this purpose, we define a supremum norm of $\|\cdot\|$ in $C(J, \mathbb{R})$ as

$$\|x\| = \sup_{t \in J} |x(t)|, \tag{3.1}$$

and for $x, y \in C(J, \mathbb{R})$

$$(xy)(t) = x(t) y(t), \tag{3.2}$$

is a multiplication in this space. We consider $C(J, \mathbb{R})$ is a Banach algebra with respect to norm $\|\cdot\|$ and multiplication (3.2). Moreover the norm $\|\cdot\|_{L^1}$ for $x \in C(J, \mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^T |x(s)| ds. \tag{3.3}$$

Now, for expressing the existence theorem for the DOFHDE (2.3), we state a fixed point theorem in the Banach algebra.

Theorem 3.1 (see [15]). *Let S be a nonempty, closed convex, and bounded subset of the Banach algebra X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is Lipschitz constant α,*

(b) *B is completely continuous,*

(c) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$,*

(d) *$\alpha M < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.*

Then the operator equation $AxBx = x$ has a solution in S.

At this point, we consider some hypotheses as follows.

(A_0) The function $x \mapsto x/f(t, x)$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

(A_1) There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x - y|, \tag{3.4}$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(A_2) There exists a function $h \in L^1(J,\mathbb{R})$ and a real nonnegative upper bound h^* such that

$$|g(t,x)| \leq h(t) \leq h^*, \tag{3.5}$$

for all $t \in J$ and $x \in \mathbb{R}$.

Theorem 3.2 (Titchmarsh theorem [16]). *Let $F(s)$ be an analytic function which has a branch cut on the real negative semiaxis. Furthermore, $F(s)$ has the following properties:*

$$\begin{aligned} F(s) &= O(1), \quad |s| \longrightarrow \infty, \\ F(s) &= O\left(\frac{1}{|s|}\right), \quad |s| \longrightarrow 0, \end{aligned} \tag{3.6}$$

for any sector $|\arg(s)| < \pi - \eta$, where $0 < \eta < \pi$. Then, the Laplace transform inversion $f(t)$ can be written as the Laplace transform of the imaginary part of the function $F(re^{-i\pi})$ as follows:

$$f(t) = \mathcal{L}^{-1}\{F(s);t\} = \frac{1}{\pi}\int_0^\infty e^{-rt}\Im\left(F\left(re^{-i\pi}\right)\right)dr. \tag{3.7}$$

Definition 3.3. Suppose that (X, d) be a metric space and let $B \subseteq C(X,\mathbb{R})$. Then, B is equicontinuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in B$ and $a, x \in X$

$$d(x,a) < \delta \Longrightarrow |f(x) - f(a)| < \epsilon. \tag{3.8}$$

Theorem 3.4 (Arzela-Ascoli theorem [17]). *Let (X, d) be a compact metric space and let $B \subset C(X,\mathbb{R})$. Then, B is compact if and only if B is closed, bounded, and equicontinuous.*

Theorem 3.5 (Lebesgue dominated convergence theorem [18]). *Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measure space (S,Σ,μ). Also, suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that*

$$|f_n(x)| \leq g(x), \tag{3.9}$$

for all numbers n in the index set of the sequence and all points x in S. Then, f is integrable and

$$\lim \int_S f_n d\mu = \int_S f d\mu. \tag{3.10}$$

4. Existence Theorem for the DOFHDEs

We apply the following lemma to prove the main existence theorem of this section.

Lemma 4.1. *Assume that hypothesis* (A_0) *holds in pervious section, then for any* $h \in L^1(J, \mathbb{R})$ *and* $0 < q < 1$, *the function* $x \in C(J, \mathbb{R})$ *is a solution of the DOFHDE* (2.3) *if and only if* x *satisfies the following equation*

$$x(t) = \frac{f(t, x(t))}{\pi} \int_0^t \mathcal{L}\left\{ \Im\left\{ \frac{1}{B(re^{-i\pi})} \right\}; t - \tau \right\} g(\tau, x(\tau)) d\tau \tag{4.1}$$

such that $0 \leq \tau \leq t \leq T$ *and*

$$B(s) = \int_0^1 b(q) s^q dq. \tag{4.2}$$

Proof. Applying the Laplace transform on both sides of (2.3) and letting

$$Y(t) = \frac{x(t)}{f(t, x(t))}, \tag{4.3}$$

we have

$$\begin{aligned} \mathcal{L}\left\{ \int_0^1 b(q) D^q Y(t) dq; s \right\} &= \mathcal{L}\{ g(t, x(t)); s \} \\ &= \int_0^1 b(q) \left[s^q Y(s) - D_t^{q-1} Y(0) \right] dq = G(s). \end{aligned} \tag{4.4}$$

Since $Y(0) = 0$, we have

$$Y(s) \left(\int_0^1 b(q) s^q dq \right) = G(s), \tag{4.5}$$

and hence,

$$Y(s) = \frac{1}{B(s)} G(s), \tag{4.6}$$

such that

$$B(s) = \int_0^1 b(q) s^q dq. \tag{4.7}$$

Now, using the inverse Laplace transform on both sides of (4.6) and applying the convolution product, we get

$$\begin{aligned}\mathcal{L}^{-1}\{Y(s)\} &= \frac{x(t)}{f(t,x(t))} = \mathcal{L}^{-1}\left\{\frac{1}{B(s)}G(s)\right\} \\ &= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{B(s)};t-\tau\right\}g(\tau,x(\tau))d\tau,\end{aligned} \tag{4.8}$$

or equivalently

$$x(t) = f(t,x(t))\int_0^t \mathcal{L}^{-1}\left\{\frac{1}{B(s)};t-\tau\right\}g(\tau,x(\tau))d\tau. \tag{4.9}$$

Since $B(s)$ is an analytic function which has a branch cut on the real negative semiaxis, according to the Titchmarsh Theorem 3.2 we get

$$x(t) = \frac{f(t,x(t))}{\pi}\int_0^t\int_0^\infty e^{-r(t-\tau)}\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}g(\tau,x(\tau))dr\,d\tau, \tag{4.10}$$

which by the Laplace transform definition, (4.1) is held. Conversely, let x satisfies (4.1), therefore, x satisfies the equivalent equation (4.9). By $t=0$ in (4.1), we have

$$\frac{x(0)}{f(0,x(0))} = 0 = \frac{0}{f(0,0)}. \tag{4.11}$$

According to hypothesis (A_0), the map $x \mapsto x/f(0,x)$ is injective in $\mathbb{R}$ and hence $x(0)=0$. Next, with dividing (4.9) by $f(t,x(t))$ and using the Laplace transform operator on both sides of this equation, (4.6) also holds. Since $Y(0)=0$, we obtain (4.4) and by applying the inverse Laplace transform, (2.3) also holds. □

Theorem 4.2. *Suppose that hypothesis (A_0)–(A_2) hold. Further, if*

$$\frac{LM\|h\|_{L^1}}{\pi} < 1, \quad M > 0, \tag{4.12}$$

then, the DOFHDE (2.3) *has a solution defined on J.*

Proof. We set $X = C(J,\mathbb{R})$ as a Banach algebra and define a subset S of X by

$$S = \{x \in X \mid \|x\| \le N\}, \tag{4.13}$$

such that

$$N = \frac{F_0 M\|h\|_{L^1}}{\pi - LM\|h\|_{L^1}}, \qquad F_0 = \sup_{t\in J}\left|f(t,0)\right|. \tag{4.14}$$

It is obvious that S is closed and if $x_1, x_2 \in \mathbb{R}$, then $\|x_1\| \le N$ and $\|x_2\| \le N$, also by properties of the norm, we get

$$\|\lambda x_1 + (1-\lambda)x_2\| \le \lambda\|x_1\| + (1-\lambda)\|x_2\| \le \lambda N + (1-\lambda)N = N. \tag{4.15}$$

Therefore, S is a convex and bounded and by applying Lemma 4.1, DOFHDE (2.3) is equivalent to (4.1).

Define operators $A : X \to X$ and $B : S \to X$ by

$$\begin{gathered} Ax(t) = f(t, x(t)), \quad t \in J, \\ Bx(t) = \frac{1}{\pi}\int_0^t \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}; t-\tau\right\} g(\tau, x(\tau))d\tau, \quad t \in J, \end{gathered} \tag{4.16}$$

thus, from (4.1), we obtain the operator equation as follows:

$$Ax(t)Bx(t) = x(t), \quad t \in J. \tag{4.17}$$

If operators A and B satisfy all the conditions of Theorem 3.1, then the operator equation (4.17) has a solution in S. For this paper, let $x, y \in X$ which by hypothesis (A_1) we have

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \le L|x(t) - y(t)| \le L\|x - y\|, \quad t \in J, \tag{4.18}$$

and if for all $x, y \in X$ take a supremum over t, then we have

$$\|Ax - Ay\| \le L\|x - y\|. \tag{4.19}$$

Therefore, A is a Lipschitz operator on X with the Lipschitz constant $L > 0$, and the condition (a) from Theorem 3.1 is held. Now, for checking the condition (b) from this theorem, first, we shall show that B is continuous on S.

Let $\{x_n\}$ be a sequence in S such that

$$\lim_{n\to\infty} x_n = x \tag{4.20}$$

with $x \in S$. By applying the Lebesgue-dominated convergence Theorem 3.5 for all $t \in J$, we get

$$\begin{aligned}\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty}\frac{1}{\pi}\int_0^t \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t-\tau\right\}g(\tau,x_n(\tau))d\tau\\ &= \frac{1}{\pi}\int_0^t \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t-\tau\right\}\lim_{n\to\infty}g(\tau,x_n(\tau))d\tau\\ &= \frac{1}{\pi}\int_0^t \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t-\tau\right\}g(\tau,x(\tau))d\tau\\ &= Bx(t).\end{aligned} \tag{4.21}$$

Thus, B is a continuous operator on S. In next stage, we shall show that B is a compact operator on S. For this paper, we shall show that $B(s)$ is a uniformly bounded and eqicontinuous set in X. Let $x \in S$, then by hypothesis (A_2) for all $t \in J$ we have

$$\begin{aligned}|Bx(t)| &= \left|\frac{1}{\pi}\int_0^t \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t-\tau\right\}g(\tau,x(\tau))d\tau\right|\\ &\le \frac{1}{\pi}\int_0^t\left|\mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t-\tau\right\}\right||h(\tau)|d\tau.\end{aligned} \tag{4.22}$$

Let $s = t-\tau$ such that $0 \le \tau \le t \le T$. Then by the existence Laplace transform theorem [19], there exists a constant $M' > 0$ such that for a constant c that $s > c$,

$$\left|\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}\right| \le M'e^{cr}. \tag{4.23}$$

Hence, we find an upper bound for the integral of (4.22) as follows:

$$\begin{aligned}\left|\mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t-\tau\right\}\right| &= \left|\int_0^\infty e^{-sr}\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}dr\right|\\ &\le \int_0^\infty e^{-sr}\left|\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}\right|dr\\ &\le \int_0^\infty M'e^{(c-s)r}dr \le \frac{M'}{|s-c|} \le M,\end{aligned} \tag{4.24}$$

such that

$$M = \sup_{0\le\tau\le t\le T}\frac{M'}{|t-\tau-c|}. \tag{4.25}$$

Finally, with respect to the inequality (4.22) we obtain

$$|Bx(t)| \le \frac{M\|h\|_{L^1}}{\pi}, \tag{4.26}$$

which by applying supremum over t, we get for all $x \in S$

$$\|Bx\| \le \frac{M}{\pi}\|h\|_{L^1}. \tag{4.27}$$

Thus, B is uniformly bounded on S.

In this stage, now we show that $B(S)$ is an equicontinuous set in X. Let $t_1, t_2 \in J$, with $t_1 < t_2$. In this respect, we have for all $x \in S$

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &= \left| \frac{1}{\pi}\int_0^{t_1} \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}; t_1-\tau\right\} g(\tau, x(\tau))d\tau \right.\\
&\quad \left. -\frac{1}{\pi}\int_0^{t_2} \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}; t_2-\tau\right\} g(\tau, x(\tau))d\tau \right|\\
&\le \left| \frac{1}{\pi}\int_0^{t_1} \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}; t_1-\tau\right\} g(\tau, x(\tau))d\tau \right.\\
&\quad \left. -\frac{1}{\pi}\int_0^{t_1} \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}; t_2-\tau\right\} g(\tau, x(\tau))d\tau \right|\\
&\quad + \left| \frac{1}{\pi}\int_0^{t_1} \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}; t_2-\tau\right\} g(\tau, x(\tau))d\tau \right.\\
&\quad \left. -\frac{1}{\pi}\int_0^{t_2} \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}; t_2-\tau\right\} g(\tau, x(\tau))d\tau \right|.
\end{aligned} \tag{4.28}$$

If we set $s_1 = t_1 - \tau$ and $s_2 = t_2 - \tau$, then by Laplace transform definition and (4.23), for $s_1 > c$ and $s_2 > c$ we can write

$$\begin{aligned}
&\left| \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}; s_1\right\} - \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}; s_2\right\} \right|\\
&\quad = \left| \int_0^{\infty} e^{-s_1 r}\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}dr - \int_0^{\infty} e^{-s_2 r}\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}dr \right|\\
&\quad \le \int_0^{\infty} \left|e^{-s_1 r} - e^{-s_2 r}\right| \left|\Im\left\{\frac{1}{B(re^{-i\pi})}\right\}\right| dr\\
&\quad \le M' \int_0^{\infty} \left(e^{(c-s_1)r} - e^{(c-s_2)r}\right)dr = M'\left(\frac{1}{s_1-c} - \frac{1}{s_2-c}\right).
\end{aligned} \tag{4.29}$$

Therefore, we have

$$\begin{aligned}&\left|\frac{1}{\pi}\int_0^{t_1}\left(\mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};s_1\right\}-\mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};s_2\right\}\right)g(\tau,x(\tau))d\tau\right|\\&\quad\le\frac{h^*}{\pi}\int_0^{t_1}M'\left(\frac{1}{t_1-\tau-c}-\frac{1}{t_2-\tau-c}\right)d\tau\\&\quad=\frac{M'h^*}{\pi}\ln\left(\frac{(c+t_1-t_2)(c-t_1)}{c(c-t_2)}\right).\end{aligned}\tag{4.30}$$

Also, by (4.24) we have

$$\begin{aligned}&\left|\frac{1}{\pi}\int_{t_2}^{t_1}\mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t_2-\tau\right\}g(\tau,x(\tau))d\tau\right|\\&\quad\le\frac{h^*}{\pi}\int_{t_2}^{t_1}\frac{M'}{t_2-\tau-c}d\tau=\frac{M'h^*}{\pi}\ln\left(\frac{c}{c+t_1-t_2}\right).\end{aligned}\tag{4.31}$$

Finally, with respect to (4.28), (4.30), and (4.31) we obtain

$$\begin{aligned}|Bx(t_1)-Bx(t_2)|&\le\frac{M'h^*}{\pi}\left(\ln\left(\frac{(c+t_1-t_2)(c-t_1)}{c(c-t_2)}\right)+\ln\left(\frac{c}{c+t_1-t_2}\right)\right)\\&=\frac{M'h^*}{\pi}\ln\left(\frac{c-t_1}{c-t_2}\right).\end{aligned}\tag{4.32}$$

Hence, for $\epsilon>0$, there exists $\delta>0$ such that if $|t_1-t_2|<\delta$, then for all $t_1,t_2\in J$ and all $x\in S$ we have

$$|Bx(t_1)-Bx(t_2)|<\epsilon,\tag{4.33}$$

which implies that $B(S)$ is an equicontinuous set in X and according to the Arzela-Ascoli Theorem 3.4, B is compact. Therefore B is continuous and compact operator on S into X and B is a completely continuous operator on S and the condition (b) from the Theorem 3.1 is held.

For checking the condition (c) of Theorem 3.1, let $x\in X$ and $y\in S$ be arbitrary such that $x=AxBy$. Then, by hypothesis (A_1) we get

$$\begin{aligned}|x(t)|&=|Ax(t)|\,|By(t)|\\&=|f(t,x(t))|\left|\frac{1}{\pi}\int_0^t\mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t-\tau\right\}g(\tau,y(\tau))d\tau\right|\\&\le\left(|f(t,x(t))-f(t,0)|+|f(t,0)|\right)\left(\frac{1}{\pi}\int_0^t\mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t-\tau\right\}|g(\tau,y(\tau))|d\tau\right)\end{aligned}$$

$$
\begin{aligned}
&\leq (L|x(t)| + F_0)\left(\frac{1}{\pi}\int_0^t \mathcal{L}\left\{\Im\left\{\frac{1}{B(re^{-i\pi})}\right\};t-\tau\right\}h(\tau)d\tau\right) \\
&\leq (L|x(t)| + F_0)\left(\frac{M\|h\|_{L^1}}{\pi}\right).
\end{aligned}
\tag{4.34}
$$

Therefore,

$$
|x(t)| \leq \frac{F_0 M\|h\|_{L^1}}{\pi - LM\|h\|_{L^1}},
\tag{4.35}
$$

which by taking a supremum over t, we obtain

$$
\|x(t)\| \leq \frac{F_0 M\|h\|_{L^1}}{\pi - LM\|h\|_{L^1}} = N.
\tag{4.36}
$$

Thus, the condition (c) of Theorem 3.1 is satisfied. If we consider

$$
\begin{gathered}
M_1 = \|B(s)\| = \sup\{\|Bx\| : x \in S\} \leq \frac{M}{\pi}\|h\|_{L^1}, \\
\alpha M_1 \leq L\left(\frac{M}{\pi}\|h\|_{L^1}\right) < 1,
\end{gathered}
\tag{4.37}
$$

the hypothesis (d) of Theorem 3.1 is satisfied.

Hence, all the conditions of Theorem 3.1 are satisfied and therefore the operator equation $AxBx = x$ has a solution in S. As a result, the DOFHDE (2.3) has a solution defined on J and proof is completed. □

5. Some Special Cases

In this section, we discuss some special cases of the density function $b(q)$ for the DOFHDE (2.3) and we find the operators A and B which introduce in Theorem 4.2. In proof of Lemma 4.1, the following equation is equivalent to the DOFHDE (2.3),

$$
x(t) = f(t, x(t))\int_0^t \mathcal{L}^{-1}\left\{\frac{1}{B(s)};t-\tau\right\}g(\tau, x(\tau))d\tau,
\tag{5.1}
$$

such that,

$$
B(s) = \int_0^1 b(q)s^q dq.
\tag{5.2}
$$

(1) Let $b(q) = 1$. Then we have

$$B(s) = \int_0^1 s^q dq = \frac{s-1}{\ln(s)}. \tag{5.3}$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{B(s)}; t-\tau\right\} = e^{t-\tau}\mathrm{Ei}(t-\tau), \tag{5.4}$$

where $\mathrm{Ei}(t)$ is the exponential integral defined by

$$\mathrm{Ei}(t) = \int_t^\infty \frac{e^{-u}}{u} du. \tag{5.5}$$

Therefore, for this case, the DOFHDE (2.3) is

$$\begin{gathered}\int_0^1 D^q\left[\frac{x(t)}{f(t,x(t))}\right]dq = g(t,x(t)), \quad t \in J, \\ x(0) = 0,\end{gathered} \tag{5.6}$$

and it is equivalent to the following equation:

$$x(t) = f(t,x(t))\int_0^t e^{t-\tau}\mathrm{Ei}(t-\tau)g(\tau,x(\tau))d\tau, \tag{5.7}$$

such that the operators A and B in Theorem 4.2 are

$$Ax(t) = f(t,x(t)), \qquad Bx(t) = \int_0^t e^{t-\tau}\mathrm{Ei}(t-\tau)g(\tau,x(\tau))d\tau. \tag{5.8}$$

(2) Two-term equation: Let $b(q) = a_1\delta(q-q_1) + a_2\delta(q-0)$, which $0 < q_1 < 1$ also, a_1 and a_2 are nonnegative constant coefficients and δ is the Dirac delta function. Then by the following inverse Laplace transform [2]:

$$\mathcal{L}^{-1}\left\{\frac{1}{B(s)}; t-\tau\right\} = \mathcal{L}^{-1}\left\{\frac{1}{a_1 s^{q_1} + a_2}; t-\tau\right\} = \frac{1}{a_1}(t-\tau)^{q_1-1}E_{q_1,q_1}\left(-\frac{a_2}{a_1}(t-\tau)^{q_1}\right), \tag{5.9}$$

where $E_{\lambda,\mu}(z)$ is the Mittag-Leffler function in two parameters

$$E_{\lambda,\mu}(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\lambda j + \mu)} \qquad \lambda,\mu > 0,\ z \in \mathbb{C} \tag{5.10}$$

we get the DOFHDE (2.3) as

$$a_1 D^{q_1}\left[\frac{x(t)}{f(t,x(t))}\right] + a_2\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J, \\ x(0) = 0. \tag{5.11}$$

It is equivalent to the following equation

$$x(t) = f(t,x(t)) \int_0^t \left(\frac{1}{a_1}(t-\tau)^{q_1-1} E_{q_1,q_1}\left(-\frac{a_2}{a_1}(t-\tau)^{q_1}\right)\right) g(\tau, x(\tau))d\tau, \tag{5.12}$$

such that the operators A and B in Theorem 4.2 are

$$Ax(t) = f(t,x(t)), \qquad Bx(t) = \frac{1}{a_1}\int_0^t \left((t-\tau)^{q_1-1} E_{q_1,q_1}\left(-\frac{a_2}{a_1}(t-\tau)^{q_1}\right)\right) g(\tau, x(\tau))d\tau. \tag{5.13}$$

(3) Three-term equation: Let,

$$b(q) = a_1\delta(q-q_1) + a_2\delta(q-q_2) + a_3\delta(q-0), \tag{5.14}$$

which $1 > q_1 > q_2 > 0$ and a_1, a_2, and a_3 are nonnegative constant coefficients and δ is the Dirac delta function. Then, by virtue of [2]

$$\mathcal{L}^{-1}\left\{\frac{1}{B(s)}; t-\tau\right\} = \mathcal{L}^{-1}\left\{\frac{1}{a_1 s^{q_1} + a_2 s^{q_2} + a_3}; t-\tau\right\} = G_3(t-\tau), \tag{5.15}$$

where

$$G_3(t-\tau) = \frac{1}{a_1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{a_3}{a_1}\right)^k (t-\tau)^{q_1(k+1)-1} E^{(k)}_{q_1-q_2, q_1+kq_2}\left(-\frac{a_2}{a_1}(t-\tau)^{q_1-q_2}\right), \tag{5.16}$$

and $E^{(k)}_{\lambda,\mu}(z)$ is the kth derivative of the Mittag-Leffler function in two parameters

$$E^{(k)}_{\lambda,\mu}(z) \equiv \frac{d^k}{dz^k}E_{\lambda,\mu}(z) = \sum_{j=0}^{\infty}\frac{(j+k)!z^j}{j!\Gamma(\lambda j + \lambda k + \mu)}, \quad k = 0,1,2,\ldots. \tag{5.17}$$

We get the DOFHDE (2.3) as

$$a_1 D^{q_1}\left[\frac{x(t)}{f(t,x(t))}\right] + a_2 D^{q_2}\left[\frac{x(t)}{f(t,x(t))}\right] + a_3\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \\ x(0) = 0. \tag{5.18}$$

It is equivalent to the following equation:

$$x(t) = f(t, x(t)) \int_0^t G_3(t-\tau) g(\tau, x(\tau)) d\tau, \tag{5.19}$$

such that the operators A and B in Theorem 4.2 are

$$Ax(t) = f(t, x(t)), \qquad Bx(t) = \int_0^t G_3(t-\tau) g(\tau, x(\tau)) d\tau. \tag{5.20}$$

(4) General Case: n-term equation: suppose that

$$b(q) = a_0\delta(q-q_0) + a_1\delta(q-q_1) + a_2\delta(q-q_2) + \cdots + a_n\delta(q-q_n), \tag{5.21}$$

which $1 > q_n > q_{n-1} > \cdots > q_0 > 0$ and a_i for $i = 0, 1, 2, \ldots, n$ are nonnegative constant coefficients. Therefore, by the following inverse Laplace transform [2], we have

$$\mathcal{L}^{-1}\left\{\frac{1}{B(s)}; t-\tau\right\} = \mathcal{L}^{-1}\left\{\frac{1}{a_0 s^{q_0} + a_1 s^{q_1} + \cdots + a_n s^{q_n}}; t-\tau\right\} = G_n(t-\tau), \tag{5.22}$$

where

$$\begin{aligned} G_n(t-\tau) = {} & \frac{1}{a_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{k_0+k_1+\cdots+k_{n-2}=m}(m; k_0, k_1, \ldots, k_{n-2}) \\ & \times \prod_{i=0}^{n-2}\left(\frac{a_i}{a_n}\right)^{k_i}(t-\tau)^{(q_n-q_{n-1})m+q_n+\sum_{j=0}^{n-2}(q_{n-1}-q_j)k_j-1} \\ & \times E^{(m)}_{q_n-q_{n-1},+q_n+\sum_{j=0}^{n-2}(q_{n-1}-q_j)k_j}\left(-\frac{a_{n-1}}{a_n}(t-\tau)^{q_n-q_{n-1}}\right). \end{aligned} \tag{5.23}$$

Thus, for this case, the DOFHDE (2.3) is

$$\begin{gathered} a_0 D^{q_0}\left[\frac{x(t)}{f(t, x(t))}\right] + a_1 D^{q_1}\left[\frac{x(t)}{f(t, x(t))}\right] + \cdots + a_n D^{q_n}\left[\frac{x(t)}{f(t, x(t))}\right] = g(t, x(t)), \\ x(0) = 0. \end{gathered} \tag{5.24}$$

It is equivalent to the following equation:

$$x(t) = f(t, x(t)) \int_0^t G_n(t-\tau) g(\tau, x(\tau)) d\tau \tag{5.25}$$

and the operators A and B in Theorem 4.2 are given by

$$Ax(t) = f(t, x(t)), \qquad Bx(t) = \int_0^t G_n(t-\tau) g(\tau, x(\tau)) d\tau. \tag{5.26}$$

6. Conclusions

In this paper, we introduced a new class; the fractional hybrid differential equations of distributed order and stated an existence theorem for it. We pointed out a fixed point theorem in the Banach algebra for the existence of solution. Basis of this theorem is on finding two operator equations which in special cases for multiterms fractional hybrid equations are given with respect to the derivatives of Mittag-Leffler function.

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