Research Article

Iterative Algorithms with Perturbations for Solving
the Systems of Generalized Equilibrium Problems
and the Fixed Point Problems of
Two Quasi-Nonexpansive Mappings

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We introduce new iterative algorithms with perturbations for finding a common element of the set of solutions of the system of generalized equilibrium problems and the set of common fixed points of two quasi-nonexpansive mappings in a Hilbert space. Under suitable conditions, strong convergence theorems are obtained. Furthermore, we also consider the iterative algorithms with perturbations for finding a common element of the solution set of the systems of generalized equilibrium problems and the common fixed point set of the super hybrid mappings in Hilbert spaces.

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and $C$ a nonempty closed convex subset of $H$ and let $T$ be a mapping of $C$ into $H$. Then, $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is said to be quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T) := \{x \in C : Tx = x\}$. It is well known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping $T$ is closed and convex; see Itoh and Takahashi [1]. A mapping $T : C \to H$ is called nonspreading [2] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad (1.1)$$

for all $x, y \in C$. We remark that nonlinear every nonspreading mappings are quasi-nonexpansive mappings if the set of fixed points is nonempty.
Recall that a mapping $\Psi : C \to H$ is said to be $\mu$-inverse strongly monotone if there exists a positive real number $\mu$ such that

$$\langle \Psi x - \Psi y, x - y \rangle \geq \mu \|\Psi x - \Psi y\|^2, \quad \forall x, y \in C. \quad (1.2)$$

If $\Psi$ is a $\mu$-inverse strongly monotone mapping of $C$ into $H$, then it is obvious that $\Psi$ is $1/\mu$-Lipschitz continuous.

Let $G : C \times C \to \mathbb{R}$ be a bifunction and $\Psi : C \to H$ be $\mu$-inverse strongly monotone mapping. The generalized equilibrium problem (for short, GEP) for $F$ and $\Psi$ is to find $z \in C$ such that

$$G(z, y) + \langle \Psi z, y - z \rangle \geq 0, \quad \forall y \in C. \quad (1.3)$$

The problem (1.3) was studied by Moudafi [3]. The set of solutions for problem (1.3) is denoted by $\text{GEP}(F, \Psi)$, that is,

$$\text{GEP}(F, \Psi) = \{ z \in C : G(z, y) + \langle \Psi z, y - z \rangle \geq 0, \forall y \in C \}. \quad (1.4)$$

If $\Psi \equiv 0$ in (1.3), then GEP reduces to the classical equilibrium problem and $\text{GEP}(G, 0)$ is denoted by $\text{EP}(G)$, that is,

$$\text{EP}(G) = \{ z \in C : G(z, y) \geq 0, \forall y \in C \}. \quad (1.5)$$

If $G \equiv 0$ in (1.3), then GEP reduces to the classical variational inequality and $\text{GEP}(0, \Psi)$ is denoted by $\text{VI}(\Psi, C)$, that is,

$$\text{VI}(\Psi, C) = \{ z \in C : \langle \Psi z, y - z \rangle \geq 0, \forall y \in C \}. \quad (1.6)$$

The problem (1.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, Min–Max problems, the Nash equilibrium problems in noncooperative games, and others; see, for example, Blum and Oettli [4] and Moudafi [3].

nonexpansive mappings and the set of solutions of a general system of variational inequalities for a cocoercive mapping in a real Hilbert space. Their results extend and improve many results in the literature.

In 1967, Wittmann [11] (see also [12]) proved the strong convergence theorem of Halpern’s type [13] \( \{x_n\} \) defined by, for any \( x_1 = x \in C \),
\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},
\]
(1.7)
where \( \{\alpha_n\} \subset (0, 1) \) satisfies \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), and \( \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty \). In [14], Kurokawa and Takahashi also studied the following Halpern’s type for nonspreading mappings in a Hilbert space; see also Hojo and Takahashi [15]. Let \( T \) be a nonspreading mapping of \( C \) into itself. Let \( u \in C \) and define two sequences \( \{x_n\} \) and \( \{z_n\} \) in \( C \) as follows: \( x_1 = x \in C \) and
\[
x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \quad \text{where} \quad z_n = \frac{1}{n} \sum_{k=0}^{n-1} Tx_k, \quad \forall n \in \mathbb{N},
\]
(1.8)
for all \( n = 1, 2, \ldots \), where \( \{\alpha_n\} \subset (0, 1) \), \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). If \( F(T) \) is nonempty, they proved that \( \{x_n\} \) and \( \{z_n\} \) converge strongly to \( P_{F(T)}u \), where \( P_{F(T)} \) is the metric projection of \( H \) onto \( F(T) \). Recently, Yao and Shahzad [16] gave the following iteration process for nonexpansive mappings with perturbation: \( x_1 \in C \) and
\[
x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(\alpha_n u_n + (1 - \alpha_n)Tx_n), \quad \forall n \in \mathbb{N},
\]
(1.9)
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \([0, 1]\), and the sequence \( \{u_n\} \subseteq H \) is a small perturbation for the \( n \)-step iteration satisfying \( \|u_n\| \to 0 \) as \( n \to \infty \). In fact, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate.

On the other hand, very recently, Chuang et al. [17] considered the following iteration process for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points for a quasi-nonexpansive mapping \( T : C \to H \) with perturbation
\[
q_1 \in H, \\
x_n \in C, \quad \text{such that} \quad G(x_n, y) + \frac{1}{r_n} \langle y - x_n, x_n - q_n \rangle \geq 0, \quad \forall y \in C,
\]
(1.10)
\[
y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\
q_{n+1} = \alpha_n u_n + (1 - \alpha_n)y_n, \quad \forall n \in \mathbb{N},
\]
where \( C \) is a nonempty closed convex subset of \( H \), \( G : C \times C \to \mathbb{R} \) is a function, \( \{\alpha_n\} \) and \( \{\beta_n\} \) are real sequences in \((0, 1)\), and \( \{u_n\} \subset H \) is a convergent sequence and \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \). They obtained a strong convergence theorem for such iterations.

In this paper, inspired and motivated by Yao and Shahzad [16], S. Takahashi and W. Takahashi [18] and Chuang et al. [17], we introduce a new iterative algorithms with perturbations for finding a common element of the set of solutions of the system of generalized equilibrium problems and the set of common fixed points of two quasi-nonexpansive
mappings in a Hilbert space. Under suitable conditions, strong convergence theorems are obtained. Furthermore, we also consider the iterative algorithms with perturbations for finding a common element of the solution set of the system of generalized equilibrium problems and the common fixed point set of the super hybrid mappings in a Hilbert space.

2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. We denote the strongly convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. In a Hilbert space, it is known that

$$
\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2,
$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$; see [19]. Furthermore, we have that for any $x, y, u, v \in H$

$$
2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.
$$

Let $C$ be a nonempty closed convex subset of $H$ and $x \in H$. We know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C}\|x - y\|$. We denote such a correspondence by $z = P_Cx$. The mapping $P_C$ is called the metric projection of $H$ onto $C$. It is known that $P_C$ is nonexpansive and

$$
\langle x - P_Cx, P_Cx - u \rangle \geq 0;
$$

for all $x \in H$ and $u \in C$; see [19, 20] for more details.

Let $C$ be a nonempty, closed and convex subset of $H$ and let $G : C \times C \to \mathbb{R}$ be a bifunction. For solving the generalized equilibrium problem, let us assume that the bifunction $G : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $G(x, x) = 0$ for all $x \in C$;

(A2) $G$ is monotone, that is, $G(x, y) + G(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$

$$
\lim_{t \downarrow 0} G(tz + (1 - t)x, y) \leq G(x, y);
$$

(A4) for each $x \in C$, $G(x, \cdot)$ is convex and lower semicontinuous.

We know the following lemma which appears implicitly in Blum and Oettli [4].

**Lemma 2.1** (see [4]). Let $C$ be a nonempty closed convex subset of $H$ and let $G$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists a unique $z \in C$ such that

$$
G(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \forall y \in C.
$$

The following lemma was also given in Combettes and Hirstoaga [5].
Lemma 2.2 (see [5]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $G : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{ z \in C : G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\} \quad (2.6)$$

for all $x \in H$. Then the following hold:

(i) $T_r$ is single-valued;

(ii) $T_r$ is firmly nonexpansive, that is, for any $x, y \in C$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \quad (2.7)$$

(iii) $(EP)$ is a closed convex subset of $C$;

(iv) $F(T_r) = EP(G)$.

Remark 2.3. For any $x \in H$ and $r > 0$, by Lemma 2.2 (i), there exists $u \in H$ such that

$$G(u, y) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \quad \forall y \in H. \quad (2.8)$$

Replacing $x$ with $x - r\Psi x \in H$ in (2.8), we have

$$G(u, y) + \langle \Psi x, y - u \rangle + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \quad \forall y \in H, \quad (2.9)$$

where $\Psi : H \to H$ is an inverse strongly monotone mapping.

Lemma 2.4 (see [21]). Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_n < \Gamma_{n+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \quad (2.10)$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(1) \leq \tau(2) \leq \cdots$ and $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}, \quad \forall n \in \mathbb{N}$.

Lemma 2.5 (see [22]). Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, $\{t_n\}$ a sequence of real numbers with $\limsup t_n \leq 0$. Suppose that

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n t_n + u_n, \quad \forall n \in \mathbb{N}. \quad (2.11)$$

Then $\lim_{n \to \infty} a_n = 0$. 

3. Main Results

Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $\Psi_i$ a $\mu_i$-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two mappings. Let $\{x_n\}$ be a sequence generated in the following manner:

\[
x_1 \in H,
G_1(u_{n,1}, y) + \langle \Psi_1 x_n, y - u_{n,1} \rangle + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle \geq 0, \quad \forall y \in C,
G_2(u_{n,2}, y) + \langle \Psi_2 x_n, y - u_{n,2} \rangle + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle \geq 0, \quad \forall y \in C,
\]

\[
\vdots
\]

\[
G_k(u_{n,k}, y) + \langle \Psi_k x_n, y - u_{n,k} \rangle + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle \geq 0, \quad \forall y \in C,
\]

(3.1)

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0, 1)$ and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Under certain appropriate assumptions imposed on the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, the strong convergence theorem of $\{x_n\}$ defined by (3.1) is studied in the following theorem.

**Theorem 3.1.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $\Psi_i$ a $\mu_i$-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two quasi-nonexpansive mappings such that $1 - T_j$ are demiclosed at zero with $\Omega := F(T_1) \cap F(T_2) \cap \bigcap_{i=1}^{k} \text{GEP}(G_i, \Psi_i) \neq \emptyset$. Let the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be defined by (3.1), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{u_n\}$ satisfy the following conditions:

(C1) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0$;

(C3) $\liminf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0$;

(C4) $\lim_{n \to \infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to $x^*$, where $x^* = P_\Omega u$. 
Proof. We first have that for all $i = 1, 2, \ldots, k$, $I - r_n \Psi_i$ is a nonexpansive mapping. Indeed, for all $x, y \in C$, we obtain
\[
\| (I - r_n \Psi_i)x - (I - r_n \Psi_i)y \|^2 = \| (x - y) - r_n (\Psi_i x - \Psi_i y) \|^2 \\
= \| x - y \|^2 - 2r_n (\Psi_i x - \Psi_i y, x - y) + r_n^2 \| \Psi_i x - \Psi_i y \|^2 \\
\leq \| x - y \|^2 - r_n (2\mu_i - r_n) \| \Psi_i x - \Psi_i y \|^2 \\
\leq \| x - y \|^2. 
\] (3.2)

Thus $I - r_n \Psi_i$ is nonexpansive for each $i \in \{1, 2, \ldots, k\}$. Now, let $w \in \Omega$ be arbitrary. By (C4), \{un\} is a bounded sequence, there exists $M \leq 0$ such that
\[
\sup_{n \in \mathbb{N}} \| u_n - w \| \leq M. \tag{3.3}
\]

For each $i = 1, 2, \ldots, k$ and $n \in \mathbb{N}$, we have from $u_{n,i} = T_{r_n_i}(x_n - r_n \Psi_i x_n)$ that
\[
\| u_{n,i} - w \| = \| T_{r_n_i}(x_n - r_n \Psi_i x_n) - T_{r_n_i}(w - r_n \Psi_i w) \| \\
\leq \| (x_n - r_n \Psi_i x_n) - (w - r_n \Psi_i w) \| \\
\leq \| x_n - w \|. \tag{3.4}
\]

which gives also that
\[
\| \omega_n - w \| \leq \frac{1}{k} \sum_{i=1}^{k} \| u_{n,i} - w \| \leq \| x_n - w \| \quad \forall w \in \Omega. \tag{3.5}
\]

Since $T_1$ is quasi-nonexpansive, we have
\[
\| y_n - w \| = \| \gamma_n \omega_n + (1 - \gamma_n) T_1 \omega_n - w \| \\
= \| \gamma_n (\omega_n - w) + (1 - \gamma_n) (T_1 \omega_n - w) \| \\
\leq \gamma_n \| \omega_n - w \| + (1 - \gamma_n) \| T_1 \omega_n - w \| \\
\leq \| \omega_n - w \|. \tag{3.6}
\]

So, we have from (3.5) and (3.6) and the quasi-nonexpansiveness of $T_2$ that
\[
\| x_{n+1} - w \| = \| \alpha_n (u_n - w) + (1 - \alpha_n) (z_n - w) \| \\
\leq \alpha_n \| u_n - w \| + (1 - \alpha_n) \| z_n - w \| \\
\leq \alpha_n \| u_n - w \| + (1 - \alpha_n) \{ \beta_n \| y_n - w \| + (1 - \beta_n) \| T_2 \omega_n - w \| \} \\
\leq \alpha_n \| u_n - w \| + (1 - \alpha_n) \{ \beta_n \| \omega_n - w \| + (1 - \beta_n) \| \omega_n - w \| \} \\
\leq \alpha_n \| u_n - w \| + (1 - \alpha_n) \| \omega_n - w \| \\
\leq \alpha_n \| u_n - w \| + (1 - \alpha_n) \| x_n - w \| \\
\leq \max \{ M, \| x_n - w \| \}. \tag{3.7}
\]
By Induction, we have that
\[ \|x_n - w\| \leq \max\{\|x_1 - w\|, M\}, \quad \forall n \in \mathbb{N}. \] (3.8)

Thus we obtain that \(\{\|x_n - w\|\}\) is bounded, so also \(\{x_n\}, \{y_n\}, \{z_n\}, \{\omega_n\}, \{T_1\omega_n\},\) and \(\{T_2\omega_n\}\) are bounded. Since \(\Omega\) is closed and convex, we can take \(x^* = P_\Omega u\). It follows that
\[ \|y_n - x^*\|^2 = \|\gamma_n(\omega_n - x^*) + (1 - \gamma_n)(T_1\omega_n - x^*)\|^2 \]
\[ = \beta_n\|y_n - x^*\|^2 + (1 - \beta_n)\|T_2\omega_n - x^*\|^2 \leq \|\omega_n - x^*\|^2 \] (3.9)

From (3.9), we have
\[ \|z_n - x^*\|^2 = \|\beta_n(y_n - x^*) + (1 - \beta_n)(T_2\omega_n - x^*)\|^2 \]
\[ = \beta_n\|y_n - x^*\|^2 + (1 - \beta_n)\|T_2\omega_n - x^*\|^2 \leq \|\omega_n - x^*\|^2 \] (3.10)

Hence we have from (3.5), (3.9), and (3.10) that
\[ \|\omega_{n+1} - x^*\|^2 \leq \|x_{n+1} - x^*\|^2 \]

\[ \leq \|\alpha_n(u_n - x^*) + (1 - \alpha_n)(z_n - x^*)\|^2 \]
\[ = \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|z_n - x^*\|^2 \leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|u_n - z_n\|^2 \]
\[ \leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|z_n - x^*\|^2 \]
\[ = \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\left\{ \beta_n\|y_n - x^*\|^2 + (1 - \beta_n)\|T_2\omega_n - x^*\|^2 \right\} \]
\[ \leq \alpha_n\|u_n - x^*\|^2 + \beta_n \left\{ \gamma_n\|\omega_n - x^*\|^2 + (1 - \gamma_n)\|T_1\omega_n - x^*\|^2 \right. \]
\[ - \gamma_n(1 - \gamma_n)\|\omega_n - T_1\omega_n\|^2 \] + \(1 - \beta_n)\|T_2\omega_n - x^*\|^2 \]
\[ - \beta_n(1 - \beta_n)\|y_n - T_2\omega_n\|^2 \]
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\[ \leq \alpha_n \|u_n - x^*\|^2 + \beta_n \left( \gamma_n \|\omega_n - x^*\|^2 + (1 - \gamma_n) \|\omega_n - x^*\|^2 \right) \\
- \gamma_n (1 - \gamma_n) \|\omega_n - T_1 \omega_n\|^2 + (1 - \beta_n) \|\omega_n - x^*\|^2 \\
- \beta_n (1 - \beta_n) \|y_n - T_2 \omega_n\|^2 \\
= \alpha_n \|u_n - x^*\|^2 + \|\omega_n - x^*\|^2 - \gamma_n (1 - \gamma_n) \|\omega_n - T_1 \omega_n\|^2 \\
- \beta_n (1 - \beta_n) \|y_n - T_2 \omega_n\|^2. \]

\((3.11)\)

We also have that

\[ \gamma_n (1 - \gamma_n) \|\omega_n - T_1 \omega_n\|^2 \leq \alpha_n \|u_n - x^*\|^2 + \|\omega_n - x^*\|^2 - \|\omega_n + x^*\|^2, \]

\((3.12)\)

\[ \beta_n (1 - \beta_n) \|y_n - T_2 \omega_n\|^2 \leq \alpha_n \|u_n - x^*\|^2 + \|\omega_n - x^*\|^2 - \|\omega_n + x^*\|^2. \]

\((3.13)\)

Furthermore, we have from \(y_n = \gamma_n \omega_n + (1 - \gamma_n) T_1 \omega_n\) that

\[ \|\omega_n - T_2 \omega_n\| \leq \|\omega_n - y_n\| + \|y_n - T_2 \omega_n\| \\
= \|\omega_n - \gamma_n \omega_n - (1 - \gamma_n) T_1 \omega_n\| + \|y_n - T_2 x_n\| \\
= (1 - \gamma_n) \|\omega_n - T_1 \omega_n\| + \|y_n - T_2 \omega_n\|. \]

\((3.14)\)

On the other hand, since \(x_{n+1} - x^* = \alpha_n (u_n - x^*) + (1 - \alpha_n) (z_n - x^*)\), we have

\[ \|\omega_{n+1} - x^*\|^2 \leq \|x_{n+1} - x^*\|^2 \\
\leq (1 - \alpha_n) \|z_n - x^*\|^2 + 2 \alpha_n \langle u_n - x^*, x_{n+1} - x^* \rangle \\
\leq (1 - \alpha_n) \|\omega_n - x^*\|^2 + 2 \alpha_n \langle u_n - u, x_{n+1} - x^* \rangle \\
= (1 - \alpha_n) \|\omega_n - x^*\|^2 + 2 \alpha_n \langle u_n - u, x_{n+1} - x^* \rangle \\
+ 2 \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\
= (1 - \alpha_n) \|\omega_n - x^*\|^2 + 2 \alpha_n \langle u_n - u, x_{n+1} - x^* \rangle \\
+ 2 \alpha_n \langle u - x^*, x_{n+1} - \omega_n \rangle + 2 \alpha_n \langle u - x^*, \omega_n - x^* \rangle. \]

\((3.15)\)

We also have that

\[ \|x_{n+1} - \omega_n\| \leq \|x_{n+1} - y_n\| + \|y_n - \omega_n\| \\
= \|\alpha_n (u_n - y_n) + (1 - \alpha_n) (z_n - y_n)\| + \| (1 - \gamma_n) (\omega_n - T_1 \omega_n)\| \\
\leq \alpha_n \|u_n - y_n\| + (1 - \alpha_n) \|\beta_n y_n + (1 - \beta_n) T_2 \omega_n - y_m\| \\
+ (1 - \gamma_n) \|\omega_n - T_1 \omega_n\| \\
= \alpha_n \|u_n - y_n\| + (1 - \alpha_n)(1 - \beta_n) \|y_n - T_2 \omega_n\| \\
+ (1 - \gamma_n) \|\omega_n - T_1 \omega_n\|. \]

\((3.16)\)
Moreover, for any $i \in \{1, 2, \ldots, k\}$, we have from $u_{n,i} = T_{r_{n,i}}(x_n - r_n \Psi_i x_n)$ that
\[
\|u_{n,i} - x^*\|^2 \leq \|(x_n - x^*) - r_n(\Psi_i x_n - \Psi_i x^*)\|^2
\]
\[
= \|(x_n - x^*)\|^2 - 2r_n(x_n - x^*, \Psi_i x_n - \Psi_i x^*) + r_n^2\|\Psi_i x_n - \Psi_i x^*\|^2
\]
\[
\leq \|(x_n - x^*)\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2. \tag{3.17}
\]

It follows that
\[
\|\omega_n - x^*\|^2 = \left\| \sum_{i=1}^k \frac{1}{k} (u_{n,i} - x^*) \right\|^2
\]
\[
\leq \frac{1}{k} \sum_{i=1}^k \|u_{n,i} - x^*\|^2 \tag{3.18}
\]
\[
\leq \|x_n - x^*\|^2 - \frac{1}{k} \sum_{i=1}^k r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2.
\]

This implies that
\[
\|x_{n+1} - x^*\|^2 = \|\alpha_n (u_n - x^*) + (1 - \alpha_n)(z_n - x^*)\|^2
\]
\[
\leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|\omega_n - x^*\|^2
\]
\[
\leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2
\]
\[
- \frac{1}{k} \sum_{i=1}^k r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2, \tag{3.19}
\]

and hence
\[
(1 - \alpha_n) \frac{1}{k} \sum_{i=1}^k r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2 \leq \alpha_n\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.20}
\]

Furthermore, we have from Lemma 2.2 that for any $i \in \{1, 2, \ldots, k\}$, we have
\[
\|u_{n,i} - x^*\|^2 \leq \langle (x_n - r_n \Psi_i x_n) - (x^* - r_n \Psi_i x^*), u_{n,i} - x^* \rangle
\]
\[
= \frac{1}{2} \left\{ \|(x_n - r_n \Psi_i x_n) - (x^* - r_n \Psi_i x^*)\|^2 + \|u_{n,i} - x^*\|^2 - \|(x_n - r_n \Psi_i x_n) - (x^* - r_n \Psi_i x^*) - (u_{n,i} - x^*)\|^2 \right\}
\]
\[
\leq \frac{1}{2} \left\{ \|x_n - x^*\|^2 + \|u_{n,i} - x^*\|^2 - \|(x_n - u_{n,i}) - r_n(\Psi_i x_n - \Psi_i x^*)\|^2 \right\} \tag{3.21}
\]
\[
= \frac{1}{2} \left\{ \|x_n - x^*\|^2 + \|u_{n,i} - x^*\|^2 - \|x_n - u_{n,i}\|^2 - r_n^2\|\Psi_i x_n - \Psi_i x^*\|^2 + 2r_n\langle x_n - u_{n,i}, \Psi_i x_n - \Psi_i x^* \rangle \right\}. \]
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This implies that

\[ \| u_{n,i} - x^* \|^2 \leq \| x_n - x^* \|^2 - \| x_n - u_{n,i} \|^2 + 2r_n \| x_n - u_{n,i} \| \| \Psi_i x_n - \Psi_i x^* \|. \] (3.22)

Then we have from (3.22) that

\[ \| \omega_n - x^* \|^2 \leq \frac{1}{k} \sum_{i=1}^{k} \| u_{n,i} - x^* \|^2 \leq \| x_n - x^* \|^2 - \frac{1}{k} \sum_{i=1}^{k} \| u_{n,i} - x_n \|^2 \] (3.23)

\[ + \frac{1}{k} \sum_{i=1}^{k} 2r_n \| x_n - u_{n,i} \| \| \Psi_i x_n - \Psi_i x^* \|. \]

Hence we have from (3.23) that

\[ \| x_{n+1} - x^* \|^2 \leq \alpha_n \| u_n - x^* \|^2 + (1 - \alpha_n) \| \omega_n - x^* \|^2 \]

\[ \leq \alpha_n \| u_n - x^* \|^2 + (1 - \alpha_n) \left( \| x_n - x^* \|^2 - \frac{1}{k} \sum_{i=1}^{k} \| u_{n,i} - x_n \|^2 \right) \]

\[ + (1 - \alpha_n) \left( \frac{1}{k} \sum_{i=1}^{k} 2r_n \| x_n - u_{n,i} \| \| \Psi_i x_n - \Psi_i x^* \| \right). \] (3.24)

It follows that

\[ (1 - \alpha_n) \frac{1}{k} \sum_{i=1}^{k} \| u_{n,i} - x_n \|^2 \leq \alpha_n \| u_n - x^* \|^2 + \| x_n - x^* \|^2 - \| x_{n+1} - x^* \|^2 \]

\[ + (1 - \alpha_n) \left( \frac{1}{k} \sum_{i=1}^{k} 2r_n \| x_n - u_{n,i} \| \| \Psi_i x_n - \Psi_i x^* \| \right). \] (3.25)

Next, we will consider the following two cases.

Case A. Putting \( \Gamma_n = \| \omega_n - x^* \|^2 \) for all \( n \in \mathbb{N} \). Suppose that \( \Gamma_{n+1} \leq \Gamma_n \) for all \( n \in \mathbb{N} \). In this case \( \lim_{n \to \infty} \Gamma_n \) exists and then \( \lim_{n \to \infty} (\Gamma_{n+1} - \Gamma_n) = 0 \). By (C1), (C3), and (3.12), we have

\[ \lim_{n \to \infty} \| \omega_n - T_1 \omega_n \| = 0. \] (3.26)

Similarly by (C1), (C2), and (3.13), we also have

\[ \lim_{n \to \infty} \| y_n - T_2 \omega_n \| = 0. \] (3.27)
So, we have from (3.14), (3.26), and (3.27) that

$$\lim_{n \to \infty} \|\omega_n - T_2 \omega_n\| = 0. \tag{3.28}$$

Since $\lim_{n \to \infty} \|\omega_n - x^\ast\|$ exists, we have from (3.11) and (3.26)

$$\lim_{n \to \infty} \|\omega_n - x^\ast\| = \lim_{n \to \infty} \|x_n - x^\ast\|. \tag{3.29}$$

We also have from (C1), (3.16), (3.26), and (3.27) that

$$\lim_{n \to \infty} \|x_{n+1} - \omega_n\| = 0. \tag{3.30}$$

Since $\lim_{n \to \infty} \|x_n - x^\ast\|$ exists we have from (C1) and (3.20) that

$$\lim_{n \to \infty} \|\Psi_i x_{n+1} - \Psi_i x^\ast\| = 0, \quad \forall i = 1, 2, \ldots, k. \tag{3.31}$$

This together with (3.25) and the existence of $\lim_{n \to \infty} \|x_n - x^\ast\|$ implies that

$$\lim_{n \to \infty} \|u_{n,i} - x_n\| = 0, \quad \forall i = 1, 2, \ldots, k, \tag{3.32}$$

which gives that

$$\|\omega_n - x_n\| \leq \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\| \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.33}$$

So, from (3.30), $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$. Furthermore, we have from (3.33) that

$$\|\omega_{n+1} - \omega_n\| \leq \|\omega_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - \omega_n\| \longrightarrow 0 \quad \text{as} \quad n \to \infty; \tag{3.34}$$

that is

$$\lim_{n \to \infty} \|\omega_{n+1} - \omega_n\| = 0. \tag{3.35}$$

Now, since $\{\omega_n\}$ is a bounded sequence, there exists a subsequence $\{\omega_{n_j}\}$ of $\{\omega_n\}$ such that

$$\limsup_{n \to \infty} \langle u - x^\ast, \omega_n - x^\ast \rangle = \lim_{j \to \infty} \langle u - x^\ast, \omega_{n_j} - x^\ast \rangle. \tag{3.36}$$

Without loss of generality, we may assume that $\omega_{n_j} \rightharpoonup v$. Since $T_1$ is demiclosed at zero and by (3.26), we conclude that $v \in F(T_1)$. Similarly, since $T_2$ is demiclosed at zero and by (3.28), we have $v \in F(T_2)$. Therefore, we get that

$$v \in F(T_1) \cap F(T_2). \tag{3.37}$$
Next, we show that $v \in \cap_{i=1}^{k} \text{GEP}(G_i, \Psi_i)$. For each $i \in \{1, 2, \ldots, k\}$, since $u_{n,i} = T_{r_n}(x_n - r_n \Psi_i x_n)$, we have

$$G_i(u_{n,i}, y) + \langle \Psi_i x_n, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, u_{n,i} - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.38}$$

From (A2), we also have

$$\langle \Psi_i x_n, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, u_{n,i} - x_n \rangle \geq G_i(y, u_{n,i}). \tag{3.39}$$

Replacing $n$ by $n_j$, we have

$$\langle \Psi_i x_{n_j}, y - u_{n_j,i} \rangle + \left( y - u_{n_j,i} \frac{u_{n_j,i} - x_{n_j}}{r_{n_j}} \right) \geq G_i(y, u_{n_j,i}). \tag{3.40}$$

Put $y_t = ty + (1-t)v$ for all $t \in (0,1]$ and $y \in C$. Since $v \in C$, then $y_t \in C$ and

$$\langle y_t - u_{n_j,i}, \Psi_i y_t \rangle \geq \langle y_t - u_{n_j,i}, \Psi_i y_t \rangle - \langle y_t - u_{n_j,i}, \Psi_i x_n \rangle$$

$$- \langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j}} \rangle + G_i(y_t, u_{n_j,i})$$

$$= \langle y_t - u_{n_j,i}, \Psi_i y_t - \Psi_i u_{n_j,i} \rangle + \langle y_t - u_{n_j,i}, \Psi_i u_{n_j,i} - \Psi_i x_{n_j} \rangle$$

$$- \langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j}} \rangle + G_i(y_t, u_{n_j,i}). \tag{3.41}$$

Since $\|u_{n_j,i} - x_{n_j}\| \to 0$ as $j \to \infty$, we obtain that $|\|\Psi_i u_{n_j,i} - \Psi_i x_{n_j}\| | \to 0$ as $j \to \infty$. Furthermore, by the monotonicity of $\Psi_i$, we obtain that

$$\langle y_t - u_{n_j,i}, \Psi_i y_t - \Psi_i u_{n_j,i} \rangle \geq 0. \tag{3.42}$$

Taking $j \to \infty$ in (3.41), we have from (A4) that

$$\langle y_t - v, \Psi_i y_t \rangle \geq G_i(y_t, v). \tag{3.43}$$

Now, from (A1), (A4), and (3.43), we also have

$$0 = G_i(y_t, y_t) \leq tG_i(y_t, y) + (1-t)G_i(y_t, v)$$

$$\leq tG_i(y_t, y) + (1-t)\langle y_t - v, \Psi_i y_t \rangle$$

$$= tG_i(y_t, y) + (1-t)\langle y - v, \Psi_i y_t \rangle, \tag{3.44}$$
which yields that
\begin{equation}
G_i(y_t, y) + (1-t)(y - v, \Psi_i y_t) \geq 0.
\end{equation} (3.45)

Taking \( t \to 0 \), we have, for each \( y \in C \)
\begin{equation}
G_i(v, y) + (y - v, \Psi_i v) \geq 0, \quad \forall i \in \{1, 2, \ldots, k\}.
\end{equation} (3.46)

This shows \( v \in \text{GEP}(G_i, \Psi_i) \), for all \( i = 1, 2, \ldots, k \). Then, \( v \in \bigcap_{i=1}^{k} \text{GEP}(G_i, \Psi_i) \). Hence we have \( v \in F(T_1) \cap F(T_2) \cap (\bigcap_{i=1}^{k} \text{GEP}(G_i, \Psi_i)) := \Omega \). So, we have from (3.36) that
\begin{equation}
\lim_{n \to \infty} \langle u - x^*, \omega_n - x^* \rangle = \langle u - x^*, v - x^* \rangle \leq 0.
\end{equation} (3.47)

By (C1), (C4), (3.15), (3.30), (3.47), and Lemma 2.5, we obtain that \( \lim_{n \to \infty} \| \omega_n - x^* \| = 0 \). Hence we have from (3.29) that \( \{x_n\} \) converges to \( x^* \), where \( x^* = P_{\Omega} u \).

Case B. Assume that there exists a subsequence \( \{\Gamma_n\}_{n \geq 0} \) of \( \{\Gamma_i\}_{n \geq 0} \) such that \( \Gamma_n < \Gamma_{n+1} \) for all \( i \in \mathbb{N} \). In this case, it follows from Lemma 2.4 that there exists a subsequence \( \{\Gamma_{\tau(n)}\} \) of \( \{\Gamma_n\} \) such that \( \Gamma_{\tau(n)+1} > \Gamma_{\tau(n)} \), where \( \tau : \mathbb{N} \to \mathbb{N} \) is defined by
\begin{equation}
\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \quad \forall n \in \mathbb{N}.
\end{equation} (3.48)

So, from (3.12), that
\begin{equation}
\| \omega_{\tau(n)+1} - x^* \|^2 - \| \omega_{\tau(n)} - x^* \|^2 + \gamma_{\tau(n)} (1 - \gamma_{\tau(n)}) \| \omega_{\tau(n)} - T_1 \omega_{\tau(n)} \|^2 \leq \alpha_{\tau(n)} \| u_{\tau(n)} - x^* \|^2.
\end{equation} (3.49)

Since \( \| \omega_{\tau(n)} - x^* \|^2 := \Gamma_{\tau(n)} < \Gamma_{\tau(n)+1} := \| \omega_{\tau(n)+1} - x^* \|^2 \), we have
\begin{equation}
\gamma_{\tau(n)} (1 - \gamma_{\tau(n)}) \| \omega_{\tau(n)} - T_1 \omega_{\tau(n)} \|^2 \leq \alpha_{\tau(n)} \| u_{\tau(n)} - x^* \|^2.
\end{equation} (3.50)

By (C1) and (C3), we have
\begin{equation}
\lim_{n \to \infty} \| \omega_{\tau(n)} - T_1 \omega_{\tau(n)} \| = 0.
\end{equation} (3.51)

By (3.15), we have
\begin{equation}
\| \omega_{\tau(n)+1} - x^* \|^2 \leq (1 - \alpha_{\tau(n)}) \| \omega_{\tau(n)} - x^* \|^2 + 2 \alpha_{\tau(n)} \langle u_{\tau(n)} - x^*, x_{\tau(n)+1} - x^* \rangle.
\end{equation} (3.52)
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Now, in view of $\Gamma_{T(n)} < \Gamma_{T(n)+1}$, we see that

$$
\left\| \omega_{T(n)} - x^* \right\|^2 \leq 2 \langle u_{T(n)} - x^*, \, x_{T(n)+1} - x^* \rangle
$$

$$
= 2 \langle u_{T(n)} - u, \, x_{T(n)+1} - x^* \rangle + 2 \langle u - x^*, \, x_{T(n)+1} - \omega_{T(n)} \rangle
$$

$$
+ 2 \langle u - x^*, \, \omega_{T(n)} - x^* \rangle.
$$

(3.53)

Furthermore, we also have from (3.13) that

$$
\beta_{T(n)} \left( 1 - \beta_{T(n)} \right) \left\| y_{T(n)} - T_2 \omega_{T(n)} \right\|^2 \leq \alpha_{T(n)} \left\| u_{T(n)} - x^* \right\|^2 + \left\| \omega_{T(n)} - x^* \right\|^2
$$

$$
- \left\| \omega_{T(n)+1} - x^* \right\|^2
$$

$$
\leq \alpha_{T(n)} \left\| u_{T(n)} - x^* \right\|^2.
$$

(3.54)

Applying (C1) and (C2) to the last inequality, we get that

$$
\lim_{n \to \infty} \left\| y_{T(n)} - T_2 \omega_{T(n)} \right\| = 0.
$$

(3.55)

By (C1), (3.16), (3.51), and (3.55), we have

$$
\lim_{n \to \infty} \left\| x_{T(n)+1} - \omega_{T(n)} \right\| = 0.
$$

(3.56)

By (3.33), we have

$$
\lim_{n \to \infty} \left\| \omega_{T(n)+1} - x_{T(n)+1} \right\| = 0.
$$

(3.57)

It follows from (3.56) and (3.57) that

$$
\lim_{n \to \infty} \left\| \omega_{T(n)+1} - \omega_{T(n)} \right\| = 0.
$$

(3.58)

Since $\{ \omega_{T(n)} \}$ is a bounded sequence, there exists a subsequence $\{ \omega_{T(n_j)} \}$ such that

$$
\limsup_{n \to \infty} \langle u - x^*, \omega_{T(n)} - x^* \rangle = \liminf_{j \to \infty} \langle u - x^*, \omega_{T(n_j)} - x^* \rangle.
$$

(3.59)

Following the same argument as the proof of Case A for $\{ \omega_{T(n_j)} \}$, we have that

$$
\limsup_{n \to \infty} \langle u - x^*, \omega_{T(n)} - x^* \rangle \leq 0.
$$

(3.60)

Using (C4), (3.53), (3.56), and (3.60), we have that

$$
\lim_{n \to \infty} \left\| \omega_{T(n)} - x^* \right\| = 0.
$$

(3.61)
By (3.58) and (3.61), we have that
\[
\lim_{n \to \infty} \| \omega_{\tau(n)+1} - x^* \| = 0.
\] (3.62)

By Lemma 2.4 (ii), we get \( \lim_{n \to \infty} \Gamma_n = 0 \); that is \( \lim_{n \to \infty} \| \omega_n - x^* \| = 0 \). We observe that
\[
\| x_{n+1} - x^* \|^2 \leq \alpha_n \| u_n - x^* \|^2 + (1 - \alpha_n) \| \omega_n - x^* \|^2.
\] (3.63)

Applying (C1), (C4), and \( \lim_{n \to \infty} \| \omega_n - x^* \|^2 = 0 \), we have immediately
\[
\lim_{n \to \infty} \| x_n - x^* \| = 0;
\] (3.64)

that is, \( \{ x_n \} \) converges strongly to \( x^* \), where \( x^* = P_{\Omega} u \). This completes the proof. \( \square \)

Setting \( \Psi_i \equiv 0 \) for all \( i = 1, 2, \ldots, k \) in Theorem 3.1, we obtain the following result.

**Corollary 3.2.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). For each \( i = 1, 2, \ldots, k \), let \( G_i: C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)-(A4). For each \( j = 1, 2 \), let \( T_j: C \to H \) be two quasi-nonexpansive mappings such that \( I - T_j \) are demiclosed at zero with \( \Omega := F(T_1) \cap F(T_2) \cap \bigcap_{i=1}^{k} EP(G_i) \neq \emptyset \). Let the sequences \( \{ x_n \}, \{ y_n \}, \) and \( \{ z_n \} \) be defined by

\[
x_1 \in H,
\]
\[
G_1(u_{n,1}, y) + \frac{1}{r_n} (y - u_{n,1}, u_{n,1} - x_n) \geq 0, \quad \forall y \in C,
\]
\[
G_2(u_{n,2}, y) + \frac{1}{r_n} (y - u_{n,2}, u_{n,2} - x_n) \geq 0, \quad \forall y \in C,
\]
\[
\vdots
\]
\[
G_k(u_{n,k}, y) + \frac{1}{r_n} (y - u_{n,k}, u_{n,k} - x_n) \geq 0, \quad \forall y \in C,
\] (3.65)

\[
\omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i},
\]
\[
y_n = \gamma_n \omega_n + (1 - \gamma_n) T_1 \omega_n,
\]
\[
z_n = \beta_n y_n + (1 - \beta_n) T_2 \omega_n,
\]
\[
x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \quad \forall n \in \mathbb{N},
\]

where \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \} \) satisfy the following conditions.

(C1) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);

(C2) \( \lim \inf_{n \to \infty} \beta_n (1 - \beta_n) > 0 \);

(C3) \( \lim \inf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0 \);

(C4) \( \lim_{n \to \infty} u_n = u \) for some \( u \in H \).
Then \( \{x_n\} \) converges strongly to \( x^* \), where \( x^* = P_{\Omega}u \).

In the next results, using Theorem 3.1, we have new strong convergence theorems for two nonexpansive mappings in a Hilbert space.

**Corollary 3.3.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). For each \( i = 1, 2, \ldots, k \), let \( G_i : C \times C \rightarrow \mathbb{R} \) be a bifunction satisfying (A1)–(A4) and \( \Psi_i \) a \( \mu_i \)-inverse strongly monotone mapping. For each \( j = 1, 2 \), let \( T_j : C \rightarrow H \) be two nonexpansive mappings such that \( \Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} \text{GEP}(G_i, \Psi_i)) \neq \emptyset \). Let the sequences \( \{x_n\} \), \( \{y_n\} \), and \( \{z_n\} \) be defined by (3.1), where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) satisfy the following conditions.

\[
(C1) \lim_{n \to \infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty; \\
(C2) \lim \inf_{n \to \infty} \beta_n (1 - \beta_n) > 0; \\
(C3) \lim \inf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0; \\
(C4) \lim_{n \to \infty} \mu_n = u \text{ for some } u \in H.
\]

Then \( \{x_n\} \) converges strongly to \( x^* \), where \( x^* = P_{\Omega}u \).

**4. Applications**

In this section, we present some convergence theorems deduced from the results in the previous section. Recall that a mapping \( T : C \rightarrow H \) is said to be nonspreading [2] if

\[
2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{4.1}
\]

for all \( x, y \in C \). Further, a mapping \( T : C \rightarrow H \) is said to be hybrid [23] if

\[
3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{4.2}
\]

for all \( x, y \in C \). These mappings are deduced from a firmly nonexpansive mapping in a Hilbert space.

A mapping \( F : C \rightarrow H \) is said to be firmly nonexpansive if

\[
\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle \tag{4.3}
\]

for all \( x, y \in C \); see, for instance, Browder [24] and Goebel and Kirk [25]. We also know that a firmly nonexpansive mapping \( F \) can be deduced from an equilibrium problem in a Hilbert space.

Recently, Kocourek et al. [26] introduced a more broad class of nonlinear mappings call generalized hybrid if there are \( \alpha, \beta \in \mathbb{R} \) such that

\[
\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2 \tag{4.4}
\]
for all \( x, y \in C \). Very recently, they defined a more broad class of mappings than the class of generalized hybrid mappings in a Hilbert space. A mapping \( S : C \to H \) is called super hybrid if there are \( \alpha, \beta, \gamma \in \mathbb{R} \) such that

\[
\alpha \| Sx - Sy \|^2 + (1 - \alpha + \gamma) \| x - Sy \|^2 \leq (\beta + (\beta - \alpha) \gamma) \| Sx - y \|^2 + (1 - \beta - (\beta - \alpha - 1) \gamma) \| x - y \|^2 \\
+ (\alpha - \beta) \gamma \| x - Sx \|^2 + \gamma \| y - Sy \|^2,
\]

(4.5)

for all \( x, y \in C \). We call such a mapping an \((\alpha, \beta, \gamma)\)-super hybrid mapping. We notice that an \((\alpha, \beta, 0)\)-super hybrid mapping is \((\alpha, \beta)\)-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. A super hybrid mapping is not quasi-nonexpansive generally. For more details, see [27]. Before proving, we need the following lemmas.

**Lemma 4.1** (see [27]). Let \( C \) be a nonempty subset of a Hilbert space \( H \) and let \( \alpha, \beta \) and \( \gamma \) be real numbers with \( \gamma \neq -1 \). Let \( S \) and \( T \) be mappings of \( C \) into \( H \) such that \( S = (1/(1+\gamma))T + (\gamma/(1+\gamma))I \). Then, \( T \) is \((\alpha, \beta, \gamma)\)-super hybrid if and only if \( S \) is \((\alpha, \beta)\)-generalized hybrid. In this case, \( F(S) = F(T) \).

**Lemma 4.2** (see [27]). Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( S : C \to H \) be a generalized hybrid mapping. Then \( S \) is demiclosed on \( C \).

Setting \( S_j := (1/(1+\gamma_j))T_j + (\gamma_j/(1+\gamma_j))I \) in Theorem 3.1, where \( T_j \) is a super hybrid mapping and \( \gamma_j \) is a real number, we obtain the following result.

**Theorem 4.3.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). For each \( i = 1, 2, \ldots, k \), let \( G_i : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)-(A4) and \( \Psi_i \) a \( \mu_i \)-inverse strongly monotone mapping. For each \( j = 1, 2 \), let \( T_j : C \to H \) be \((\alpha_j, \beta_j, \gamma_j)\)-super hybrid mappings such that \( \Omega := F(T_1) \cap F(T_2) \cap (r_{i=1}^k \text{GEP}(G_i, \Psi_i)) \neq \emptyset \). Let the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) be defined by

\[
x_1 \in H,
\]

\[
G_1(u_{n,1}, y) + (\Psi_1 x_n, y - u_{n,1}) + \frac{1}{r_{n}} (y - u_{n,1}, u_{n,1} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
G_2(u_{n,2}, y) + (\Psi_2 x_n, y - u_{n,2}) + \frac{1}{r_{n}} (y - u_{n,2}, u_{n,2} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
\vdots
\]

\[
G_k(u_{n,k}, y) + (\Psi_k x_n, y - u_{n,k}) + \frac{1}{r_{n}} (y - u_{n,k}, u_{n,k} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
\omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i},
\]

\[
y_n = \gamma_n \omega_n + (1 - \gamma_n) \left( \frac{1}{1 + \gamma_1} T_1 \omega_n + \frac{\gamma_1}{1 + \gamma_1} \omega_n \right),
\]

\[
z_n = \beta_n y_n + (1 - \beta_n) \left( \frac{1}{1 + \gamma_2} T_2 \omega_n + \frac{\gamma_2}{1 + \gamma_2} \omega_n \right),
\]

\[
x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \quad \forall n \in \mathbb{N},
\]
where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \((0,1)\) and \( \{u_n\} \subset H \) is a sequence and \( \{r_n\} \subset [a,2\mu) \) for some \( a > 0 \) and for all \( i \in \{1,2,\ldots,k\} \). Suppose the following conditions are satisfied.

(C1) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty; \)

(C2) \( \lim \inf_{n \to \infty} \beta_n (1 - \beta_n) > 0; \)

(C3) \( \lim \inf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0; \)

(C4) \( \lim_{n \to \infty} u_n = u \) for some \( u \in H. \)

Then \( \{x_n\} \) converges strongly to \( x^* \), where \( x^* = P_{\Omega} u. \)

**Proof.** For each \( j = 1,2 \), setting

\[
S_j = \frac{1}{1 + \gamma_j} T_j + \frac{\gamma_j}{1 + \gamma_j} I, \tag{4.7}
\]

we have from Lemma 4.1 that each \( S_j \) is a generalized hybrid mapping and \( F(S_j) = F(T_j) \). Since \( F(S_j) \neq \emptyset \), we have that each \( S_j \) is quasi-nonexpansive. Following the proof of Theorem 3.1 and applying Lemma 4.2, we have the desired result. This completes the proof.

Setting \( \Psi \equiv 0 \) in Theorem 4.3, we obtains the following result.

**Corollary 4.4.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). For each \( i = 1,2,\ldots,k \), let \( G_i : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)–(A4). For each \( j = 1,2 \), let \( T_j : C \to H \) be \((\alpha_j, \beta_j, \gamma_j)\)-super hybrid mappings such that \( \Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} EP(G_i)) \neq \emptyset \). Let the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) be defined by

\[
x_1 \in H, \quad G_1(u_{n,1}, y) + \frac{1}{r_n} (y - u_{n,1}, u_{n,1} - x_n) \geq 0, \quad \forall y \in C, \tag{4.8}
\]

\[
G_2(u_{n,2}, y) + \frac{1}{r_n} (y - u_{n,2}, u_{n,2} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
\vdots
\]

\[
G_k(u_{n,k}, y) + \frac{1}{r_n} (y - u_{n,k}, u_{n,k} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
\omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i},
\]

\[
y_n = \gamma_n \omega_n + (1 - \gamma_n) \left( \frac{1}{1 + \gamma_1} T_1 \omega_n + \frac{\gamma_1}{1 + \gamma_1} \omega_n \right),
\]

\[
z_n = \beta_n y_n + (1 - \beta_n) \left( \frac{1}{1 + \gamma_2} T_2 \omega_n + \frac{\gamma_2}{1 + \gamma_2} \omega_n \right),
\]

\[
x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \quad \forall n \in \mathbb{N},
\]
where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \((0, 1)\) and \(\{u_n\} \subset H\) is a sequence and \(\{r_n\} \subset [a, \infty)\) for some \(a > 0\). Suppose the following conditions are satisfied.

\[
\begin{align*}
(C1) \lim_{n \to \infty} \alpha_n &= 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty; \\
(C2) \liminf_{n \to \infty} \beta_n (1-\beta_n) &> 0; \\
(C3) \liminf_{n \to \infty} \gamma_n (1-\gamma_n) &> 0; \\
(C4) \lim_{n \to \infty} u_n = u \text{ for some } u \in H.
\end{align*}
\]

Then \(\{x_n\}\) converges strongly to \(x^*\), where \(x^* = P_H u\).

In Corollary 4.4, put \(G_i(x, y) = 0\) for all \(x, y \in C\) and \(r_n = 1\) for all \(n \in \mathbb{N}\). Then we have that \(u_{n,i} = x_n\) for all \(i = 1, 2, \ldots, k\), which gives that \(\omega_n = (1/k) \sum_{i=1}^{k} u_{n,i} = x_n\). Thus we obtain the following results from Corollary 4.4.

**Corollary 4.5.** Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\). For each \(j = 1, 2\), let \(T_j : C \to H\) be \((\alpha_j, \beta_j, \gamma_j)\)-super hybrid mappings such that \(F(T_1) \cap F(T_2) \neq \emptyset\). Let the sequences \(\{x_n\}, \{y_n\}, \text{ and } \{z_n\}\) be defined by

\[
\begin{align*}
x_1 &\in H, \\
y_n &= \gamma_n x_n + (1-\gamma_n) \left( \frac{1}{1+\gamma_1} T_1 x_n + \frac{\gamma_1}{1+\gamma_1} x_n \right), \\
z_n &= \beta_n y_n + (1-\beta_n) \left( \frac{1}{1+\gamma_2} T_2 x_n + \frac{\gamma_2}{1+\gamma_2} x_n \right), \\
x_{n+1} &= \alpha_n u_n + (1-\alpha_n) z_n, \quad \forall n \in \mathbb{N},
\end{align*}
\]

where \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) are sequences in \((0, 1)\) and \(\{u_n\} \subset H\) is a sequence. Suppose the following conditions are satisfied.

\[
\begin{align*}
(C1) \lim_{n \to \infty} \alpha_n &= 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty; \\
(C2) \liminf_{n \to \infty} \beta_n (1-\beta_n) &> 0; \\
(C3) \liminf_{n \to \infty} \gamma_n (1-\gamma_n) &> 0; \\
(C4) \lim_{n \to \infty} u_n = u \text{ for some } u \in H.
\end{align*}
\]

Then \(\{x_n\}\) converges strongly to \(x^*\), where \(x^* = P_{F(T_1)\cap F(T_2)} u\).

In Corollary 4.5, put \(T_1 = I\), the identity mapping, and \(T_2 := T\), an \((\alpha, \beta, \gamma)\)-super hybrid mapping. Thus we obtain the following results.

**Corollary 4.6.** Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\). Let \(T\) be an \((\alpha, \beta, \gamma)\)-super hybrid mapping such that \(F(T) \neq \emptyset\). Let the sequences \(\{x_n\}, \{y_n\}, \text{ and } \{z_n\}\) be defined by

\[
\begin{align*}
x_1 &\in H, \\
z_n &= \beta_n x_n + (1-\beta_n) \left( \frac{1}{1+\gamma} T x_n + \frac{\gamma}{1+\gamma} x_n \right), \\
x_{n+1} &= \alpha_n u_n + (1-\alpha_n) z_n, \quad \forall n \in \mathbb{N},
\end{align*}
\]
where \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are sequences in \((0, 1)\) and \( \{ u_n \} \subset H \) is a sequence. Suppose the following conditions are satisfied.

\[
\begin{align*}
(C1) \quad & \lim_{n \to \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty; \\
(C2) \quad & \lim \inf_{n \to \infty} \beta_n (1 - \beta_n) > 0; \\
(C3) \quad & \lim_{n \to \infty} u_n = u \quad \text{for some} \quad u \in H.
\end{align*}
\]

Then \( \{ x_n \} \) converges strongly to \( x^* \), where \( x^* = P_{F(T)} u \).

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**References**


