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Research Article

The Existence of Fixed Points for Nonlinear Contractive Maps in Metric Spaces with w-Distances

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Some fixed point theorems for (φ, ψ, p)-contractive maps and (φ, k, p)-contractive maps on a complete metric space are proved. Presented fixed point theorems generalize many results existing in the literature.

1. Introduction and Preliminaries

Branciari [1] established a fixed point result for an integral type inequality, which is a generalization of Banach contraction principle. Kada et al. [2] introduced and studied the concept of w-distance on a metric space. They give examples of w-distances and improved Caristi's fixed point theorem, Ekeland's ϵ-variational's principle, and the nonconvex minimization theorem according to Takahashi (see many useful examples and results on w-distance in [2–5] and in references therein). Kada et al. [2] defined the concept of w-distance in a metric space as follows.

Definition 1.1 (see [2]). Let X be a metric space endowed with a metric d. A function $p : X \times X \to [0, \infty)$ is called a w-distance on X if it satisfies the following properties:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$,

(2) p is lower semicontinuous in its second variable, that is, if $x \in X$ and $y_n \to y$ in X then $p(x, y) \leq \liminf_{n \to \infty} p(x, y_n)$,

(3) for each $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

We denote by Φ the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following hypotheses:

(c1) φ is continuous and nondecreasing,

(c2) $\varphi(t) = 0$ if and only if $t = 0$.

We denote by Ψ the set of functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying the following hypotheses:

(h1) ψ is right continuous and nondecreasing,

(h2) $\psi(t) < t$ for all $t > 0$.

Let p be a w-distance on metric space (X,d), $\varphi \in \Phi$ and $\psi \in \Psi$. A map T from X into itself is a (φ,ψ,p)-contractive map on X if for each $x, y \in X$, $\varphi p(Tx,Ty) \le \psi\varphi p(x,y)$.

The following lemmas are used in the next section.

Lemma 1.2 (see [3]). *If $\psi \in \Psi$, then $\lim_{n\to\infty}\psi^n(t) = 0$ for each $t > 0$, and if $\varphi \in \Phi$, $\{a_n\} \subseteq [0,\infty)$ and $\lim_{n\to\infty}\varphi(a_n) = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 1.3 (see [2]). *Let (X,d) be a metric space and let p be a w-distance on X.*

(i) *If $\{x_n\}$ is a sequence in X such that $\lim_n p(x_n,x) = \lim_n p(x_n,y) = 0$, then $x = y$. In particular, if $p(z,x) = p(z,y) = 0$, then $x = y$.*

(ii) *If $p(x_n,y_n) \le \alpha_n$ $p(x_n,y) \le \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,\infty)$ converging to 0, then $\{y_n\}$ converges to y.*

(iii) *Let p be a w-distance on metric space (X,d) and $\{x_n\}$ a sequence in X such that for each $\varepsilon > 0$ there exist $N_\varepsilon \in N$ such that $m > n > N_\varepsilon$ implies $p(x_n,x_m) < \varepsilon$ (or $\lim_{m,n\to\infty} p(x_n,x_m) = 0$), then $\{x_n\}$ is a Cauchy sequence.*

Note that if $p(a,b) = p(b,a) = 0$ and $p(a,a) \le p(a,b) + p(b,a) = 0$, then $p(a,a) = 0$ and, by Lemma 1.3, $a = b$.

In [3], Razani et al. proved a fixed point theorem for (φ,ψ,p)-contractive mappings, which is a new version of the main theorem in [1], by considering the concept of the w-distance.

The main aim of this paper is to present some generalization fixed point Theorems by Kada et al. [2], Hicks and Rhoades [6] and several other results with respect to (φ,ψ,p)-contractive maps on a complete metric space.

2. (φ,ψ,p)-Contractive Maps

In the next theorem we state one of the main results of this paper generalizing Theorem 4 of [2]. In what follows, we use φp to denote the composition of φ with p.

Theorem 2.1. *Let p be a w-distance on complete metric space (X,d), $\varphi \in \Phi$ and $\psi \in \Psi$. Suppose $T : X \to X$ is a map that satisfies*

$$\varphi p\left(Tx,T^2x\right) \le \psi(\varphi p(x,Tx)), \tag{2.1}$$

for each $x \in X$ and that

$$\inf\{p(x,y) + p(x,Tx) : x \in X\} > 0 \tag{2.2}$$

for every $y \in X$ with $y \neq Ty$. Then there exists $u \in X$ such that $u = Tu$. Moreover, if $v = Tv$, then $p(v,v) = 0$.

Proof. Fix $x \in X$. Set $x_{n+1} = Tx_n$ with $x_0 = x$. Then by (2.1)

$$\begin{aligned} \varphi p(x_n, x_{n+1}) &\leq \psi\varphi p(x_{n-1}, x_n) \\ &\leq \psi^2 \varphi p(x_{n-2}, x_{n-1}) \\ &\leq \cdots \leq \psi^n(\varphi p(x_0, x_1)), \end{aligned} \tag{2.3}$$

thus $\lim_n \varphi p(x_n, x_{n+1}) = 0$ and Lemma 1.2 implies

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0, \tag{2.4}$$

and similarly

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{2.5}$$

Now we proof that $\{x_n\}$ is a Cauchy sequence. By triangle inequality, continuity of φ and (2.4), we have

$$\varphi p(x_n, x_{n+2}) \leq \psi\varphi[p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})] \longrightarrow 0, \tag{2.6}$$

as $n \to \infty$ and so $\lim_{n\to\infty} \varphi p(x_n, x_{n+2}) = 0$ which concludes

$$\lim_{n\to\infty} p(x_n, x_{n+2}) = 0. \tag{2.7}$$

By induction, for any $k > 0$ we have

$$\lim_{n\to\infty} p(x_n, x_{n+k}) = 0. \tag{2.8}$$

So, by Lemma 1.3, $\{x_n\}$ is a Cauchy sequence, and since X is complete, there exists $u \in X$ such that $x_n \to u$ in X.

Now we prove that u is a fixed point of T.

From (2.8), for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon$ implies $p(x_{N_\varepsilon}, x_n) < \varepsilon$ but $x_n \to u$ and $p(x, \cdot)$ is lower semicontinuous, thus

$$p(x_{N_\varepsilon}, u) \leq \liminf_{n\to\infty} p(x_{N_\varepsilon}, x_n) \leq \varepsilon. \tag{2.9}$$

Therefore, $p(x_{N_\varepsilon}, u) \leq \varepsilon$. Set $\varepsilon = 1/k$, $N_\varepsilon = n_k$ and we have

$$\lim_{k \to \infty} p(x_{n_k}, u) = 0. \tag{2.10}$$

□

Now, assume that $u \neq Tu$. Then by hypothesis, we have

$$0 < inf\{p(x,u) + p(x,Tx) : x \in X\} \leq \inf\{p(x_n,u) + p(x_n,x_{n+1}) : n \in \mathbb{N}\} \longrightarrow 0 \tag{2.11}$$

as $n \to \infty$ by (2.4) and (2.10). This is a contradiction. Hence $u = Tu$.

If $v = Tv$, we have

$$\varphi p(v,v) = \varphi p\left(Tv, T^2 v\right) \leq \psi\varphi p(v,Tv) = \psi\varphi p(v,v) < \varphi p(v,v). \tag{2.12}$$

This is a contradiction. So $\varphi p(v,v) = 0$, and by hypothesis $p(v,v) = 0$.

Here we give a simple example illustrating Theorem 2.1. In this example, we will show that Theorem 4 in [2] cannot be applied.

Example 2.2. Let $X = \{(1/n) \mid n \in \mathbb{N}\} \cup \{0\}$, which is a complete metric space with usual metric d of reals. Moreover, by defining $p(x,y) = y$, p is a w-distance on (X,d). Let $T : X \to X$ be a map as $T(1/n) = 1/(n+1)$, $T0 = 0$. Suppose $\varphi(t) = t^{1/t}$ is a continuous and strictly nondecreasing map and $\psi(t) = (1/3)t$, for any $t > 0$. We have

$$\sup_{x \in X} \frac{p(Tx, T^2x)}{p(x,Tx)} = 1, \tag{2.13}$$

and so there is not any $r \in [0,1)$ such that $p(Tx,T^2x) \leq rp(x,Tx)$, and hence Theorem 4 in [2] dose not work. But

$$\begin{aligned} \varphi p\left(Tx, T^2x\right) = p\left(Tx,T^2x\right)^{1/p(Tx,T^2x)} &= \left(\frac{1}{n+2}\right)^{n+2} \leq \frac{1}{3}\left(\frac{1}{n+1}\right)^{n+1} \\ &= \frac{1}{3} p(x,Tx)^{1/p(x,Tx)} = \psi\varphi p(x,Tx), \end{aligned} \tag{2.14}$$

because for any $n \in \mathbb{N}$ we have $((n+1)/(n+2))^{n+1} 1/(n+2) \leq 1/3$. Also for any $n \in \mathbb{N}$ we have $1/n \neq T(1/n)$. So for arbitrary $n \in \mathbb{N}$, $\inf\{p(1/m,1/n) + p(1/m,1/(m+1)) : m \in \mathbb{N}\} = 1/n > 0$, hence T is satisfied in Theorem 2.1. We note that 0 is a fixed point for T.

The next examples show the role of the conditions (2.1) and (2.2).

Example 2.3. Let $X = [-1,1]$, $d(x,y) = |x-y|$, and define $p : X \to X$ by $p(x,y) = |3x - 3y|$, where $x, y \in X$. Set $\psi(t) = rt$ and $\varphi(t) = t$ for all $t \in [0,\infty)$. Let us define $T : X \to X$ by $T0 = 1$ and $Tx = x/10$ if $x \neq 0$. We have

$$\varphi p\left(T0, T^2 0\right) = p\left(T0,T^2 0\right) = p\left(1, \frac{1}{10}\right) = 3 - \frac{3}{10} \leq 3 = \frac{1}{3} p(0,T0) = \psi\varphi p(0,T0). \tag{2.15}$$

If $x \neq 0$, then

$$\varphi p\left(Tx, T^2x\right) = p\left(Tx, T^2x\right) = p\left(\frac{x}{10}, \frac{x}{100}\right) = \frac{1}{10}\left|3x - \frac{3x}{10}\right| \leq \frac{1}{3}p(x, Tx) = \psi\varphi p(x, Tx) \tag{2.16}$$

and hence (2.1) holds.

Now, we remark that $0 \neq T(0)$, and

$$\inf_{n\in\mathbb{N}} p(T^n(x), 0) + p(T^n(x), TT^n(x)) = 0 \quad \text{for every } x \in X. \tag{2.17}$$

Thus, the condition (2.2) is not satisfied, and there is no $z \in X$ with $Tz = z$. In this case we observe that Theorem 2.1 is invalid without condition (2.2).

Example 2.4. Let $X = [2, \infty) \cup \{0, 1\}$, $d(x, y) = |x - y|$, $x, y \in X$, and set $p = d$. Let ψ, φ be as Example 2.3. Let us define $T : X \to X$ by $T0 = 1$ and $Tx = 0$ if $x \neq 0$. Clearly, T has no fixed point in X. Now, for each $x \in X$ and that

$$\inf\left\{d(x, y) + d(x, Tx) : x \in X\right\} > 0 \tag{2.18}$$

for every $y \in X$ with $y \neq Ty$, so condition (2.2) is satisfied. But, for $x = 0$, $d(Tx, T^2x) > rd(x, Tx)$ for any $r \in [0, 1)$. Hence, condition (2.1) dose not hold. We note that Theorem 2.1 dose not work without condition (2.1).

Suppose $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue-integrable mapping which is summable and $\int_0^{\varepsilon} \theta(\eta)d\eta > 0$, for each $\varepsilon > 0$. Now, in the next corollary, set $\varphi(t) = \int_0^t \theta(\eta)d\eta$ and $\psi(t) = ct$, where $c \in [0, 1[$. Then, $\varphi \in \Phi$ and $\psi \in \Psi$. Hence we can conclude the following corollary as a special case.

Corollary 2.5. *Let T be a selfmap of a complete metric space (X, d) satisfying*

$$\int_0^{d(Tx, T^2x)} \theta(t)dt \leq c\int_0^{d(x, Tx)} \theta(t)dt \tag{2.19}$$

for all $x \in X$. Suppose that

$$\inf\left\{d(x, y) + d(x, Tx) : x \in X\right\} > 0 \quad \textit{for every } y \in X \tag{2.20}$$

with $y \neq Ty$. Then there exists a $u \in X$ such that $Tu = u$.

Note that Corollary 2.5 is invalid without condition (2.20). For example, take $X = \{0\} \cup \{1/2^n : n \geq 1\}$, which is a complete metric space with usual metric d of reals. Define $T : X \to X$ by $T(0) = 1/2$ and $T(1/2^n) = 1/2^{n-1}$ for $n \geq 1$. Set $\varphi(t) \equiv 1$. It is easy to check that $\int_0^{d(Tx, T^2x)} \varphi(t)dt \leq (1/2)\int_0^{d(x, Tx)} \varphi(t)dt$, for any $x \in X$; however, $y \neq Ty$ for any $y \in X$ and $\inf\{d(x, y) + d(x, Tx) : x \in X\} = 0$. Clearly, T has got no fixed point in X.

Remark 2.6. From Theorem 2.1, we can obtain Theorem 4 in [2] as a special case. For this, in the hypotheses of Theorem 2.1, set $\psi(t) = rt$ and $\varphi(t) = t$ for all $t \in [0, \infty)$.

Corollary 2.7. *Let p be a w-distance on complete metric space (X, d), $\varphi \in \Phi$ and $\psi \in \Psi$. Suppose T is a continuous mapping for X into itself such that (2.1), is satisfied. Then there exists $u \in X$ such that $u = Tu$. Moreover, if $v = Tv$, then $p(v, v) = 0$.*

Proof. Assume that there exists $y \in X$ with $y \neq Ty$ and $\inf\{p(x, y) + p(x, Tx) : x \in X\} = 0$. Then there exists a sequence $\{x_n\}$ such that

$$p(x_n, y) + p(x_n, Tx_n) \longrightarrow 0 \tag{2.21}$$

as $n \to \infty$. Hence $p(x_n, y) \to 0$ and $p(x_n, Tx_n) \to 0$ as $n \to \infty$. Lemma 1.3 implies that $Tx_n \to y$ as $n \to \infty$. Now by assumption

$$\varphi p\left(Tx_n, T^2x_n\right) \leq \psi(\varphi p(x_n, Tx_n)) \tag{2.22}$$

and so $\varphi p(Tx_n, T^2x_n) \to 0$ as $n \to \infty$. By Lemma 1.2, $p(Tx_n, T^2x_n) \to 0$ as $n \to \infty$. We also have

$$p\left(x_n, T^2x_n\right) \leq p(x_n, Tx_n) + p\left(Tx_n, T^2x_n\right), \tag{2.23}$$

hence $p(x_n, T^2x_n) \to 0$ as $n \to \infty$. By Lemma 1.3, we conclude that $\{T^2x_n\}$ converges to y. Since T is continuous, we have

$$Ty = T\left(\lim_{n\to\infty} Tx_n\right) = \lim_{n\to\infty} T^2x_n = y. \tag{2.24}$$

This is a contradiction. Therefore, if $y \neq Ty$, then $\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0$. So, Theorem 2.1 gives desired result. □

In Example 2.3, T is satisfied in condition (2.1), but it is not continuous. So, the hypotheses in Corollary 2.7are not satisfied . We note that T has no fixed point.

It is an obvious fact that, if $f : X \to X$ is a map which has a fixed point $x \in X$, then x is also a fixed point of f^n for every natural number n. However, the converse is false. If a map satisfies $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, where $F(f)$ denotes a set of all fixed points of f, then it is said to have property P [7, 8]. The following theorem extends and improves Theorem 2 of [7].

Theorem 2.8. *Let (X, d) be a complete metric space with w-distance p on X. Suppose $T : X \to X$ satisfies*

(i)

$$\varphi p\left(Tx, T^2x\right) \leq \psi\varphi p(x, Tx), \quad \forall x \in X, \tag{2.25}$$

or

(ii) *with strict inequality, $\psi \equiv 1$ and for all $x \in X$, $x \neq Tx$. If $F(T) \neq \emptyset$, then T has property P.*

Proof. We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $u \in F(T^n)$. Suppose that T satisfies (i). Then,

$$\varphi p(u, Tu) = \varphi p(T^n u, TT^n u) \leq \psi\varphi p\left(T^{n-1}u, TT^{n-1}u\right) \leq \cdots \leq \psi^n \varphi p(u, Tu), \tag{2.26}$$

and so $p(u, Tu) = 0$. Now from

$$\varphi p(u, u) = \psi\varphi p(u, T^n u) \leq \sum_{i=0}^{n-1} \psi\varphi p\left(T^i u, T^{i+1} u\right) = 0, \tag{2.27}$$

we have $p(u, u) = 0$. Hence, by Lemma 1.3, we have $u = Tu$, and $u \in F(T)$. Suppose that T satisfies (ii). If $Tu = u$, then there is nothing to prove. Suppose, if possible, that $Tu \neq u$. Then a repetition of the argument for case (i) leads to $\varphi p(u, Tu) < \psi\varphi p(u, Tu)$, that is a contradiction. Therefore, in all cases, $u = Tu$ and $F(T^n) = F(T)$. □

The following theorem extends Theorem 2.1 of [6]. A function G mapping X into the real is T-orbitally lower semicontinuous at z if $\{x_n\}$ is a sequence in $O(x, \infty)$ and $x_n \to z$ implies that $G(p) \leq \liminf_{n\to\infty} G(x_n)$.

Theorem 2.9. *Let (X, d) be a complete metric space with w-distance p on X. Suppose $T : X \to X$ and there exists an x such that*

$$\varphi p\left(Ty, T^2 y\right) \leq \psi\varphi p(y, Ty), \quad \forall y \in O(x, \infty). \tag{2.28}$$

Then,

(i) $\lim T^n x = z$ *exists,*

(ii)

$$\varphi p(T^n x, z) \leq \frac{\psi^n}{1-\psi} \varphi p(x, Tx) \quad \textit{for } n \geq 1, \tag{2.29}$$

(iii) $p(z, Tz) = 0$ *if and only if* $G(x) = p(x, Tx)$ *is T-orbitally lower semicontinuous at z.*

Proof. Observe that (i) and (ii) are immediate from the proof of Theorem 2.1. We prove (iii). It is clear that $p(z, Tz) = 0$ impling $G(x)$ is T-orbitally lower semicontinuous at z.

$x_n = T^n x \to z$ and G is T-orbitally lower semicontinuous at x implies

$$0 \leq \varphi p(z, Tz) = \varphi G(z) \leq \liminf_{n\to\infty} \varphi G(x_n) = \liminf_{n\to\infty} \psi\varphi p(x_n, Tx_n) \leq \liminf_{n\to\infty} \psi^n \varphi p(x, Tx) = 0. \tag{2.30}$$

So, $p(z, Tz) = 0$. □

The mapping T is orbitally lower semicontinuous at $u \in X$ if $\lim_{k\to\infty} T^{n_k} x = u$ implies that $\lim_{k\to\infty} T^{n_k+1} x = Tu$. In the following, we improve Theorem 2 of [9] that it is correct form Theorem 1 of [7].

Theorem 2.10. *Let p be a w-distance on complete metric space $(X, d), \varphi \in \Phi$ and $\psi \in \Psi$. Suppose $T : X \to X$ is orbitally lower semicontinuous map on X that satisfies*

$$\varphi p\left(Tx, T^2x\right) \leq \psi(\varphi p(x, Tx)) \tag{2.31}$$

for each $x \in X$. Then there exists $u \in X$ such that $u \in F(T)$. Moreover, if $v = Tv$, then $p(v, v) = 0$.

Proof. Observe that the sequence $\{x_n\}$ is a Cauchy sequence immediate from the proof of Theorem 2.1 and so there exists a point u in X such that $x_n \to u$ as $n \to \infty$. Since T is orbitally lower semicontinuous at u, we have $p(u, Tu) \leq \liminf_{n\to\infty} p(x_n, x_{n+1}) = 0$. Now, we have

$$\varphi p(u, Tu) \leq \varphi \liminf_{n\to\infty} p(x_n, x_{n+1}) = \varphi(0) = 0, \tag{2.32}$$

and so $p(u, Tu) = 0$. Similarly, $p(Tu, u) = 0$. Hence, $u \in F(T)$. By Theorem 2.1 we can conclude that if $v = Tv$, then $p(v, v) = 0$. □

The following example shows that Theorem 2 in [9] cannot be applicable. So our generalization is useful.

Example 2.11. Let = $[0, \infty)$ be a metric space with metric d defined by $d(x, y) = (40/3)|x - y|, x, y \in X$, which is complete. We define $p : X \to X$ by $p(x, y) = (1/3)|y|$. Let φ be as defined before in Corollary 2.5 and $\psi(t) = (1/10)t, t > 0$. Assume that $T : X \to X$ by $Tx = x/10$ for any $x \in X$. We have, $d(Tx, T^2x) = (4/3)d(x, Tx),\ x \in X$, and so Theorem 2 in [9] dose not work. But

$$\varphi p\left(Tx, T^2x\right) \leq \psi(\varphi p(x, Tx)) \tag{2.33}$$

for each $x \in X$. Hence by Theorem 2.10 there exists a fixed point for T. We note that 0 is fixed point for T.

3. (φ, k, p)-Contractive Maps

In this section we obtain fixed points for (φ, k, p)-contractive maps (i.e., (φ, ψ, p)-contractive maps that $\psi(t) = k$ for all $t \in [0, \infty)$, where $k \in [0, 1)$).

In 1969, Kannan [10] proved the following fixed point theorem. Contractions are always continuous and Kannan maps are not necessarily continuous.

Theorem 3.1 (see [10]). *Let (X, d) be a complete metric space. Let T be a Kannan mapping on X, that is, there exists $k \in [0, 1/2)$ such that*

$$d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty)) \tag{3.1}$$

for all $x, y \in X$. Then, T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}_{n\geq 1}$ converges to the fixed point.

In the next theorem, we generalize this theorem as follows.

Theorem 3.2. *Let (X,d) be a complete metric space. Let T be a (φ,k)-Kannan mapping on X, that is, there exists $k \in [0,1/2)$ such that*

$$\varphi d(Tx,Ty) \le k(\varphi d(x,Tx) + \varphi d(y,Ty)) \tag{3.2}$$

for all $x,y \in X$. Then, T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}_{n\ge 1}$ converges to the fixed point.

Proof. Let $x \in X$ and define $x_{n+1} = T^n x$ for any $n \in N$, and set $r = k/(1-k)$. Then, $r \in [0,1)$,

$$\varphi d\left(Tx,T^2x\right) \le k\left(\varphi d(x,Tx) + \varphi d\left(Tx,T^2x\right)\right) \tag{3.3}$$

and so

$$\varphi d\left(Tx,T^2x\right) \le r\varphi d(x,Tx). \tag{3.4}$$

Then, from the proof of Theorem 2.1, $\lim T^n x = z$ exists. From (3.4), we have

$$\varphi d(T^n x,Tz) \le r\varphi d\left(T^{n-1}x,z\right) \le \frac{r^n}{1-r}\varphi d(x,Tx) \quad \text{for } n \ge 1. \tag{3.5}$$

Thus, $\lim T^n x = Tz$, and so $z = Tz$. Clearly, z is unique. This completes the proof. □

The set of all subadditive functions φ in Φ is denoted by Φ'. In the following theorems, we generalize Theorems 3.4 and 3.5 due to Suzuki and Takahashi [4].

Theorem 3.3. *Let p be a w-distance on complete metric space $(X,d), \varphi \in \Phi'$ and T be a selfmap. Suppose there exists $k \in [0,1/2)$ such that*

(i) $\varphi p(Tx,T^2x) \le k\varphi p(x,T^2x)$ *for each* $x \in X$,

(ii) $\inf\{p(x,z) + p(x,Tx) : x \in X\} > 0$ *for every* $z \in X$ *with* $z \ne Tz$.

Then T has a fixed point in X. Moreover, if v is a fixed point of T, then $p(v,v) = 0$.

Proof. Fix $x \in X$. Define $x_0 = x$ and $x_n = T^n x_0$ for every $n \in \mathbb{N}$. Put $r = k/(1-k)$. Then, $0 \le r < 1$. By hypothesis, since $\varphi \in \Phi'$, we have

$$\varphi p(x_n,x_{n+1}) \le k\varphi p(x_{n-1},x_{n+1}) \le k\varphi p(x_{n-1},x_n) + k\varphi p(x_n,x_{n+1}), \tag{3.6}$$

for all $n \in \mathbb{N}$. It follows that

$$\varphi p(x_n,x_{n+1}) \le r\varphi p(x_{n-1},x_n) \le \cdots \le r^n \varphi p(x_0,x_1), \tag{3.7}$$

for all $n \in \mathbb{N}$. Using the similar argument as in the proof of Theorem 2.1, we can prove that the sequence $\{u_n\}$ is Cauchy and so there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Also, we have $u \in F(T)$. Since

$$\varphi p(v, v) = \varphi p\left(Tv, T^2 v\right) \leq k\varphi p\left(v, T^2 v\right) = k\varphi p(v, v), \tag{3.8}$$

we have $\varphi p(v, v) = 0$ and so $p(v, v) = 0$. The proof is completed. □

Corollary 3.4. *Let p be a w-distance on complete metric space (X, d), $\varphi \in \Phi'$ and let T be a continuous map. Suppose there exists $k \in [0, 1/2)$ such that*

$$\varphi p\left(Tx, T^2 x\right) \leq k\varphi p\left(x, T^2 x\right), \tag{3.9}$$

for each $x \in X$.

Then T has a fixed point in X. Moreover, if v is a fixed point of T, then $p(v, v) = 0$.

Proof. It suffices to show that $\inf\{p(x, z) + p(x, Tx) : x \in X\} > 0$ for every $u \in X$ with $u \neq Tu$. Assume that there exists $u \in X$ with $u \neq Tu$ and $\inf\{p(x, u) + p(x, Tx) : x \in X\} = 0$. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}[p(x_n, u) + p(x_n, Tx_n)] = 0$. It follows that $p(x_n, u) \to 0$ and $p(x_n, Tx_n) \to 0$ as $n \to \infty$. Hence, $Tx_n \to u$. On the other hand, since $\varphi \in \Phi'$ and (3.9), we have

$$\varphi p\left(x_n, T^2 x_n\right) \leq \varphi p(x_n, Tx_n) + \varphi p\left(Tx_n, T^2 x_n\right) \leq \varphi p(x_n, Tx_n) + k\varphi p\left(x_n, T^2 x_n\right), \tag{3.10}$$

and hence

$$\varphi p\left(x_n, T^2 x_n\right) \leq \frac{1}{1-k}\varphi p(x_n, Tx_n), \tag{3.11}$$

for all $n \in \mathbb{N}$. Thus, $p(x_n, T^2 x_n) \to 0$ as $n \to \infty$. Therefore, $T^2 x_n \to u$. Since $T : X \to X$ is continuous, we have

$$T(u) = T\left(\lim_{n\to\infty} Tx_n\right) = \lim_{n\to\infty} T^2 x_n = u, \tag{3.12}$$

which is a contradiction. Therefore, using Theorem 3.3, $p(v, v) = 0$. This completes the proof. □

Question 1. Can we generalize Theorems 3.2, 3.3, and Corollary 3.4 for (φ, ψ, p)-contractive maps?

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