Research Article

α-Well-Posedness for Quasivariational Inequality Problems

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We introduce and study the concepts of α-well-posedness and L-α-well-posedness for quasivariational inequality problems having a unique solution and the concepts of α-well-posedness in the generalized sense and L-α-well-posedness in the generalized sense for quasivariational inequality problems having more than one solution. We present some necessary and/or sufficient conditions for the various kinds of well-posedness to occur. Our results generalize and strengthen previously known results for quasivariational inequality problems.

1. Introduction

Let $E$ be a reflexive real Banach space and let $K$ be a nonempty closed convex subset of $E$. Let $S$ be a set-valued mapping from $K$ to $K$ and let $A$ be an operator from $E$ to the dual space $E^*$. Bensoussan and Lions [1], Baiocchi and Capelo [2], and Mosco [3] considered the following quasivariational inequality (in short, (QVIP)), which is to find a point $u_0 \in K$ such that

$$ u_0 \in S(u_0), \quad \langle Au_0, u_0 - v \rangle \leq 0, \quad \forall v \in S(u_0). \quad (1.1) $$

The interest in quasivariational inequality problems lies in the fact that many economic or engineering problems are modeled through them, as explained in [4, 5] where a wide bibliography on variational inequalities, quasivariational inequality problems, and related problems is contained. Moreover, under suitable assumptions, a quasivariational inequality is equivalent to a generalized Nash equilibrium problem [3].

On the other hand, well-posedness plays a crucial role in the stability theory for optimization problems, which guarantees that, for an approximating solution sequence, there exists a subsequence which converges to a solution [6]. The study of well-posedness for
scalar minimization problems started from Tikhonov [7] and Levitin and Polyak [8]. Since the convergence of numerical methods for quasivariational inequality Problems can be obtained also with the aid of well-posedness theory, Lignola [9] introduced and investigated the concepts of well-posedness and L-well-posedness for quasivariational inequalities having a unique solution and the concepts of well-posedness and L-well-posedness in the generalized sense for quasivariational inequality problems having more than one solution.

In this paper, inspired by the above concepts of well-posedness for (QVIP), we introduce and study the concepts of \(\alpha\)-well-posedness and L-\(\alpha\)-well-posedness for quasivariational inequality Problems having a unique solution and the concepts of \(\alpha\)-well-posedness in the generalized sense and L-\(\alpha\)-well-posedness in the generalized sense for quasivariational inequality Problems having more than one solution. The results in this paper generalize and improve the results in [9, 10].

2. Preliminaries

Denote by \(\Gamma\) the solution set of (QVIP). Let \(\alpha > 0\). In order to investigate the \(\alpha\)-well-posed for (QVIP), we need the following definitions.

First we recall the notion of Mosco convergence [11]. A sequence \((H_n)_n\) of subsets of \(E\) Mosco converges to a set \(H\) if

\[
H = \liminf_n H_n = \mathcal{w}-\limsup_n H_n,
\]

where \(\liminf_n H_n\) and \(\mathcal{w}-\limsup_n H_n\) are, respectively, the Painlevé-Kuratowski strong limit inferior and weak limit superior of a sequence \((H_n)_n\), that is,

\[
\begin{align*}
\liminf_n H_n &= \{ y \in E : \exists y_n \in H_n, \ n \in N, \ \text{with} \ y_n \to y \}, \\
\mathcal{w}-\limsup_n H_n &= \{ y \in E : \exists n_k \uparrow \infty, \ n_k \in N, \ \exists y_{n_k} \in H_{n_k}, \ k \in N, \ \text{with} \ y_{n_k} \rightharpoonup y \}.
\end{align*}
\]

(2.2)

where “\(\rightharpoonup\)” means weak convergence, “\(\to\)” means strong convergence.

If \(H = \liminf_n H_n\), we call the sequence \((H_n)_n\) of subsets of \(E\) Lower Semi-Mosco which converges to a set \(H\).

It is easy to see that a sequence \((H_n)_n\) of subsets of \(E\) Mosco converges to a set \(H\) which implies that the sequence \((H_n)_n\), also Lower Semi-Mosco, converges to the set \(H\), but the converse is not true in general.

We will use the usual abbreviations usc and lsc for “upper semicontinuous” and “lower semicontinuous,” respectively. Let \(E\) be a reflexive real Banach space with dual \(E^*\). An operator \(A : E \to E^*\) will be called hemicontinuous if it is continuous from every segment of \(E\) to \(E^*\) endowed with the weak topology. \(A : E \to E^*\) will be called monotone if \(\langle Au - Av, u - v \rangle \geq 0\) for every \(u, v \in E\). \(A : E \to E^*\) will be called pseudomonotone if \(\langle Au, u - v \rangle \leq 0 \Rightarrow \langle Av, u - v \rangle \leq 0\) for every \(u, v \in E\).
Definition 2.1. A sequence \((u_n)_n\) is an \(\alpha\)-approximating sequence for \((QVIP)\) if

(i) \((u_n) \in K\), for all \(n \in N\);

(ii) there exists a sequence \((\varepsilon_n)_n\), \(\varepsilon_n > 0\), decreasing to 0 such that

\[
d(u_n, S(u_n)) \leq \varepsilon_n,
\]

that is, \(u_n \in B(S(u_n), \varepsilon_n)\), \(\forall n \in N\),

\[
\langle Au_n, u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n,
\]

\(\forall v \in S(u_n)\), \(\forall n \in N\). \hfill (2.3)

Definition 2.2. A quasivariational inequality (QVIP) is said to be \(\alpha\)-well-posed (resp., \(\alpha\)-well-posed in the generalized sense) if it has a unique solution \(u_0\) and every \(\alpha\)-approximating sequence \((u_n)_n\) strongly converges to \(u_0\) (resp., if the solution set \(\Gamma\) of (QVIP) is nonempty and for every \(\alpha\)-approximating sequence \((u_n)_n\) has a subsequence which strongly converges to a point of \(\Gamma\)).

Definition 2.3. A sequence \((u_n)_n\) is an \(L\)-\(\alpha\)-approximating sequence for \((QVIP)\) if:

(i) \((u_n) \in K\), for all \(n \in N\);

(ii) there exists a sequence \((\varepsilon_n)_n\), \(\varepsilon_n > 0\), decreasing to 0 such that \(d(u_n, S(u_n)) \leq \varepsilon_n\), and

\[
\langle Au_n, u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n,
\]

\(\forall v \in S(u_n)\), \(\forall n \in N\). \hfill (2.4)

Definition 2.4. A quasivariational inequality (QVIP) is said to be \(L\)-\(\alpha\)-well-posed (resp., \(L\)-\(\alpha\)-well-posed in the generalized sense) if it has a unique solution \(u_0\) and every \(L\)-\(\alpha\)-approximating sequence \((u_n)_n\) strongly converges to \(u_0\) (resp., if the solution set \(\Gamma\) of (QVIP) is nonempty and for every \(L\)-\(\alpha\)-approximating sequence \((u_n)_n\) has a subsequence which strongly converges to a point of \(\Gamma\)).

It is worth noting that if \(\alpha = 0\), then the definitions of \(\alpha\)-well-posedness, \(\alpha\)-well-posedness in the generalized sense, \(L\)-\(\alpha\)-well-posedness, and \(L\)-\(\alpha\)-well-posedness in the generalized sense for (QVIP), respectively, reduce to those of the well-posedness, well-posedness in the generalized sense, \(L\)-well-posedness, and \(L\)-well-posedness in the generalized sense for (QVIP) in [9]. We also note that Definition 2.2 generalizes and extends \(\alpha\)-well-posedness and \(\alpha\)-well-posedness in the generalized sense of variational inequalities in [10] which are related to the continuously differentiable gap function of variational inequality Problems introduced by Fukushima [12].

We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.5 (see [13]). Let \((H_n)_n\) be a sequence of nonempty subsets of the space \(E\) such that

(i) \(H_n\) is convex for every \(n \in N\);

(ii) \(H_0 \subseteq \liminf H_n\);

(iii) there exists \(m \in N\) such that \(\text{int} \bigcap_{n \geq m} H_n \neq \emptyset\).

Then, for every \(u_0 \in \text{int} H_0\), there exists a positive real number \(\delta\) such that \(B(u_0, \delta) \subseteq H_n\), for all \(n \geq m\).

If \(E\) is a finite dimensional space, then assumption (iii) can be replaced by

(iii)' \(\text{int} H_0 \neq \emptyset\).
The following Lemmas 2.6 and 2.7 play important roles in this paper. Now we present a Minty type lemma for quasivariational inequalities as follows.

**Lemma 2.6.** Suppose that set-valued mapping $S$ is nonempty convex-valued, the operator $A$ is hemicontinuous and monotone, $u_0 \in S(u_0)$. Then the following conditions are equivalent:

(i) $(Au_0, u_0 - v) - (\alpha/2)\|u_0 - v\|^2 \leq 0$, for all $v \in S(u_0),
(ii) (Av, u_0 - v) - (\alpha/2)\|u_0 - v\|^2 \leq 0$, for all $v \in S(u_0).

**Proof.** We first prove that (ii) implies (i). Let $v$ be an arbitrary point of $S(u_0)$. For every number $t \in [0, 1]$, since the set-valued mapping $S$ is convex-valued and $u_0 \in S(u_0)$, the point $v_t = tv + (1 - t)u_0$ belongs to $S(u_0)$. It follows from (ii) that

$$
(Av_t, u_0 - v_t) - \frac{\alpha}{2}\|u_0 - v_t\|^2 \leq 0.
$$

From the definition of $v_t$, one has

$$
\lim_{t \to 0} \left( (Av_t, u_0 - v) - \frac{\alpha}{2}t\|u_0 - v\|^2 \right) \leq 0,
$$

and it follows from the hemicontinuity of $A$ that

$$
(Au_0, u_0 - v) \leq 0, \quad (2.7)
$$

then

$$
(Au_0, u_0 - v) - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0, \quad \forall v \in S(u_0).
$$

The converse is an easy consequence of monotonicity of $A$.

**Lemma 2.7.** Assume that set-valued mapping $S$ is nonempty convex-valued, then $u_0 \in \Gamma$ if and only if the following conditions hold:

$$
u_0 \in S(u_0), \quad (Au_0, u_0 - v) - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0, \quad \forall v \in S(u_0).
$$

**Proof.** The necessity is clearly held. Now we prove the sufficiency. Let for all $v \in S(u_0)$, for all $t \in [0, 1]$, $v_t = tv + (1 - t)u_0$. Since $S$ is convex-valued, $v_t \in S(u_0)$, one has

$$
(Au_0, u_0 - v_t) - \frac{\alpha}{2}\|u_0 - v_t\|^2 \leq 0, \quad \forall t \in (0, 1],
$$

which implies that

$$
(Au_0, u_0 - v) - t\frac{\alpha}{2}\|u_0 - v\|^2 \leq 0, \quad \forall t \in (0, 1], \forall v \in S(u_0).
$$

\[\square\]
The above inequality implies, for $t$ converging to zero, that $u_0$ is a solution of (QVIP). This completes the proof. $\Box$

### 3. Case of a Unique Solution

In this section, we investigate some metric characterizations of $\alpha$-well-posedness and $L$-$\alpha$-well-posedness for (QVIP).

For any $\varepsilon > 0$, we consider the set

$$
Q_\varepsilon = \left\{ u \in K : u \in B(S(u), \varepsilon), \langle Au, u - v \rangle - \frac{\alpha}{2} \| u - v \|^2 \leq \varepsilon, \forall v \in S(u) \right\}
$$

$$
L_\varepsilon = \left\{ u \in K : u \in B(S(u), \varepsilon), \langle Av, u - v \rangle - \frac{\alpha}{2} \| u - v \|^2 \leq \varepsilon, \forall v \in S(u) \right\}.
$$

### Theorem 3.1

Let the same assumptions be as in Lemma 2.7. Then, one has

(a) (QVIP) is $\alpha$-well-posed if and only if the solution set $\Gamma$ of (QVIP) is nonempty and $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon = 0$;

(b) moreover, if $A : E \to E^*$ is pseudomonotone, then (QVIP) is $L$-$\alpha$-well-posed if and only if the solution set $\Gamma$ of (QVIP) is nonempty and $\lim_{\varepsilon \to 0} \text{diam } L_\varepsilon = 0$.

**Proof.** We only prove (a). The proof of (b) is similar and is omitted here. Suppose that (QVIP) is $\alpha$-well-posed, then $\Gamma \neq \emptyset$. It follows from Lemma 2.7 that $Q_\varepsilon \neq \emptyset$. Suppose by contradiction that there exists a real number $\beta$, such that $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon > \beta > 0$, then there exists $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, and $(w_n)_n, (z_n)_n \in Q_{\varepsilon_n}$, such that $\|w_n - z_n\| > \beta$, for all $n \in N$. Since the sequences $(w_n)_n, (z_n)_n$ are both $\alpha$-approximating sequences for (QVIP), $(w_n)_n$ and $(z_n)_n$ strongly converge to the unique solution $u_0$, and this gives a contradiction. Therefore, $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon = 0$.

Conversely, let $(u_n)_n, u_n \in K$, be an $\alpha$-approximating sequence for (QVIP). Then there exists a sequence $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, such that

$$
d(u_n, S(u_n)) \leq \varepsilon_n, \quad \forall n \in N,
$$

$$
\langle Au_n, u_n - v \rangle - \frac{\alpha}{2} \| u_n - v \|^2 \leq \varepsilon_n, \quad \forall v \in S(u_n), \forall n \in N.
$$

that is, $u_n \subset Q_{\varepsilon_n}$, for all $n \in N$. It is easy to see $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon = 0$ and $\Gamma \neq \emptyset$ implying that $\Gamma$ is a singleton point set. Indeed, if there exist two different solutions $z_1, z_2$, then from Lemma 2.7, we know that $z_1, z_2 \in Q_\varepsilon$, for all $\varepsilon > 0$. Thus, $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon \geq \|z_1 - z_2\| \neq 0$, a contradiction. Let $u_0$ be the unique solution of (QVIP). It follows from Lemma 2.7 that $u_0 \in Q_{\varepsilon_n}$. Thus, $\lim_{n \to 0} \|u_n - u_0\| \leq \lim_{n \to 0} \text{diam } Q_{\varepsilon_n} = 0$. So $(u_n)_n$ strongly converge to $u_0$. Therefore, (QVIP) is $\alpha$-well-posed. $\Box$

### Theorem 3.2

Let $\alpha > 0$ and the following assumptions hold:

(i) the set-valued mapping $S$ is nonempty convex-valued, and, for each sequence $(u_n)_n$ in $K$ converges to $u_0$, the sequence $(S(u_n))_n$ Lower Semi-Mosco converging to $S(u_0)$;
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(ii) for every converging sequence \( (h_n)_n \), there exists \( m \in N \), such that

\[
\text{int} \bigcap_{n \geq m} S(h_n) \neq \emptyset; \quad (3.3)
\]

(iii) the operator \( A \) is hemicontinuous and monotone on \( K \).

Then, (QVIP) is \( \alpha \)-well-posed if and only if

\[
Q_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0} \text{diam} Q_\varepsilon = 0. \quad (3.4)
\]

Proof. The necessity has been proved in Theorem 3.1(a).

Conversely, assume that (3.4) holds. It is easy to see that (3.4) implies that the solution set \( \Gamma \) of (QVIP) is a singleton point set. Let \( (u_n)_n \) be an \( \alpha \)-approximating sequence for (QVIP), that is, there exists a sequence \( \varepsilon_n > 0 \), with \( \varepsilon_n \to 0 \), such that

\[
d(u_n, S(u_n)) \leq \varepsilon_n, \quad \forall n \in N,
\]

\[
\langle Au_n, u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n, \quad \forall v \in S(u_n), \forall n \in N. \quad (3.5)
\]

Therefore, \( u_n \in Q_{\varepsilon_n} \), for all \( n \in N \). In light of (3.4), \( (u_n)_n \) is a Cauchy sequence and strongly converges to a point \( u_0 \in K \). In order to obtain that \( u_0 \) solves (QVIP), we start to prove that \( u_0 \in S(u_0) \). For each \( n \in N \), choose \( u'_n \in S(u_n) \), such that \( \|u_n - u'_n\| < d(u_n, S(u_n)) + \varepsilon_n \leq 2\varepsilon_n \).

It follows from \( u_n \to u_0 \) and \( \varepsilon_n \to 0 \) that \( u'_n \to u_0 \). It follows from the assumption (i) that \( \lim_n \inf S(u_n) = S(u_0) \). Thus, \( u_0 \in S(u_0) \).

To complete the proof, consider an arbitrary point \( v \in S(u_0) \). By Lower Semi-Mosco convergence again, we have \( S(u_0) \subseteq \lim_n \text{inf} S(u_n) \). Also observe that assumption (ii) applied to the constant sequence \( h_n = u_0 \), for all \( n \in N \), implies that \( \text{int} S(u_0) \neq \emptyset \). From Lemma 2.5, it follows that if \( v \in \text{int} S(u_0) \), then there exist \( m \in N \) and \( \delta > 0 \) such that \( \text{int} B(v, \delta) \subseteq S(u_n) \), for all \( n > m \). Thus, \( v \in S(u_n) \) for \( n \) sufficiently large. Notice the \( A \) is monotone and the sequence \( (u_n)_n \) is an \( \alpha \)-approximating sequence for (QVIP), then we have

\[
\langle Av, u_0 - v \rangle = \lim_n \langle Av, u_n - v \rangle \leq \liminf_n \langle Au_n, u_n - v \rangle \leq \lim_n \left( \varepsilon_n + \frac{\alpha}{2} \|u_n - v\|^2 \right) = \frac{\alpha}{2} \|u_0 - v\|^2.
\]

(3.6)

If \( v \in S(u_0) \setminus \text{int} S(u_0) \), let \( (v_n)_n \) be a sequence converging to \( v \), whose point belongs to a segment contained in \( \text{int} S(u_0) \). Since \( v_n \in \text{int} S(u_0) \), for all \( n \in N \), one has

\[
\langle Av_n, u_0 - v_n \rangle \leq \frac{\alpha}{2} \|u_0 - v_n\|^2. \quad (3.7)
\]

Since the hemicontinuity of \( A \),

\[
\langle Av, u_0 - v \rangle \leq \frac{\alpha}{2} \|u_0 - v\|^2, \quad \forall v \in S(u_0). \quad (3.8)
\]
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It follows from Lemma 2.6 that

\[ \langle Au_0, u_0 - v \rangle - \frac{\alpha}{2} \| u_0 - v \|^2 \leq 0, \quad \forall v \in S(u_0), \]  

(3.9)

then, by Lemma 2.7, we obtain that \( u_0 \) solves (QVIP). This completes the proof. \( \square \)

Now, we present a result in which assumption (ii) of above theorem is dropped, while the continuity assumption on the operator \( A \) is strengthened.

**Theorem 3.3.** Let the following assumptions hold:

(i) the set-valued mapping \( S \) is nonempty convex-valued, and, for each sequence \( (u_n)_n \) in \( K \) converging to \( u_0 \), the sequence \( (S(u_n))_n \) Lower Semi-Mosco converges to \( S(u_0) \);

(ii) the operator \( A \) is \( (s,w) \)-continuous on \( K \).

Then, (QVIP) is \( \alpha \)-well-posed if and only if (3.4) holds.

**Proof.** The necessity follows from Theorem 3.1 and Lemma 2.7.

Conversely, let \( (u_n)_n \) be an \( \alpha \)-approximating sequence for (QVIP) and (3.4) holds. From (3.4) and the proof of Theorem 3.2, we can obtain that \( (u_n)_n \) strongly converges to \( u_0 \), with \( u_0 \in S(u_0) \). Since Lower Semi-Mosco convergence implies for every \( v \in S(u_0) \), there exists sequence \( (v_n)_n \) strongly converging to \( v \) such that \( v_n \in S(u_n) \). Since the operator \( A \) is \( (s,w) \)-continuous and \( (u_n)_n \) is an \( \alpha \)-approximating sequence for (QVIP), we have

\[ \langle Au_0, u_0 - v \rangle = \lim_n \langle Au_n, u_n - v_n \rangle \leq \lim_n \left( \varepsilon_n + \frac{\alpha}{2} \| u_n - v_n \|^2 \right) = \frac{\alpha}{2} \| u_0 - v \|^2. \]  

(3.10)

By Lemma 2.7, we obtain that \( u_0 \) solves (QVIP). This completes the proof. \( \square \)

**Theorem 3.4.** Let the following assumptions hold:

(i) the set-valued mapping \( S \) is nonempty convex-valued, and, for each sequence \( (u_n)_n \) in \( K \) converging to \( u_0 \), the sequence \( (S(u_n))_n \) Lower Semi-Mosco converging to \( S(u_0) \);

(ii) for every converging sequence \( (h_n)_n \), there exists \( m \in N \), such that

\[ \text{int} \bigcap_{n \geq m} S(h_n) \neq \emptyset; \]  

(3.11)

(iii) the operator \( A \) is hemicontinuous and monotone on \( K \).

Then, (QVIP) is \( L \)-\( \alpha \)-well-posed if and only if

\[ L_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0} \text{diam} L_\varepsilon = 0. \]  

(3.12)

**Proof.** Assume that (QVIP) is \( L \)-\( \alpha \)-well-posed, then it follows from the monotonicity of \( A \) that \( \emptyset \neq L_\varepsilon \), for all \( \varepsilon > 0 \). It follows from Theorem 3.1(b) that the necessity can be completed.

Assume that (3.12) holds. Let \( (u_n)_n \) be an \( L \)-\( \alpha \)-approximating sequence for (QVIP), then there exists a sequence \( \varepsilon_n > 0 \), with \( \varepsilon_n \to 0 \), such that \( u_n \in L_{\varepsilon_n} \), for all \( n \in N \). Following
the same argument as the proof of Theorem 3.1, it is easy to see \( \lim_{\varepsilon \to 0} \operatorname{diam} L_\varepsilon = 0 \) and \( \Gamma \neq \emptyset \) imply that \( \Gamma \) is a singleton point set. In light of the assumption, \((u_n)_n\) is a Cauchy sequence and strongly converges to a point \( u_0 \in K \) and \( u_0 \in S(u_0) \). Let \( v \in \operatorname{int} S(u_0) \) and using Lemma 2.5, one has \( v \in S(u_n) \), for \( n \) sufficiently large. Then, we get

\[
\langle Av, u_0 - v \rangle - \frac{\alpha}{2} \| u_0 - v \|^2 = \lim_{n} \left( \langle Av, u_n - v \rangle - \frac{\alpha}{2} \| u_n - v \|^2 \right) \leq \lim_{n} \varepsilon_n = 0. \tag{3.13}
\]

If \( v \in S(u_0) \setminus \operatorname{int} S(u_0) \), let a sequence \( v_n \) converges to \( v \), whose points belong to a segment contained in \( \operatorname{int} S(u_0) \). Since

\[
\langle Av_n, u_0 - v_n \rangle - \frac{\alpha}{2} \| u_0 - v_n \|^2 \leq 0 \tag{3.14}
\]

and the operator \( A \) is hemicontinuous, one gets

\[
\langle Av, u_0 - v \rangle - \frac{\alpha}{2} \| u_0 - v \|^2 \leq 0. \tag{3.15}
\]

According to Lemmas 2.6 and 2.7, \( u_0 \) is the solution of (QVIP).

\[ \square \]

**Theorem 3.5.** Let the following assumptions hold:

(i) the set-valued mapping \( S \) is nonempty convex-valued, and, for each sequence \((u_n)_n\) in \( K \) converging to \( u_0 \), the sequence \((S(u_n))_n\) Lower Semi-Mosco converges to \( S(u_0) \);

(ii) the operator \( A \) is \((s, w)\)-continuous and monotone on \( K \).

Then, (QVIP) is L-\( \alpha \)-well-posed if and only if (3.12) holds.

**Proof.** Assume (3.12) holds. Let \((u_n)_n\) be an L-\( \alpha \)-approximating sequence for (QVIP), then there exists a sequence \( \varepsilon_n > 0 \), with \( \varepsilon_n \to 0 \), such that \((u_n)_n \subset L_{\varepsilon_n} \) for all \( n \in \mathbb{N} \). Since \( \lim_{\varepsilon \to 0} \operatorname{diam} L_\varepsilon = 0 \), \((u_n)_n\) is a Cauchy sequence and converges to \( u_0 \). As the similar proof of Theorem 3.2, \( u_0 \in S(u_0) \). Let \( v \in S(u_0) \). Since Lower Semi-Mosco convergence implies for every \( v \in S(u_0) \), there exists a sequence \((v_n)_n\) converging to \( v \), such that \( v_n \in S(u_n) \). Since \( A \) is \((s, w)\)-continuous and \((u_n)_n\) is an L-\( \alpha \)-approximating sequence for (QVIP), one has

\[
\langle Av, u_0 - v \rangle - \frac{\alpha}{2} \| u_0 - v \|^2 = \lim_{n} \left( \langle Av_n, u_n - v_n \rangle - \frac{\alpha}{2} \| u_n - v_n \|^2 \right) \leq \lim_{n} \varepsilon_n = 0. \tag{3.16}
\]

Applying Lemmas 2.6 and 2.7, we have that (QVIP) is L-\( \alpha \)-well-posed.

The necessity can be completed as Theorem 3.3. \( \square \)

**4. \( \alpha \)-Well-Posedness in the Generalized Sense**

In this section, we introduce and investigate some metric characterizations of \( \alpha \)-well-posedness in the generalized sense and L-\( \alpha \)-well-posedness in the generalized sense for (QVI).
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Definition 4.1 (see [11]). Let \((X,d)\) be a metric space and let \(A, B\) be nonempty subsets of \(X\). The Hausdorff distance \(H(\cdot, \cdot)\) between \(A\) and \(B\) is defined by
\[
H(A, B) = \max \{e(A, B), e(B, A)\},
\]
where \(e(A, B) = \sup_{a \in A} d(a, B)\) with \(d(a, B) = \inf_{b \in B} \|a - b\|\).

Definition 4.2 (see [11]). Let \(A\) be a nonempty subset of \(X\). The measure of noncompactness \(\mu\) of the set \(A\) is defined by
\[
\mu(A) = \inf \left\{ \varepsilon > 0 : \ A \subseteq \bigcup_{i=1}^{n} A_i, \ \text{diam} A_i < \varepsilon, \ i = 1, 2, \ldots, n \right\},
\]
where \(\text{diam}\) means the diameter of a set.

Theorem 4.3. Let the same assumptions be as in Lemma 2.7. Then, one has the following.

(a) \((\text{QVIP})\) is \(\alpha\)-well-posed in the generalized sense if and only if the solution set \(\Gamma\) of \((\text{QVIP})\) is nonempty compact and \(e(Q_{\varepsilon}, \Gamma) \to 0\), as \(\varepsilon \to 0\).

(b) Moreover, if \(A\) is pseudomonotone, then \((\text{QVIP})\) is \(L\alpha\)-well-posed in the generalized sense if and only if the solution set \(\Gamma\) of \((\text{QVIP})\) is nonempty compact and \(e(L_{\varepsilon}, \Gamma) \to 0\), as \(\varepsilon \to 0\).

Proof. We only prove (a), the proof of (b) is similar and is omitted here. Assume that \((\text{QVIP})\) is \(\alpha\)-well-posed in the generalized sense, then the \(\Gamma\) is nonempty and compact. It follows from Lemma 2.7 that \(Q_{\varepsilon} \neq \emptyset\). Now we prove \(e(Q_{\varepsilon}, \Gamma) \to 0\), as \(\varepsilon \to 0\). Suppose by contradiction that there exists \(\beta > 0\), \(\varepsilon_{n} \to 0\), and \(w_{n} \in Q_{\varepsilon_{n}}\), such that \(d(w_{n}, \Gamma) \geq \beta\). It follows from \(w_{n} \in Q_{\varepsilon_{n}}\) that \((w_{n})_{n}\) is an \(\alpha\)-approximating sequence for \((\text{QVIP})\). \((\text{QVIP})\) is \(\alpha\)-well-posedness in the generalized sense, then there exists a subsequence \((w_{n_{k}})_{k}\) of \((w_{n})_{n}\) strongly converging to a point of \(\Gamma\). This contradicts \(d(w_{n_{k}}, \Gamma) \geq \beta\). Thus, \(e(Q_{\varepsilon_{n}}, \Gamma) \to 0\), as \(\varepsilon \to 0\).

For the converse, let \((u_{n})_{n}\) be an \(\alpha\)-approximating sequence for \((\text{QVIP})\), then \(u_{n} \in Q_{\varepsilon_{n}}\). It follows from \(e(Q_{\varepsilon_{n}}, \Gamma) \to 0\) that there exists a sequence \(z_{n} \in \Gamma\), such that \(d(u_{n}, z_{n}) \to 0\). Since \(\Gamma\) is compact, there exists a subsequence \((z_{n_{k}})_{k}\) of \((z_{n})_{n}\) strongly converging to \(u_{0} \in \Gamma\). Thus there exists the corresponding subsequence \((u_{n_{k}})_{k}\) of \((u_{n})_{n}\) strongly converging to \(u_{0}\). Therefore, \((\text{QVIP})\) is \(\alpha\)-well-posed in the generalized sense. \(\square\)

Theorem 4.4. (a) If \((\text{QVIP})\) is \(\alpha\)-well-posed in the generalized sense, then
\[
Q_{\varepsilon} \neq \emptyset, \ \forall \varepsilon > 0, \ \lim_{\varepsilon \to 0} \mu(Q_{\varepsilon}) = 0.
\]

(b) If (4.3) and the following assumptions hold:

(i) the set-valued mapping \(S\) is nonempty convex-valued, and, for each sequence \((u_{n})_{n}\) in \(K\) converges to \(u_{0}\), the sequence \((S(u_{n}))_{n}\) Lower Semi-Mosco converging to \(S(u_{0})\);

(ii) the operator \(A\) is \((s, w)\)-continuous on \(K\),

then, \((\text{QVIP})\) is \(\alpha\)-well-posed in the generalized sense.
Proof. (a) Suppose that (QVIP) is $\alpha$-well-posed in the generalized sense. So $Q_{\varepsilon} \neq \emptyset$, for all $\varepsilon > 0$. By Theorem 4.3(a), $\Gamma$ is nonempty compact and $e(Q_{\varepsilon}, \Gamma) \to 0$, as $\varepsilon \to 0$. For any $\varepsilon > 0$, we have

$$H(Q_{\varepsilon}, \Gamma) = \max \{e(Q_{\varepsilon}, \Gamma), e(\Gamma, Q_{\varepsilon})\} = e(Q_{\varepsilon}, \Gamma), \quad (4.4)$$

and since $\Gamma$ is compact, $\mu(\Gamma) = 0$. For every $n \in N$, the following relation holds [14]:

$$\mu(Q_{\varepsilon}) \leq 2H(Q_{\varepsilon}, \Gamma) + \mu(\Gamma) = 2H(Q_{\varepsilon}, \Gamma) = 2e(Q_{\varepsilon}, \Gamma). \quad (4.5)$$

It follows from $e(Q_{\varepsilon}, \Gamma) \to 0$, as $\varepsilon \to 0$, that $\lim_{\varepsilon \to 0} \mu(Q_{\varepsilon}) = 0$.

(b) Assume that (4.3) holds. Then, for any $\varepsilon > 0$, $\text{cl}(Q_{\varepsilon})$ is nonempty closed and increasing with $\varepsilon > 0$. By (4.3), $\lim_{\varepsilon \to 0} \mu(\text{cl}(Q_{\varepsilon})) = \lim_{\varepsilon \to 0} \mu(Q_{\varepsilon}) = 0$, where $\text{cl}(Q_{\varepsilon})$ is the closure of $Q_{\varepsilon}$. By the generalized Cantor theorem [11, page 412], we know that

$$\lim_{\varepsilon \to 0} H(\text{cl}(Q_{\varepsilon}), \Delta) = 0, \quad \text{as } \varepsilon \to 0, \quad (4.6)$$

where $\Delta = \bigcap_{\varepsilon > 0} \text{cl}(Q_{\varepsilon})$ is nonempty compact.

Now we show that

$$\Gamma = \Delta. \quad (4.7)$$

It follows from Lemma 2.7 that $\Gamma \subseteq \Delta$. So we need to prove that $\Delta \subseteq \Gamma$. Indeed, let $u_0 \in \Delta$. Then, $d(u_0, Q_{\varepsilon}) = 0$ for every $\varepsilon > 0$. Given $\varepsilon_n > 0$, $\varepsilon_n \to 0$, for every $n$, there exists $u_n \in Q_{\varepsilon_n}$ such that $d(u_0, u_n) < \varepsilon_n$. Hence, $u_n \to u_0$ and

$$d(u_n, S(u_n)) \leq \varepsilon_n, \quad (4.8)$$

$$\langle Au_n, u_n - v \rangle \leq \varepsilon_n + \frac{\alpha}{2} \|u_n - v\|^2, \quad \forall v \in S(u_n). \quad (4.9)$$

It follows from (4.8), $u_n \to u_0$, and the proof of Theorem 3.2 that $u_0 \in S(u_0)$.

Since Lower Semi-Mosco convergence implies that, for every $v \in S(u_0)$, there exists a sequence $v_n \in S(u_n)$, for all $n \in N$, such that $\lim_n v_n = v$ in the strongly topology.

Since the operator $A$ is $(s, \omega)$-continuous on $K$, hence

$$\langle Au_0, u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 = \lim_n \left[\langle Au_n, u_n - v_n \rangle - \frac{\alpha}{2} \|u_n - v_n\|^2 \right] \leq \lim_n \varepsilon_n = 0. \quad (4.10)$$

By Lemma 2.7, we know $u_0 \in \Gamma$. Thus, $\Delta \subseteq \Gamma$. It follows from (4.6) and (4.7) that $\lim_{\varepsilon \to 0} e(Q_{\varepsilon}, \Gamma) = 0$. It follows from the compactness of $\Gamma$ and Theorem 4.3(a) that (QVIP) is $\alpha$-well-posed in the generalized sense. The proof is completed. \qed

**Theorem 4.5.** Let $K$ be a nonempty, compact, and convex subset of $E$, let the set-valued mapping $S$ be nonempty convex-valued, and, for each sequence $(u_n)_n$ in $K$ converging to $u_0$, the sequence $(S(u_n))_n$ Lower Semi-Mosco converges to $S(u_0)$, and the operator $A$ is $(s, \omega)$-continuous on $K$. Then, (QVIP) is $\alpha$-well-posed in the generalized sense.
Theorem 4.6. Let the following assumptions hold:

(i) the set-valued mapping $S$ is nonempty convex-valued, and, for each sequence $(u_n)_n$ in $K$ converging to $u_0$, the sequence $(S(u_n))_n$ Lower Semi-Mosco converges to $S(u_0)$;

(ii) the operator $A$ is $(s,w)$-continuous and monotone on $K$.

Then, (QVIP) is $L$-$\alpha$-well-posed in the generalized sense if and only if

$$L_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0^+} \mu(L_\varepsilon) = 0. \tag{4.11}$$

Proof. Assume that (QVIP) is $L$-$\alpha$-well-posed in the generalized sense. It follows from Lemma 2.7 and the monotonicity of $A$ that $\Gamma \subseteq L_{\varepsilon}$, for all $\varepsilon > 0$. And so $L_{\varepsilon} \neq \emptyset$, for each $\varepsilon > 0$. By Theorem 4.3(b), we can get $e(L_{\varepsilon}, \Gamma) \to 0$ as $\varepsilon \to 0$. From the proof of Theorem 4.4, we also obtain

$$\mu(L_{\varepsilon}) \leq 2H(L_{\varepsilon}, \Gamma) + \mu(\Gamma) = 2H(L_{\varepsilon}, \Gamma) = 2e(L_{\varepsilon}, \Gamma). \tag{4.12}$$

Thus, $\lim_{\varepsilon \to 0^+} \mu(L_{\varepsilon}) = 0$.

Conversely, assume (4.11) holds. Then, for any $\varepsilon > 0$, $cl(L_{\varepsilon})$ is nonempty closed and increasing with $\varepsilon > 0$. By (4.11), $\lim_{\varepsilon \to 0^+} \mu(cl(L_{\varepsilon})) = \lim_{\varepsilon \to 0^+} \mu(L_{\varepsilon}) = 0$, where $cl(L_{\varepsilon})$ is the closure of $L_{\varepsilon}$. By the generalized Cantor theorem [11, page 412], we know that

$$\lim_{\varepsilon \to 0^+} H(cl(L_{\varepsilon}), \Delta) = 0, \quad \text{as } \varepsilon \to 0^+, \tag{4.13}$$

where $\Delta = \bigcap_{\varepsilon > 0} cl(L_{\varepsilon})$ is nonempty compact.

Now we show that

$$\Gamma = \Delta. \tag{4.14}$$

It follow from Lemma 2.7 and the monotonicity of $A$ that $\Gamma \subseteq \Delta$. So we need to prove that $\Delta \subseteq \Gamma$. Indeed, let $u_0 \in \Delta$. Then $d(u_0, L_{\varepsilon}) = 0$ for every $\varepsilon > 0$. Given $\varepsilon_n > 0, \varepsilon_n \to 0$, for every $n$, there exists $u_n \in L_{\varepsilon_n}$ such that $d(u_0, u_n) < \varepsilon_n$. Hence, $u_n \to u_0$ and

$$d(u_n, S(u_n)) \leq \varepsilon_n, \tag{4.15}$$

$$(Av, u_n - v) \leq \varepsilon_n + \frac{\alpha}{2} \|u_n - v\|^2, \quad \forall v \in S(u_n). \tag{4.16}$$

It follows from (4.15), $x_n \to x_0$, and the proof of Theorem 3.2 that $u_0 \in S(u_0)$.

Since $S(u_n)$ Lower Semi-Mosco converges to $S(u_0)$, for every $v \in S(u_0)$, there exists a sequence $v_n \in S(u_n)$, for all $n \in N$, such that $\lim_n v_n = v$ in the strong topology.
Since the operator $A$ is $(s, w)$-continuous on $K$, hence

$$\langle Av, u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 = \lim_{n} \left( \langle Av_n, u_n - v_n \rangle - \frac{\alpha}{2} \|u_n - v_n\|^2 \right) \leq \lim_{n} \varepsilon_n = 0. \quad (4.17)$$

By Lemma 2.6 we know that $u_0 \in S(u_0)$, such that

$$\langle Au_0, u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 \leq 0, \quad \forall v \in S(u_0). \quad (4.18)$$

It follows from Lemma 2.7 that $u_0 \in \Gamma$. Thus, $\Delta \subseteq \Gamma$. It follows from (4.13) and (4.14) that $\lim_{n \to 0} e(L_{\alpha}, \Gamma) = 0$. It follows from the compactness of $\Gamma$ and Theorem 4.3(b) that (QVIP) is $L$-$\alpha$-well-posed in the generalized sense. The problem is completed. \hfill \Box

**Remark 4.7.** It is easy to see that if $\alpha = 0$, then by the main results in our paper, we can recover the corresponding results in [9] with the weaker condition that $S(x_n)$ Lower Semi-Mosco converges to $S(x_0)$ instead of the condition that $S$ is $(s, w)$-closed and $(s, w)$-subcontinuous, and $(s, s)$-lower semicontinuous.

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