Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 152023, 18 pages
doi:10.1155/2012/152023

Research Article

A Viscosity Approximation Scheme for Finding Common Solutions of Mixed Equilibrium Problems, a Finite Family of Variational Inclusions, and Fixed Point Problems in Hilbert Spaces

Bin-chao Deng,[1] Tong Chen,[1] and Baogui Xin[1, 2]

[1] *School of Management, Tianjin University, Tianjin 300072, China*
[2] *School of Economics and Management, Shandong University of Science and Technology, Qingdao 266510, China*

Correspondence should be addressed to Bin-chao Deng, dbchao1985@yahoo.com.cn

Received 16 February 2012; Accepted 10 April 2012

Academic Editor: Yeong-Cheng Liou

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We introduce an iterative method for finding a common element of set of fixed points of nonexpansive mappings, the set of solutions of a finite family of variational inclusion with set-valued maximal monotone mappings and inverse strongly monotone mappings, and the set of solutions of a mixed equilibrium problem in Hilbert spaces. Under suitable conditions, some strong convergence theorems for approximating this common elements are proved. The results presented in the paper improve and extend the main results of Plubtemg and Sripard and many others.

1. Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C\times C$ into $\mathbb{R}$ which is the set of real numbers. The equilibrium problem for $F : C\times C \to \mathbb{R}$ is to find $x\in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. The mixed equilibrium problem for two bifunction of $F_1, F_2 : C \times C \to \mathbb{R}$ is to find $x\in C$ such that

$$F_1(x,y) + F_2(x,y) + \langle Ax, x-y\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

In the sequel we will indicate by $\mathrm{MEP}(F_1, F_2, A)$ the set of solution of our mixed equilibrium problem. If $A = 0$ we denote $\mathrm{MEP}(F_1, F_2, 0)$ with $\mathrm{MEP}(F_1, F_2)$.

In 2005, Combettes and Hirstoaga [1] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem. Let $A : C \to H$ be a nonlinear mapping. The classical variational inequality which is denoted by $\mathrm{VI}(A, C)$ is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The variational inequality has been extensively studied in the literature; see, for example, [2, 3] and the reference therein. Recall that mapping T of C into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

A mapping $f : C \to C$ is called contractive if there exists a constant $\beta \in (0, 1)$ such that

$$\|fx - fy\| \leq \beta\|x - y\|, \quad \forall x, y \in C. \tag{1.5}$$

We denote by $F_{ix}(T)$ the set of fixed points of T.

Some methods have been proposed to solve the equilibrium problem and fixed point problem of nonexpansive mapping; see, for instance, [2, 4–6] and the references therein. In 2007, Plubtieng and Punpaeng [6] introduced the following iterative scheme. Let $x_1 \in H$ and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tu_n, \quad \forall n \in \mathbb{N}. \end{gathered} \tag{1.6}$$

They proved that if the sequences $\{\alpha_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)z, z - x \rangle \geq 0, \quad \forall x \in F_{ix}(T) \cap \mathrm{EP}(F), \tag{1.7}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F_{ix}(T) \cap \mathrm{EP}(F)} \frac{1}{2}\langle Ax, x \rangle - h(x), \tag{1.8}$$

where h is a potential function for γf.

Let $A : H \to H$ be a single-valued nonlinear mapping, and let $M : H \to 2^H$ be a set-valued mapping. We consider the following variational inclusion, which is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{1.9}$$

where θ is the zero vector in H. The set of solutions of problem (1.9) is denoted by $I(A, M)$. Let $A_i : H \to H$, $i = 1, 2, \dots N$ be single-valued nonlinear mappings, and let $M_i : H \to 2^H$, $i = 1, 2, \dots N$, be set-valued mappings. If $A \equiv 0$, then problem (1.9) becomes the variational inclusion problem introduced by Rockafellar [7]. If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty]$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.10}$$

then the variational inclusion problem (1.9) is equivalent to variational inequality problem (1.3). It is known that (1.9) provides a convenient framework for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, and game theory. Also various types of variational inclusions problems have been extended and generalized (see [8] and the references therein). We introduce following a finite family of variational inclusions, which are to find a point $u \in H$ such that

$$\theta \in A_i(u) + M_i(u), \quad i = 1, 2, \dots N, \tag{1.11}$$

where θ is the zero vector in H. The set of solutions of problem (1.11) is denoted by $\bigcap_{i=1}^{N} I(A_i, M_i)$. The formulation (1.11) extends this formalism to a finite family of variational inclusions covering, in particular, various forms of feasibility problems (see, e.g., [9]).

In 2009, Plubtemg and Sripard [10] introduced the following iterative scheme for finding a common element of set of solutions to the problem (1.9) with multivalued maximal monotone mapping and inverse-strongly monotone mapping, the set solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in Hilbert spaces. Starting with an arbitrary $x_1 \in H$, define sequence $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \quad \forall n > 0, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S_n y_n, \end{gathered} \tag{1.12}$$

for all $n \in N$, where $\lambda \in (0, 2\alpha]$, $\{\alpha_n\} \subset [0, 1]$, and $\{r_n\} \subset (0, \infty)$; B is a strongly positive bounded linear operator on H, and $\{S_n\}$ is a sequence of nonexpansive mappings on H. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ generated by (1.12) converge strongly to $z \in \bigcap_{i=1}^{\infty} F_{ix}(S_i) \cap I(A, M) \cap \mathrm{EP}(F)$, where $z = P_{\bigcap_{i=1}^{\infty} F_{ix}(S_i) \cap I(A,M) \cap \mathrm{EP}(F)} f(z)$.

In 2011, Yao et al. [11] considered the following iterative method for finding a common element of set of solutions to the problem (1.9) with multi-valued maximal monotone mapping and inverse-strongly monotone mapping, the set solutions of a mixed equilibrium problem, and the set of fixed points of an infinite family of nonexpansive mappings in Hilbert spaces. Let $F : H \times H \to \mathbb{R}$ be a bifunction, A be a strongly positive bounded linear operator, and $B_1, B_2 : H \to H$ be inverse strongly monotone and let inverse strongly monotone, and

$\varphi : H \to \mathbb{R}$ be a lower semicontinuous and convex function. For any initial x_0 is selected in H arbitrarily

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rB_1 x_n)\rangle \geq 0, \quad \forall y \in H,$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n J_{R,\lambda}(u_n - \lambda B_2 u_n), \quad n \geq 0, \tag{1.13}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ and W_n is an infinite family of nonexpansive mappings. They study the strong convergence of the iterative algorithm (1.13).

Motivated and inspired by Saeidi [12], Aoyama et al. [13], Plubieng and Punpaeng [6], Plubtemg and Sripard [10], Peng et al. [14], and Yao et al. [11], we introduce an iterative scheme for finding a common element of the set of solutions of a finite family of variational inclusion problems (1.11) with multi-valued maximal monotone mappings and inverse-strongly monotone mappings, the set of solutions of a mixed equilibrium problem, and the set of fixed points of nonexpansive mappings in Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by

$$F_1(u_n, y) + F_2(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H,$$
$$y_n = J_{M_N,\lambda_{N,n}}(I - \lambda_{N,n}A_N)\cdots J_{M_1,\lambda_{1,n}}(I - \lambda_{1,n}A_1)u_n,$$
$$x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n B)S_n y_n, \tag{1.14}$$

for all $n \in N$, where $\lambda_{i,n} \in (0, 2\alpha_i]$, $i \in \{1,2,\ldots,N\}$, $\{\epsilon_n\} \subset [0,1]$, and $\{r_n\} \subset (0,\infty)$, B is a strongly positive bounded linear operator on H, and $\{S_n\}$ is a sequence of nonexpansive mappings on H. Under suitable conditions, some strong convergence theorems for approximating to this common elements are proved. Our results extend and improve some corresponding results in [10, 11, 14] and the references therein.

2. Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in next section.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. It is well known that for all $x, y \in H$ and $\lambda \in [0,1]$, there holds

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.1}$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$. such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H into C. We know that P_C is nonexpansive. It is also known that $P_C x \in C$ and

$$\langle x - P_C x, P_C x - z\rangle \geq 0, \quad \forall x \in H, \ z \in C. \tag{2.2}$$

It is easy to see that (2.2) is equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|P_C x - z\|^2, \quad \forall x \in H, \ z \in C. \tag{2.3}$$

For solving the mixed equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y); \tag{2.4}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.1 (see [15]). *Let C be a convex closed subset of a Hilbert spaces H.*
Let $F_1 : C \times C \to R$ be a bifunction such that

(f1) $F_1(x, x) = 0$ *for all* $x \in C$;

(f2) F_1 *is monotone and upper hemicontinuous in the first variable;*

(f3) F_1 *is lower semicontinuous and convex in the second variable.*
Let $F_2 : C \times C \to R$ be a bifunction such that

(h1) $F_2(x, x) = 0$ *for all* $x \in C$;

(h2) F_2 *is monotone and weakly upper semicontinuous in the first variable;*

(h3) F_2 *is convex in the second variable.*

Moreover let us suppose that

(H) *for fixed $r > 0$ and $x \in C$ there exists a bounded set $K \subset C$ and $a \in K$ such that for all $z \in C \setminus K, -F_1(a, z) + F_2(z, a) + (1/r)\langle a - z, z - x\rangle < 0$.*

For $r > 0$ and $x \in H$, let $T_r : H \to C$ be a mapping defined by

$$T_r(x) = \left\{ y \in C : F_1(z, y) + F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall z \in C \right\}. \tag{2.5}$$

Called resolvent of F_1 and F_2.
Then,

(1) $T_r x \neq \emptyset$;

(2) T_r *is a single value;*

(3) T_r *is firmly nonexpansive;*

(4) $\mathrm{MEP}(F_1, F_2) = F_{ix}(T_r x)$ *and it is closed and convex.*

Recall that a mapping $A : H \to H$ is called α-inverse-strongly monotone, if there exists a positive number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \quad \forall u, v \in H. \tag{2.6}$$

Let I be the identity mapping on H. It is well known that if $A : H \to H$ is α-inverse-strongly monotone, then A is $1/\alpha$-Lipschitz continuous and monotone mapping. In addition, if $0 < \lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping.

A set-valued $M : H \to 2^H$ is called monotone, if for all $x, y \in H, f \in Mx$, and $g \in My$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if its graph $G(M) : \{(x, f) \in H \times H \mid f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(H)$ implies $f \in Mx$.

Let the set-valued $M : H \to 2^H$ be maximal monotone. we define the resolvent operator $J_{M,\lambda}$ associated with M and λ as follows:

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H, \tag{2.7}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive, and 1-inverse-strongly monotone, see for example [16] and that a solution of problem (1.9) is a fixed point of the operator $J_{M,\lambda}(I - \lambda A)$ for all $\lambda > 0$, see for instance, [17]. Furthermore, a solution of a finite family of variational inclusion problems (1.11) is a common fixed point of $J_{M_k,\lambda}(I - \lambda A_k)$, $k \in \{1, \ldots, N\}$, $\lambda > 0$.

Lemma 2.2 (see [16]). *Let $M : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ a Lipschitz-continuous mapping. Then the mapping $S = M + A : H \to 2^H$ is a maximal monotone mapping.*

Lemma 2.3 (see [18]). *Let H be a Hilbert space, C a nonempty closed subset of H, $f : H \to H$ a contraction with coefficient $0 < \alpha < 1$, and B a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Then,*

(1) *if $0 < \gamma < \overline{\gamma}/\alpha$, then $\langle x - y, (B - \gamma f)x - (B - \gamma f)y\rangle \geq (\overline{\gamma} - \gamma\alpha)\|x - y\|^2$, $x, y \in H$;*

(2) *if $0 < \rho < \|B\|^{-1}$, then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.4. *For all $x, y \in H$, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.8}$$

Lemma 2.5 (the resolvent identity). *Let E be a Banach space, for $\lambda > 0$, $\mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big). \tag{2.9}$$

Lemma 2.6. *Let H be a Hilbert space. Let $A_i : H \to H$, $i = 1,2,\ldots,N$ be α_i-inverse-strongly monotone mappings, $M_i : H \to 2^H$, $i = 1,2,\ldots,N$ maximal monotone mappings, and $\{\omega_n\}$ be a bounded sequence in H. Assume $\lambda_{j,n} > 0$, $j = 1,2,\ldots,N$, satisfy*

(H1) $\lim_{n\to\infty} \sum_{n=1}^{\infty} |\lambda_{j,n} - \lambda_{j,n+1}| < \infty$,

(H2) $\liminf_{n\to\infty} \lambda_{j,n} > 0$.

Set $\Theta_n^k = J_{M_k,\lambda_{k,n}}(I - \lambda_{k,n}A_k)\cdots J_{M_1,\lambda_{1,n}}(I - \lambda_{1,n}A_1)$ for $k \in \{1,2,\ldots,N\}$ and $\Theta_n^0 = I$ for all n. Then, for $k \in \{1,2,\ldots,N\}$,

$$\sum_{i=1}^{\infty} \left\| \Theta_{i+1}^k \omega_i - \Theta_i^k \omega_i \right\| < \infty. \tag{2.10}$$

Proof. From Lemma 2.5, we have, for all $k \in \{1,2,\ldots,N\}$,

$$\begin{aligned} &\left\| J_{M_k,\lambda_{k,n+1}}(I - \lambda_{k,n+1}A_k)\omega_n - J_{M_k,\lambda_{k,n}}(I - \lambda_{k,n}A_k)\omega_n \right\| \\ &\quad\le \left| 1 - \frac{\lambda_{k,n}}{\lambda_{k,n+1}} \right| \left(\left\| J_{M_k,\lambda_{k,n+1}}(I - \lambda_{k,n+1}A_k)\omega_n \right\| + \|\omega_n\| \right). \end{aligned} \tag{2.11}$$

Furthermore, from the definition of Θ_n^k, it follows

$$\Theta_n^k = J_{M_k,\lambda_{k,n}}(I - \lambda_{k,n}A_k)\Theta_n^{k-1}. \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$\begin{aligned} \left\| \Theta_{n+1}^k \omega_n - \Theta_n^k \omega_n \right\| &\le \left\| J_{M_k,\lambda_{k,n+1}}(I - \lambda_{k,n+1}A_k)\Theta_{n+1}^{k-1}\omega_n - J_{M_k,\lambda_{k,n}}(I - \lambda_{k,n}A_k)\Theta_n^{k-1}\omega_n \right\| \\ &\le \left\| J_{M_k,\lambda_{k,n+1}}(I - \lambda_{k,n+1}A_k)\Theta_{n+1}^{k-1}\omega_n - J_{M_k,\lambda_{k,n+1}}(I - \lambda_{k,n+1}A_k)\Theta_n^{k-1}\omega_n \right\| \\ &\quad + \left\| J_{M_k,\lambda_{k,n+1}}(I - \lambda_{k,n+1}A_k)\Theta_n^{k-1}\omega_n - J_{M_k,\lambda_{k,n}}(I - \lambda_{k,n}A_k)\Theta_n^{k-1}\omega_n \right\| \\ &\le \left\| \Theta_{n+1}^{k-1}\omega_n - \Theta_n^{k-1}\omega_n \right\| + \left| 1 - \frac{\lambda_{k,n}}{\lambda_{k,n+1}} \right| \\ &\quad \times \left(\left\| J_{M_k,\lambda_{k,n+1}}(I - \lambda_{k,n+1}A_k)\Theta_n^{k-1}\omega_n \right\| + \|\omega_n\| \right) \\ &\le \left\| \Theta_{n+1}^{k-1}\omega_n - \Theta_n^{k-1}\omega_n \right\| + \left| 1 - \frac{\lambda_{k,n}}{\lambda_{k,n+1}} \right| M_1 \\ &\le \cdots \\ &\le \left\| \Theta_{n+1}^{0}\omega_n - \Theta_n^{0}\omega_n \right\| + \sum_{l=1}^{k} \left| 1 - \frac{\lambda_{l,n}}{\lambda_{l,n+1}} \right| M_1 \\ &= \sum_{l=1}^{k} \left| 1 - \frac{\lambda_{l,n}}{\lambda_{l,n+1}} \right| M_1, \end{aligned} \tag{2.13}$$

where $M_1 = \sup\{\|\omega_n\| + \sum_{k=1}^{N} \|J_{M_k,\lambda_{k,n+1}}(I - \lambda_{k,n+1}A_k)\Theta_n^{k-1}\omega_n\|\}$. According to (H1) and (H2), then (2.11) holds. □

Lemma 2.7 (see [19]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n$ and $\limsup_{n\to\infty}\beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n \tag{2.14}$$

for all $n \geq 0$ and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.15}$$

Then $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

Lemma 2.8 (see [20]). *Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \quad n \geq 0, \tag{2.16}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

3. Main Results

Theorem 3.1. *Let H be a real Hilbert space, let F_1 and F_2 be bifunction $H\times H \to \mathbb{R}$ satisfying (A1)–(A4), and let $\{S_n\}$ be a sequence of nonexpansive mappings on H. Let $A_i : H \to H$, $i = 1,2,\ldots,N$, be α_i-inverse-strongly monotone mappings, and let $M_i : H \to 2^H$, $i = 1,2,\ldots,N$ be maximal monotone mappings such that $\Omega := (\bigcap_{n=1}^{\infty} F_{ix}(S_n)) \cap \mathrm{MEP}(F_1,F_2) \cap (\bigcap_{i=1}^{N} I(A_i,M_i)) \neq \emptyset$. Let f be a contraction of H into itself with a constant $\alpha \in (0,1)$, and let B be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{gathered} F_1(u_n,y) + F_2(u_n,y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ y_n = J_{M_N,\lambda_{N,n}}(I - \lambda_{N,n}A_N)\cdots J_{M_1,\lambda_{1,n}}(I - \lambda_{1,n}A_1)u_n, \\ x_{n+1} = \epsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n B)S_n y_n, \end{gathered} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\beta_n \in (0,1)$, $\lambda_{i,n} \in (0, 2\alpha_i]$, $i \in \{1,2,\ldots,N\}$, satisfy (H1)-(H2) and $\{\epsilon_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

(C1) $\lim_{n\to\infty}\epsilon_n = 0$;

(C2) $\sum_{n=1}^{\infty}\epsilon_n = \infty$;

(C3) $\sum_{n=1}^{\infty}|\epsilon_{n+1} - \epsilon_n| < \infty$;

(C4) $\liminf_{n\to\infty} r_n > 0$;

(C5) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(C6) $\lim_{n\to\infty} \beta_n = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ *for any bounded subset* K *of* H*. Let* S *be a mapping of* H *into itself defined by* $Sx = \lim_{n\to\infty} S_n x$*, for all* $x \in H$ *and suppose that* $F_{ix}(S) = \bigcap_{n=1}^{\infty} F_{ix}(S_n)$*. Then,* $\{x_n\}$*,* $\{y_n\}$*, and* $\{u_n\}$ *converge strongly to* z*, where* $z = P_\Omega(I - B + \gamma f)(z)$ *is a unique solution of the variational inequality*

$$\langle (B - \gamma f)z, z - x \rangle \le 0 \quad x \in \Omega. \tag{3.2}$$

Proof. Since $\epsilon_n \to 0$, we will assume that $\epsilon_n \le (1-\beta_n)\|B\|^{-1}$ and $1 - \epsilon_n(\overline{\gamma} - \alpha\gamma) > 0$. Observe that, if $\|u\| = 1$, then

$$\langle ((1-\beta_n)I - \epsilon_n B)u, u \rangle = (1-\beta_n) - \epsilon_n \langle Bu, u \rangle \ge (1 - \beta_n - \epsilon_n \|B\|) \ge 0. \tag{3.3}$$

By Lemma 2.3 we have

$$\|(1-\beta_n)I - \epsilon_n B\| \le (1-\beta_n) - \epsilon_n \overline{\gamma}. \tag{3.4}$$

Moreover, using the definition of Θ_n^k in Lemma 2.6, we have $y_n = \Theta_n^N u_n$. We divide the proof into several steps.

Step 1. The sequence $\{x_n\}$ is bounded.

Since $\epsilon_n \to 0$, we may assume that $\epsilon_n \le \|B\|^{-1}$ for all n. Let $p \in \Omega$. Using the fact that $J_{M_k,\lambda_{k,n}}(I - \lambda_{k,n}A_k)$, $k \in \{1, 2, \dots, N\}$, is nonexpansive and $p = J_{M_k,\lambda_{k,n}}(I - \lambda_{k,n}A_k)p$, we have

$$\|y_n - p\| = \left\|\Theta_n^N u_n - \Theta_n^N p\right\| \le \|u_n - p\| \le \|T_r x_n - T_r p\| \le \|x_n - p\|, \tag{3.5}$$

for all $n \ge 1$. Then, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \Big\|((1-\beta_n)I - \epsilon_n B)\left(S_n\Theta_n^N u_n - S_n\Theta_n^N p\right) + \epsilon_n (f(x_n) - f(p)) \\
&\quad + \epsilon_n(\gamma f(p) - Bp) + \beta_n(x_n - p)\Big\| \\
&\le (1 - \epsilon_n(\overline{\gamma} - \alpha\gamma))\|x_n - p\| + \epsilon_n(\overline{\gamma} - \alpha\gamma)\frac{\|\gamma f(p) - Bp\|}{\overline{\gamma} - \alpha\gamma}.
\end{aligned} \tag{3.6}$$

It follow from (3.6) and induction that

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \alpha\gamma}\right\}, \quad \forall n \ge 1. \tag{3.7}$$

Hence $\{x_n\}$ is bounded and therefore $\{u_n\}$, $\{y_n\}$, $\{f(x_n)\}$, and $\{S_n y_n\}$ are also bounded.

Step 2. We show that $\|x_{n+1} - x_n\| \to 0$.

Define $x_{n+1} = \beta_n x_n + (1-\beta_n)v_n$ for each $n \geq 0$. From the definition of v_n, we obtain

$$\begin{aligned}
v_{n+1} - v_n &= \frac{1}{1-\beta_{n+1}}\left(x_{n+2} - \beta_{n+1}x_{n+1}\right) - \frac{1}{1-\beta_n}\left(x_{n+1} - \beta_n x_n\right) \\
&= \frac{\epsilon_{n+1}\gamma f(x_{n+1}) + \left((1-\beta_{n+1})I - \epsilon_{n+1}B\right)S_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\epsilon_n\gamma f(x_n) + \left((1-\beta_n)I - \epsilon_n B\right)S_n y_n}{1-\beta_n} \\
&= \frac{\epsilon_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} - \frac{\epsilon_n\gamma f(x_n)}{1-\beta_n} + S_{n+1}y_{n+1} - S_n y_n + \frac{\epsilon_n}{1-\beta_n}BS_n y_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}BS_{n+1}y_{n+1} \\
&= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\gamma f(x_{n+1}) - BS_{n+1}y_{n+1}\right) + \frac{\epsilon_n}{1-\beta_n}\left(\gamma f(x_n) - BS_n y_n\right) \\
&\quad + S_{n+1}y_{n+1} - S_{n+1}y_n + S_{n+1}y_n - S_n y_n.
\end{aligned} \tag{3.8}$$

It follows that

$$\begin{aligned}
\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\| &\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \left\|BS_{n+1}y_{n+1}\right\|\right) \\
&\quad + \frac{\epsilon_n}{1-\beta_n}\left(\left\|\gamma f(x_n)\right\| + \left\|BS_n y_n\right\|\right) + \left\|S_{n+1}y_{n+1} - S_{n+1}y_n\right\| \\
&\quad + \left\|S_{n+1}y_n - S_n y_n\right\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \left\|BS_{n+1}y_{n+1}\right\|\right) \\
&\quad + \frac{\epsilon_n}{1-\beta_n}\left(\left\|\gamma f(x_n)\right\| + \left\|BS_n y_n\right\|\right) + \left\|y_{n+1} - y_n\right\| \\
&\quad + \left\|S_{n+1}y_n - S_n y_n\right\| - \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.9}$$

By the suppose of $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$, we obtain

$$\lim_{n\to\infty} \left\|S_{n+1}y_n - S_n y_n\right\| = 0. \tag{3.10}$$

From Lemma 2.6, we obtain

$$\lim_{n\to\infty} \|\Theta_{n+1}^N u_{n+1} - \Theta_n^N u_{n+1}\| = 0. \tag{3.11}$$

By Θ_n^N and T_r being nonexpansive, we have

$$\begin{aligned}
\left\|y_{n+1} - y_n\right\| &= \left\|\Theta_{n+1}^N u_{n+1} - \Theta_n^N u_n\right\| \\
&\leq \left\|\Theta_{n+1}^N u_{n+1} - \Theta_n^N u_{n+1}\right\| + \left\|\Theta_n^N u_{n+1} - \Theta_n^N u_n\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\Theta_{n+1}^{N} u_{n+1}-\Theta_{n}^{N} u_{n+1}\right\|+\left\|u_{n+1}-u_{n}\right\| \\
&\leq \left\|\Theta_{n+1}^{N} u_{n+1}-\Theta_{n}^{N} u_{n+1}\right\|+\left\|T_{r} x_{n+1}-T_{r} x_{n}\right\| \\
&\leq \left\|\Theta_{n+1}^{N} u_{n+1}-\Theta_{n}^{N} u_{n+1}\right\|+\left\|x_{n+1}-x_{n}\right\| .
\end{aligned}
\tag{3.12}
$$

Substituting (3.12) into (3.9), we get

$$
\begin{aligned}
\left\|v_{n+1}-v_{n}\right\|-\left\|x_{n+1}-x_{n}\right\| \leq & \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f\left(x_{n+1}\right)\right\|+\left\|B S_{n+1} y_{n+1}\right\|\right) \\
& +\frac{\epsilon_{n}}{1-\beta_{n}}\left(\left\|\gamma f\left(x_{n}\right)\right\|+\left\|B S_{n} y_{n}\right\|\right)+\left\|\Theta_{n+1}^{N} u_{n+1}-\Theta_{n}^{N} u_{n+1}\right\| \\
& +\left\|S_{n+1} y_{n}-S_{n} y_{n}\right\| .
\end{aligned}
\tag{3.13}
$$

By (3.10), (3.11), and the conditions (C1) and (C6), we imply that

$$
\lim_{n \rightarrow \infty}\left(\left\|v_{n+1}-v_{n}\right\|-\left\|x_{n+1}-x_{n}\right\|\right)=0 . \tag{3.14}
$$

Hence, by Lemma 2.7, we have $\lim_{n \rightarrow \infty}\|v_{n}-x_{n}\|=0$. Consequently, it follows that

$$
\lim_{n \rightarrow \infty}\left\|x_{n+1}-x_{n}\right\|=\lim_{n \rightarrow \infty}\left(1-\beta_{n}\right)\left\|v_{n}-x_{n}\right\|=0 . \tag{3.15}
$$

From (3.11), (3.12), and (3.15), we also imply that

$$
\lim_{n \rightarrow \infty}\left\|y_{n+1}-y_{n}\right\|=0 . \tag{3.16}
$$

Step 3. We now show that

$$
\lim_{n \rightarrow \infty}\left\|\Theta_{n}^{k} u_{n}-\Theta_{n}^{k-1} u_{n}\right\|=0, \quad k=1,2, \ldots, N . \tag{3.17}
$$

Indeed, let $p \in \Omega$. It follows from the firmly nonexpansiveness of $J_{M_k,\lambda_{k,n}}(I-\lambda_{k,n}A_k)$ that

$$
\begin{aligned}
\left\|\Theta_{n}^{k} u_{n}-p\right\|^{2} &=\left\|J_{M_{k}, \lambda_{k, n}}\left(I-\lambda_{k, n} A_{k}\right) \Theta_{n}^{k-1} u_{n}-J_{M_{k}, \lambda_{k, n}}\left(I-\lambda_{k, n} A_{k}\right) p\right\|^{2} \\
&\leq\left\langle\Theta_{n}^{k} u_{n}-p, \Theta_{n}^{k-1} u_{n}-p\right\rangle \\
&=\frac{1}{2}\left(\left\|\Theta_{n}^{k} u_{n}-p\right\|^{2}+\left\|\Theta_{n}^{k-1} u_{n}-p\right\|^{2}\left\|\Theta_{n}^{k} u_{n}-\Theta_{n}^{k-1} u_{n}\right\|^{2}\right),
\end{aligned}
\tag{3.18}
$$

for each $k \in \{1,2,\ldots,N\}$. Thus we get

$$
\left\|\Theta_{n}^{k} u_{n}-p\right\|^{2} \leq\left\|\Theta_{n}^{k-1} u_{n}-p\right\|^{2}-\left\|\Theta_{n}^{k} u_{n}-\Theta_{n}^{k-1} u_{n}\right\|^{2}, \tag{3.19}
$$

which implies that, for each $k \in \{1, 2, \ldots, N\}$,

$$\begin{aligned}
\|y_n - p\|^2 = \left\|\Theta_n^N u_n - p\right\|^2 &\leq \left\|\Theta_n^0 u_n - p\right\|^2 - \sum_{k=1}^{N} \left\|\Theta_n^k u_n - \Theta_n^{k-1} u_n\right\|^2 \\
&\leq \|u_n - p\|^2 - \left\|\Theta_n^k u_n - \Theta_n^{k-1} u_n\right\|^2 \\
&\leq \|x_n - p\|^2 - \left\|\Theta_n^k u_n - \Theta_n^{k-1} u_n\right\|^2.
\end{aligned} \tag{3.20}$$

Set $\theta_n = \gamma f(x_n) - BS_n y_n$, and let $\lambda > 0$ be a constant such that

$$\lambda > \sup_{n,k} \{\|\theta_n\|, \|x_k - p\|\}. \tag{3.21}$$

Using Lemma 2.2 and noting that $\|\cdot\|^2$ is convex, we derive, from (3.20),

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\beta_n)(S_n y_n - p) + \beta_n(x_n - p) + \epsilon_n \theta_n\|^2 \\
&\leq \|(1-\beta_n)(S_n y_n - p) + \beta_n(x_n - p)\|^2 + 2\epsilon_n \langle \theta_n, x_{n+1} - p\rangle \\
&\leq (1-\beta_n)\|S_n y_n - p\|^2 + \beta_n \|x_n - p\|^2 + 2\lambda^2 \epsilon_n \\
&\leq (1-\beta_n)\|y_n - p\|^2 + \beta_n \|x_n - p\|^2 + 2\lambda^2 \epsilon_n \\
&\leq (1-\beta_n)\left(\|x_n - p\|^2 - \left\|\Theta_n^k u_n - \Theta_n^{k-1} u_n\right\|^2\right) + \beta_n \|x_n - p\|^2 + 2\lambda^2 \epsilon_n \\
&\leq \|x_n - p\|^2 - (1-\beta_n)\left\|\Theta_n^k u_n - \Theta_n^{k-1} u_n\right\|^2 + 2\lambda^2 \epsilon_n.
\end{aligned} \tag{3.22}$$

It follows, by Step 2 and condition (C1), that

$$\begin{aligned}
\left\|\Theta_n^k u_n - \Theta_n^{k-1} u_n\right\|^2 &\leq \frac{1}{1-\beta_n}\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\lambda^2 \epsilon_n\right) \\
&\leq \frac{1}{1-\beta_n}\left(2\lambda \|x_n - x_{n+1}\| + 2\lambda^2 \epsilon_n\right) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.
\end{aligned} \tag{3.23}$$

Step 4. We will prove $\lim_{n\to\infty} \|u_n - x_n\| = 0$.

We note from (3.1)

$$\begin{aligned}
\|x_n - S_n y_n\| &\leq \|x_n - S_{n-1} y_{n-1}\| + \|S_{n-1} y_{n-1} - S_{n-1} y_n\| + \|S_{n-1} y_n - S_n y_n\| \\
&\leq \epsilon_{n-1} \|\gamma f(x_{n-1}) - BS_{n-1} y_{n-1}\| + \|y_{n-1} - y_n\| \\
&\quad + \beta_n \|x_{n-1} - S_{n-1} y_{n-1}\| + \sup\{\|S_{n+1} z - S_n z\| : z \in \|y_n\|\}.
\end{aligned} \tag{3.24}$$

Since $\epsilon_n \to 0, \beta_n \to 0$ as $n \to \infty$, $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$ and $\sup\{\|S_{n+1} z - S_n z\| : z \in \{y_n\}\} \to 0$, we get

$$\|x_n - S_n y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

Let $v \in \Omega$. Since $u_n = T_{r_n} x_n$, it follows from Lemma 2.1 that

$$\begin{aligned} \|u_n - v\|^2 &= \|T_{r_n} x_n - T_{r_n} v\|^2 \le \langle T_{r_n} x_n - T_{r_n} v, x_n - v\rangle = \langle u_n - v, x_n - v\rangle \\ &\le \frac{1}{2}\left(\|u_n - v\|^2 + \|x_n - v\|^2 - \|u_n - x_n\|^2\right), \end{aligned} \tag{3.26}$$

and hence $\|u_n - v\|^2 \le \|x_n - v\|^2 - \|u_n - x_n\|^2$. Therefore, using Lemma 2.6 and (3.22), we have

$$\begin{aligned} \|x_{n+1} - v\|^2 &\le (1-\beta_n)\left\|y_n - p\right\|^2 + \beta_n\left\|x_n - p\right\|^2 + 2\lambda^2\epsilon_n \\ &\le (1-\beta_n)\left\|u_n - p\right\|^2 + \beta_n\left\|x_n - p\right\|^2 + 2\lambda^2\epsilon_n \\ &\le (1-\beta_n)\left(\|x_n - v\|^2 - \|u_n - x_n\|^2\right) + \beta_n\left\|x_n - p\right\|^2 + 2\lambda^2\epsilon_n \\ &\le \|x_n - v\|^2 - (1-\beta_n)\|u_n - x_n\|^2 + 2\lambda^2\epsilon_n, \end{aligned} \tag{3.27}$$

and hence

$$\begin{aligned} \|u_n - x_n\|^2 &\le \frac{1}{1-\beta_n}\left(\|x_n - v\|^2 - \|x_{n+1} - v\|^2 + 2\lambda^2\epsilon_n\right) \\ &\le \frac{1}{1-\beta_n}\left\{\|x_n - x_{n+1}\|(\|x_n - v\| - \|x_{n+1} - v\|) + 2\lambda^2\epsilon_n\right\}. \end{aligned} \tag{3.28}$$

Since $\{x_n\}$ is bounded, $\lim_{n\to\infty}\epsilon_n = 0$, $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$, and $\lambda > \sup_{n,k}\{\|\theta_n\|, \|x_k - p\|\}$, it follows that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.29}$$

Next we will prove $\lim_{n\to\infty}\|u_n - y_n\| = 0$.

$$\begin{aligned} \left\|u_n - y_n\right\| &= \left\|\Theta_n^N u_n - u_n\right\| \\ &\le \left\|\Theta_n^N u_n - \Theta_n^{N-1} u_n\right\| + \left\|\Theta_n^{N-1} u_n - \Theta_n^{N-2} u_n\right\| \\ &\quad + \cdots + \left\|\Theta_n^2 u_n - \Theta_n^1 u_n\right\| + \left\|\Theta_n^1 u_n - \Theta_n^0 u_n\right\| + \|u_n - u_n\|. \end{aligned} \tag{3.30}$$

From (2.10), we obtain

$$\lim_{n\to\infty}\left\|u_n - y_n\right\| = 0. \tag{3.31}$$

In addition, according to $\|x_n - y_n\| \le \|x_n - u_n\| + \|u_n - y_n\|$, we have

$$\lim_{n\to\infty}\left\|x_n - y_n\right\| = 0. \tag{3.32}$$

It follows from (3.25), (3.32), and the inequality $\|y_n - S_n y_n\| \le \|y_n - x_n\| + \|x_n - S_n y_n\|$ that $\lim_{n\to\infty}\|y_n - S_n y_n\| = 0$. Since

$$\begin{aligned} \|Sy_n - y_n\| &\le \|Sy_n - S_n y_n\| + \|S_n y_n - y_n\| \\ &\le \sup\{\|Sz - S_n z\| : z \in \{y_n\}\} + \|S_n y_n - y_n\|, \end{aligned} \tag{3.33}$$

for all $n \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty}\|Sy_n - y_n\| = 0. \tag{3.34}$$

Step 5. We show $\omega \in (\bigcap_{n=1}^{\infty} F_{ix}(S_n)) \cap \text{MEP}(F_1, F_2) \cap (\bigcap_{i=1}^{N} I(A_i, M_i))$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to ω. From (3.29), we obtain $\{u_{n_i}\}$ which converges weakly to ω. From (3.32), it follows that $y_{n_i} \rightharpoonup \omega$. We show $\omega \in \text{MEP}(F_1, F_2)$. According to (3.1) and (A2), we obtain

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F_1(y, u_n) + F_2(y, u_n), \tag{3.35}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F_1(y, u_n) + F_2(y, u_n). \tag{3.36}$$

Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup \omega$, from (A4), it follows that $0 \ge F_1(y, \omega) + F_2(y, \omega)$ for all $y \in H$. For t with $0 < t \le 1$ and $y \in H$, let $y_t = ty + (1-t)\omega$, then we obtain $0 \ge F_1(y_t, \omega) + F_2(y_t, \omega)$. So, from (A1) and (A4) we have

$$\begin{aligned} 0 &= F_1(y_t, y_t) + F_2(y_t, y_t) \\ &\le tF_1(y_t, y) + (1-t)F_1(y_t, \omega) + tF_2(y_t, y) + (1-t)F_2(y_t, \omega) \\ &\le F_1(y_t, y) + F_2(y_t, y) \end{aligned} \tag{3.37}$$

and hence $0 \le F_1(y_t, y) + F_2(y_t, y)$. From (A3), we have $0 \le F_1(\omega, y) + F_2(\omega, y)$ for all $y \in H$. Therefore, $\omega \in \text{MEP}(F_1, F_2)$.

Next, we show $\omega \in \bigcap_{n=1}^{\infty} F_{ix}(S_n)$. Assume $\omega \notin \bigcap_{n=1}^{\infty} F_{ix}(S_n)$; then we have $\omega \ne S\omega$. It follows by the Opial's condition and (3.34) that

$$\begin{aligned} \liminf_{n\to\infty}\|y_n - \omega\| &< \liminf_{n\to\infty}\|y_n - S\omega\| \\ &\le \liminf_{n\to\infty}\{\|y_n - Sy_n\| + \|Sy_n - S\omega\|\} \\ &\le \liminf_{n\to\infty}\|y_n - \omega\|. \end{aligned} \tag{3.38}$$

This is a contradiction. Hence $\omega \in \bigcap_{n=1}^{\infty} F_{ix}(S_n)$.

We now show that $\omega \in \bigcap_{i=1}^{N} I(A_i, M_i)$. In fact, since A_i is α_i-inverse-strongly monotone, A_i, $i = 1, 2, \ldots, N$, is a $1/\alpha_i$-Lipschitz continuous monotone mapping and $D(A_i) = H$,

$i = 1, 2, \ldots, N$. It follows from Lemma 2.2 that $M_i + A_i$, $i = 1, 2, \ldots, N$, is maximal monotone. Let $(p, g) \in G(M_i + A_i)$, $i = 1, 2, \ldots, N$, that is, $g - A_i p \in (M_i p)$, $i = 1, 2, \ldots, N$. Since $\Theta_n^k u_n = J_{M_k, \lambda_{k,n}}(I - \lambda_{k,n} A_k)\Theta_n^{k-1} u_n$, we have $\Theta_n^{k-1} u_n - \lambda_{k,n} A_k \Theta_n^{k-1} u_n \in (I + \lambda_{k,n} M_k)(\Theta_n^k u_n)$, that is,

$$\frac{1}{\lambda_{n,k}}\left(\Theta_n^{k-1} u_n - \Theta_n^k u_n - \lambda_{N,n} A_k \Theta_n^{k-1} u_n\right) \in M_k\left(\Theta_n^k u_n\right). \tag{3.39}$$

By the maximal monotonicity of $M_i + A_i i = 1, 2, \ldots, N$, we have

$$\left\langle p - \Theta_n^k u_n, g - A_k p - \frac{1}{\lambda_{k,n}}\left(\Theta_n^{k-1} u_n - \Theta_n^k u_n - \lambda_{k,n} A_k \Theta_n^{k-1} u_n\right)\right\rangle \geq 0, \tag{3.40}$$

which implies

$$\begin{aligned}
\langle p - \Theta_n^k u_n, g\rangle &\geq \left\langle p - \Theta_n^k u_n, A_k p + \frac{1}{\lambda_{n,k}}\left(\Theta_n^{k-1} u_n - \Theta_n^k u_n - \lambda_{k,n} A_k \Theta_n^{k-1} u_n\right)\right\rangle \\
&= \left\langle p - \Theta_n^k u_n, A_k p - A_k \Theta_n^k u_n + A_k \Theta_n^k u_n - A_k \Theta_n^{k-1} u_n + \frac{1}{\lambda_{k,n}}\left(\Theta_n^{k-1} u_n - \Theta_n^k u_n\right)\right\rangle \\
&\geq 0 + \left\langle p - \Theta_n^k u_n, A_k \Theta_n^k u_n - A_k \Theta_n^{k-1} u_n\right\rangle + \left\langle p - \Theta_n^k u_n, \frac{1}{\lambda_{k,n}}\left(\Theta_n^{k-1} u_n - \Theta_n^k u_n\right)\right\rangle,
\end{aligned} \tag{3.41}$$

for $k \in \{1, 2, \ldots, N\}$. From (3.17), it follows $\lim_{n\to\infty}\|\Theta_n^k u_n - \Theta_n^{k-1} u_n\| = 0$, especially, $\Theta_{n_i}^k u_{n_i} \rightharpoonup \omega$. Since A_k, $k = 1, \ldots, N$, are Lipschitz continuous operators, we have $\|A_k \Theta_n^{k-1} u_n - A_k \Theta_n^k u_n\| \to 0$. So, from (3.41), we have

$$\lim_{i\to\infty}\left\langle p - \Theta_{n_i}^k u_{n_i}, g\right\rangle = \langle p - \omega, g\rangle \geq 0. \tag{3.42}$$

Since $A_k + M_k$, $k \in \{1, 2, \ldots, N\}$ is maximal monotone, this implies that $0 \in (M_k + A_k)(\omega)$, $k \in \{1, 2, \ldots, N\}$, that is, $\omega \in \bigcap_{i=1}^N I(A_i, M_i)$. So, we obtain result.

Step 6. We show that

$$\limsup_{n\to\infty}\langle (B - \gamma f)z, z - x_n\rangle \leq 0, \tag{3.43}$$

where $z = P_\Omega(I - B + \gamma f)(z)$ is unique solution of the variational inequality (3.2).

To show this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\langle (B - \gamma f)z, z - x_{n_i}\rangle = \limsup_{n\to\infty}\langle (B - \gamma f)z, z - x_n\rangle. \tag{3.44}$$

By the proof of Step 5, we obtain that

$$\limsup_{n\to\infty}\langle (B - \gamma f)z, z - x_n\rangle = \lim_{i\to\infty}\langle (B - \gamma f)z, z - x_{n_i}\rangle = \langle (B - \gamma f)z, z - \omega\rangle \leq 0. \tag{3.45}$$

Step 7. We prove that $x_n \to \omega$.

By using Lemmas 2.3 and 2.4, we have

$$\begin{aligned}\|x_{n+1}-\omega\|^2 &= \left\|\left((1-\beta_n)I-\epsilon_n B\right)(S_n y_n-\omega)+\beta_n(x_n-\omega)+\epsilon_n(\gamma f(x_n)-B\omega)\right\|^2\\ &\le \left\|\left((1-\beta_n)I-\epsilon_n B\right)(S_n y_n-\omega)+\beta_n(x_n-\omega)\right\|^2+2\epsilon_n\langle \gamma f(x_n)-B\omega, x_{n+1}-\omega\rangle\\ &\le \left\|(1-\beta_n)I-\epsilon_n B)(S_n y_n-\omega)\right\|^2+\beta_n\|x_n-\omega\|^2\\ &\quad+2\epsilon_n\langle \gamma f(x_n)-f(\omega), x_{n+1}-\omega\rangle+2\epsilon_n\langle \gamma f(\omega)-B\omega, x_{n+1}-\omega\rangle\\ &\le \left((1-\beta_n)-\epsilon_n\overline{\gamma}\right)\left\|S_n y_n-\omega\right\|^2+\beta_n\|x_n-\omega\|^2\\ &\quad+2\epsilon_n\gamma\alpha\|x_n-\omega\|\|x_{n+1}-\omega\|+2\epsilon_n\langle \gamma f(\omega)-B\omega, x_{n+1}-\omega\rangle\\ &\le \left((1-\beta_n)-\epsilon_n\overline{\gamma}\right)\|x_n-\omega\|^2+\beta_n\|x_n-\omega\|^2\\ &\quad+\epsilon_n\gamma\alpha\left(\|x_n-\omega\|^2+\|x_{n+1}-\omega\|^2\right)+2\epsilon_n\langle \gamma f(\omega)-B\omega, x_{n+1}-\omega\rangle\\ &\le \left(1-\epsilon_n(\overline{\gamma}-\gamma\alpha)\right)\|x_n-\omega\|^2+\epsilon_n\gamma\alpha\|x_{n+1}-\omega\|^2+2\epsilon_n\langle \gamma f(\omega)-B\omega, x_{n+1}-\omega\rangle.\end{aligned} \tag{3.46}$$

It follows that

$$\|x_{n+1}-\omega\|^2 \le \left(1-\frac{(\overline{\gamma}-\alpha\gamma)\epsilon_n}{1-\alpha\gamma\epsilon_n}\right)\|x_n-\omega\|^2+\frac{2\epsilon_n}{1-\alpha\gamma\epsilon_n}\langle \gamma f(\omega)-B\omega, x_{n+1}-\omega\rangle. \tag{3.47}$$

Now, from conditions (C1), (C2), and (C6), Step 6 and Lemma 2.8, we obtain $\lim_{n\to\infty}\|x_n-\omega\|=0$. Namely, $x_n \to \omega$ in norm.

□

Corollary 3.2. *Let H be a real Hilbert space, let F be a bifunction $H\times H \to \mathbb{R}$ satisfying (A1)–(A4), and let $\{S_n\}$ be a sequence of nonexpansive mappings on H. Let $A_i : H \to H$, $i=1,2,\dots,N$ be α_i- inverse-strongly monotone mappings and $M_i : H \to 2^H$, $i=1,2,\dots,N$, maximal monotone mappings such that $\Omega := (\bigcap_{n=1}^{\infty} F_{ix}(S_n))\cap EP(F)\cap(\bigcap_{i=1}^{N} I(A_i,M_i))\neq\emptyset$. Let f be a contraction of H into itself with a constant $\alpha\in(0,1)$, and let B be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma}>0$ and $0<\gamma<\overline{\gamma}/\alpha$. Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be sequences generated by $x_1\in H$ and*

$$\begin{gathered} F(u_n,y)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y\in H,\\ y_n = J_{M_N,\lambda_{N,n}}(I-\lambda_{N,n}A_N)\cdots J_{M_1,\lambda_{1,n}}(I-\lambda_{1,n}A_1)u_n,\\ x_{n+1}=\epsilon_n\gamma f(x_n)+\beta_n x_n+\left((1-\beta_n)I-\epsilon_n B\right)S_n y_n, \end{gathered} \tag{3.48}$$

for all $n\in\mathbb{N}$, where $\beta_n\in(0,1)\lambda_{i,n}\in(0,2\alpha_i]$, $i\in\{1,2,\dots,N\}$, satisfy (H1)-(H2) and $\{\epsilon_n\}\subset[0,1]$, and $\{r_n\}\subset(0,\infty)$ satisfy

(C1) $\lim_{n\to\infty}\epsilon_n=0$;

(C2) $\sum_{n=1}^{\infty}\epsilon_n=\infty$;

(C3) $\sum_{n=1}^{\infty}|\epsilon_{n+1}-\epsilon_n|<\infty$;

(C4) $\liminf_{n\to\infty} r_n > 0$;

(C5) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(C6) $\lim_{n\to\infty} \beta_n = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ *for any bounded subset* K *of* H*. Let* S *be a mapping of* H *into itself defined by* $Sx = \lim_{n\to\infty} S_n x$*, for all* $x \in H$*, and suppose that* $F_{ix}(S) = \bigcap_{n=1}^{\infty} F_{ix}(S_n)$*. Then,* $\{x_n\}$*,* $\{y_n\}$*, and* $\{u_n\}$ *converge strongly to* z*, where* $z = P_\Omega(I - B + \gamma f)(z)$ *is a unique solution of the variational inequality*

$$\langle (B - \gamma f)z, z - x \rangle \le 0 \quad x \in \Omega. \tag{3.49}$$

Acknowledgment

This work is supported in part by China Postdoctoral Science Foundation (Grant no. 20100470783).

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