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Research Article

Positive Solutions of a Fractional Boundary Value Problem with Changing Sign Nonlinearity

Yongqing Wang,[1] Lishan Liu,[1, 2] and Yonghong Wu[2]

[1] *School of Mathematical Sciences, Qufu Normal University, Shandong, Qufu 273165, China*
[2] *Department of Mathematics and Statistics, Curtin University of Technology, Perth, WA 6845, Australia*

Correspondence should be addressed to Lishan Liu, lls@mail.qfnu.edu.cn

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We discuss the existence of positive solutions of a boundary value problem of nonlinear fractional differential equation with changing sign nonlinearity. We first derive some properties of the associated Green function and then obtain some results on the existence of positive solutions by means of the Krasnoselskii's fixed point theorem in a cone.

1. Introduction

Recently, much attention has been paid to the existence of solutions for fractional differential equations due to its wide range of applications in engineering, economics, and many other fields, and for more details see, for instance, [1–17] and the references therein. In most of the works in literature, the nonlinearity needs to be nonnegative to get positive solutions [10–17]. In particular, by using the Krasnosel'skii fixed-point theorem and the Leray-Schauder nonlinear alternative, Bai and Qiu [14] consider the positive solution for the following boundary value problem:

$$\begin{gathered} {}^cD^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{gathered} \tag{P}$$

where $2 < \alpha \leq 3$ is a real number, ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative, $f : (0, 1] \times [0, \infty) \to [0, \infty)$ is continuous and singular at $t = 0$.

To the best of our knowledge, there are only very few papers dealing with the exis-tence of positive solutions of semipositone fractional boundary value problems due to the difficulties in finding and analyzing the corresponding Green function. The purpose of this paper is to establish the existence of positive solutions to the following nonlinear fractional differential equation boundary value problem:

$$
\begin{gathered}
{}^cD_{0+}^{\alpha}u(t) + \lambda f(t,u(t)) = 0, \quad 0 < t < 1, \\
u(0) = u'(1) = u''(0) = 0,
\end{gathered} \tag{1.1}
$$

where $2 < \alpha \leq 3$ is a real number, ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative, λ is a positive parameter, and f may change sign and may be singular at $t = 0, 1$. In this paper, by a positive solution to (1.1), we mean a function $u \in C[0,1]$, which is positive on $(0,1]$ and satisfies (1.1).

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. We also develop some properties of the associated Green function. In Section 3, we discuss the existence of positive solutions of the semipositone BVP (1.1). In Section 4, we give two examples to illustrate the application of our main results.

2. Basic Definitions and Preliminaries

In this section, we present some preliminaries and lemmas that are useful to the proof of our main results. For the convenience of the reader, we also present here some necessary definitions from fractional calculus theory. These definitions can be found in the recent literature.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$
I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)ds \tag{2.1}
$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2. The Caputo's fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$
{}^cD_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds, \tag{2.2}
$$

where $n - 1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.3 (see [14]). *Given $y(t) \in C(0,1) \cap L(0,1)$, the unique solution of the problem*

$$
\begin{gathered}
{}^cD_{0+}^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1, \\
u(0) = u'(1) = u''(0) = 0
\end{gathered} \tag{2.3}
$$

is

$$u(t)=\int_0^1 G(t,s)y(s)ds, \tag{2.4}$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(\alpha-1)t(1-s)^{\alpha-2}, & 0\le t\le s\le 1,\\ (\alpha-1)t(1-s)^{\alpha-2}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1.\end{cases} \tag{2.5}$$

Lemma 2.4. *The function $G(t,s)$ has the following properties:*

(1) $G(t,s)\le (1/\Gamma(\alpha-1))\ t(1-s)^{\alpha-2}$, *for* $t,s\in[0,1]$,

(2) $G(t,s)\le (1/\Gamma(\alpha-1))(\alpha-2+s)(1-s)^{\alpha-2}$, *for* $t,s\in[0,1]$,

(3) $G(t,s)\ge (1/\Gamma(\alpha))(\alpha-2+s)t(1-s)^{\alpha-2}$, *for* $t,s\in[0,1]$.

Proof. It is obvious that (1) holds. In the following, we will prove (2) and (3).

(i) When $0\le s\le t\le 1$, as $2<\alpha\le 3$, we have

$$\frac{\partial G(t,s)}{\partial t}=\frac{(1-s)^{\alpha-2}-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\ge 0, \tag{2.6}$$

therefore

$$G(t,s)\le G(1,s)=\frac{(\alpha-2+s)}{\Gamma(\alpha)}(1-s)^{\alpha-2}\le\frac{1}{\Gamma(\alpha-1)}(\alpha-2+s)(1-s)^{\alpha-2}. \tag{2.7}$$

On the other hand, since $0<\alpha-2\le 1$, we have

$$\begin{aligned}G(t,s)&=\frac{(\alpha-1)t(1-s)^{\alpha-2}-(t-s)(t-s)^{\alpha-2}}{\Gamma(\alpha)}\\&\ge\frac{(\alpha-1)t(1-s)^{\alpha-2}-(t-s)(1-s)^{\alpha-2}}{\Gamma(\alpha)}\\&=\frac{(\alpha-2)t+s}{\Gamma(\alpha)}(1-s)^{\alpha-2}\ge\frac{(\alpha-2)t+st}{\Gamma(\alpha)}(1-s)^{\alpha-2}\\&=\frac{1}{\Gamma(\alpha)}(\alpha-2+s)t(1-s)^{\alpha-2}.\end{aligned} \tag{2.8}$$

(ii) When $0\le t\le s\le 1$, we have

$$G(t,s)=\frac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}\le\frac{(\alpha-1)s(1-s)^{\alpha-2}}{\Gamma(\alpha)}\le\frac{1}{\Gamma(\alpha-1)}(\alpha-2+s)(1-s)^{\alpha-2}. \tag{2.9}$$

On the other hand, as $\alpha - 1 \geq \alpha - 2 + s$ for $0 \leq s \leq 1$, we have

$$G(t,s) = \frac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)} \geq \frac{1}{\Gamma(\alpha)}(\alpha-2+s)t(1-s)^{\alpha-2}. \tag{2.10}$$

The proof is completed. □

Remark 2.5. By Lemma 2.4, there exists $K > 0$ such that the positive solution u in [14] satisfies

$$u(t) \geq \frac{t}{\alpha-1}\|u\|, \qquad u(t) \leq Kt, \tag{2.11}$$

where $\|u\| = \max_{0\leq t\leq 1}|u(t)|$.

Proof. In [14], the positive solution of (P) is equivalent to the fixed point of A in Q, where $Q = \{u(t) \in C[0,1] : u(t) \geq 0\}$ and

$$Au(t) = \int_0^1 G(t,s)f(s,u(s))ds. \tag{2.12}$$

For any $u \in Q$, by (1) of Lemma 2.4, we have

$$Au(t) \leq \frac{t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}f(s,u(s))ds. \tag{2.13}$$

On the other hand, by (2), (3) of Lemma 2.4, we get

$$\begin{aligned} Au(t) &\geq \frac{t}{\Gamma(\alpha)}\int_0^1 (\alpha-2+s)(1-s)^{\alpha-2}f(s,u(s))ds, \\ Au(t) &\leq \frac{1}{\Gamma(\alpha-1)}\int_0^1 (\alpha-2+s)(1-s)^{\alpha-2}f(s,u(s))ds, \end{aligned} \tag{2.14}$$

which implies $Au(t) \geq (t/(\alpha-1))\|Au(t)\|$.

If u is a positive solution of (P), then u is a fixed point of A in Q, therefore

$$u(t) \geq \frac{t}{\alpha-1}\|u\|, \qquad u(t) \leq Kt, \tag{2.15}$$

where $K = (1/(\Gamma(\alpha-1)))\int_0^1 (1-s)^{\alpha-2}f(s,u(s))ds$. The proof is completed. □

For the convenience of presentation, we list here the hypotheses to be used later.

(H_1) $f \in C((0,1) \times [0,+\infty), (-\infty,+\infty))$ and satisfies

$$-r(t) \leq f(t,x) \leq z(t)g(x), \tag{2.16}$$

where $r, z \in C((0,1),[0,+\infty))$, $g \in C([0,+\infty),[0,+\infty))$.

(H_2) $0 < \int_0^1 r(s)ds < +\infty$, $0 < \int_0^1 (\alpha-2+s)(1-s)^{\alpha-2}(z(s)+r(s))ds < +\infty$.

(H_3) There exists $[a,b] \subset (0,1)$ such that

$$\liminf_{u\to+\infty} \min_{t\in[a,b]} \frac{f(t,u)}{u} = +\infty. \tag{2.17}$$

(H_4) There exists $[c,d] \subset (0,1)$ such that

$$\begin{gathered} \liminf_{u\to+\infty} \min_{t\in[c,d]} f(t,u) > \frac{2(\alpha-1)^2 \int_0^1 (1-s)^{\alpha-2} r(s)ds}{\int_c^d (\alpha-2+s)(1-s)^{\alpha-2}ds}, \\ \lim_{u\to+\infty} \frac{g(u)}{u} = 0. \end{gathered} \tag{2.18}$$

Lemma 2.6. *Assume that (H_1) and (H_2) hold, then the boundary value problem*

$$\begin{gathered} {}^cD_{0+}^{\alpha} u(t) + r(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{gathered} \tag{2.19}$$

has a unique solution $\omega(t) = \int_0^1 G(t,s)r(s)ds$ with

$$\omega(t) \le t \int_0^1 \frac{1}{\Gamma(\alpha-1)} (1-s)^{\alpha-2} r(s)ds, \quad t \in [0,1]. \tag{2.20}$$

Proof. By Lemma 2.3, we have that $\omega(t) = \int_0^1 G(t,s)r(s)ds$ is the unique solution of (2.19). By (1) of Lemma 2.4, we have

$$\omega(t) = \int_0^1 G(t,s)r(s)ds \le t \int_0^1 \frac{1}{\Gamma(\alpha-1)} (1-s)^{\alpha-2} r(s)ds. \tag{2.21}$$

The proof is completed. □

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Define a cone P by

$$P = \left\{ u(t) \in E : u(t) \ge \frac{t}{\alpha-1}\|u\| \right\}. \tag{2.22}$$

Set $B_r = \{u(t) \in E : \|u\| < r\}$, $P_r = P \cap B_r$, $\partial P_r = P \cap \partial B_r$.

Next we consider the following boundary value problem:

$$
\begin{gathered}
{}^{c}D_{0+}^{\alpha}u(t)+\lambda\left[f\left(t,[u(t)-\lambda\omega(t)]^{+}\right)+r(t)\right]=0,\quad 0<t<1,\\
u(0)=u'(1)=u''(0)=0,
\end{gathered}
\tag{2.23}
$$

where $\lambda > 0$, $\omega(t)$ is defined in Lemma 2.6, $[u(t)-\lambda\omega(t)]^{+}=\max\{u(t)-\lambda\omega(t),0\}$.

Let

$$
Tu(t)=\lambda\int_{0}^{1}G(t,s)\left[f\left(s,[u(s)-\lambda\omega(s)]^{+}\right)+r(s)\right]ds. \tag{2.24}
$$

It is easy to check that u is a solution of (2.23) if and only if u is a fixed point of T.

Lemma 2.7. *$T : P \to P$ is a completely continuous operator.*

Proof. For any $u \in P$, Lemma 2.4 implies that

$$
Tu(t)\ge\frac{\lambda t}{\Gamma(\alpha)}\int_{0}^{1}(\alpha-2+s)(1-s)^{\alpha-2}\left[f\left(s,[u(s)-\lambda\omega(s)]^{+}\right)+r(s)\right]ds. \tag{2.25}
$$

On the other hand

$$
Tu(t)\le\frac{\lambda}{\Gamma(\alpha-1)}\int_{0}^{1}(\alpha-2+s)(1-s)^{\alpha-2}\left[f\left(s,[u(s)-\lambda\omega(s)]^{+}\right)+r(s)\right]ds. \tag{2.26}
$$

Then $Tu(t)\ge(t/(\alpha-1))\|Tu(t)\|$, which implies $T : P \to P$.

According to the Ascoli-Arzela theorem, we can easily get that $T : P \to P$ is a completely continuous operator. The proof is completed. □

Lemma 2.8 (see [18]). *Let E be a real Banach space, and let $P \subset E$ be a cone. Assume that Ω_1 and Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(1) $\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|, u \in P \cap \partial\Omega_2$, *or*

(2) $\|Tu\| \ge \|u\|, u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence of Positive Solutions

Theorem 3.1. *Suppose that (H_1)–(H_3) hold. Then there exists $\lambda^* > 0$ such that the boundary value problem (1.1) has at least one positive solution for any $\lambda \in (0, \lambda^*)$.*

Proof. Choose $r_1 > ((\alpha-1)/\Gamma(\alpha-1))\int_0^1 (1-s)^{\alpha-2} r(s)ds$. Let

$$\lambda^* = \min\left\{1, \frac{r_1\Gamma(\alpha-1)}{\left[g^*(r_1)+1\right]\int_0^1(\alpha-2+s)(1-s)^{\alpha-2}(z(s)+r(s))ds}\right\}, \tag{3.1}$$

where

$$g^*(r) = \max_{x\in[0,r]} g(x). \tag{3.2}$$

In the rest of the proof, we suppose $\lambda \in (0, \lambda^*)$.

For any $u \in \partial P_{r_1}$, noting that

$$u(t) \geq \frac{t}{\alpha-1} r_1, \quad t \in [0,1] \tag{3.3}$$

and using (2.20), we have

$$0 \leq t\left[\frac{r_1}{\alpha-1} - \int_0^1 \frac{1}{\Gamma(\alpha-1)}(1-s)^{\alpha-2} r(s)ds\right] \leq u(t) - \lambda\omega(t) \leq r_1. \tag{3.4}$$

Therefore,

$$\begin{aligned}
Tu(t) &= \lambda\int_0^1 G(t,s)\left[f\left(s,[u(s)-\lambda\omega(s)]^+\right)+r(s)\right]ds \\
&\leq \frac{\lambda}{\Gamma(\alpha-1)}\int_0^1(\alpha-2+s)(1-s)^{\alpha-2}\left[z(s)g\left([u(s)-\lambda\omega(s)]^+\right)+r(s)\right]ds \\
&\leq \frac{\lambda}{\Gamma(\alpha-1)}\left[g^*(r_1)+1\right]\int_0^1(\alpha-2+s)(1-s)^{\alpha-2}[z(s)+r(s)]ds \\
&< \frac{\lambda^*}{\Gamma(\alpha-1)}\left[g^*(r_1)+1\right]\int_0^1(\alpha-2+s)(1-s)^{\alpha-2}[z(s)+r(s)]ds \leq r_1.
\end{aligned} \tag{3.5}$$

Thus,

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial P_{r_1}. \tag{3.6}$$

Now choose a real number

$$L > \frac{4\Gamma(\alpha)}{\lambda a\int_a^b(\alpha-2+s)(1-s)^{\alpha-2}ds}. \tag{3.7}$$

By (H_3), there exists a constant $N > 0$ such that for any $t \in [a,b]$, $x \geq N$, we have

$$f(t,x) > Lx. \tag{3.8}$$

Select

$$r_2 > \max\left\{ r_1, \frac{2(\alpha-1)}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} r(s) ds, \frac{4N}{a} \right\}. \tag{3.9}$$

Then for any $u \in \partial P_{r_2}$, we have $u(t) - \lambda\omega(t) \geq 0$, $t \in [0,1]$. Moreover, by the selection of r_2 we have

$$\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} r(s) ds < \frac{r_2}{2(\alpha-1)}. \tag{3.10}$$

Thus for any $t \in [a,b]$, as $1 < \alpha - 1 \leq 2$, we get

$$u(t) - \lambda\omega(t) \geq t\left[\frac{r_2}{\alpha-1} - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} r(s) ds \right] > \frac{ar_2}{2(\alpha-1)} \geq \frac{ar_2}{4}. \tag{3.11}$$

Noting that $r_2 > 4N/a$, we have

$$u(t) - \lambda\omega(t) > \frac{ar_2}{4} > N, \quad t \in [a,b]. \tag{3.12}$$

Hence we get

$$\begin{aligned} Tu(1) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-2+s)(1-s)^{\alpha-2} \left[f\left(s, [u(s) - \lambda\omega(s)]^+\right) + r(s) \right] ds \\ &\geq \frac{\lambda}{\Gamma(\alpha)} \int_a^b (\alpha-2+s)(1-s)^{\alpha-2} f\left(s, [u(s) - \lambda\omega(s)]^+\right) ds \\ &\geq \frac{\lambda L}{\Gamma(\alpha)} \int_a^b (\alpha-2+s)(1-s)^{\alpha-2} [u(s) - \lambda\omega(s)] ds \\ &\geq \frac{\lambda L a r_2}{4\Gamma(\alpha)} \int_a^b (\alpha-2+s)(1-s)^{\alpha-2} ds > r_2. \end{aligned} \tag{3.13}$$

Thus,

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial P_{r_2}. \tag{3.14}$$

By Lemma 2.8, T has a fixed point u such that $r_1 \leq \|u\| \leq r_2$. Since $\|u\| \geq r_1$, by (3.4) we have $u(t) - \lambda\omega(t) > 0, t \in (0,1]$. Let $\overline{u}(t) = u(t) - \lambda\omega(t)$. As $\omega(t)$ is the solution of (2.19) and $u(t)$ is the solution of (2.23), $\overline{u}(t)$ is a positive solution of the singular semipositone boundary value problem (1.1). The proof is completed. □

Theorem 3.2. *Suppose that* (H_1), (H_2), *and* (H_4) *hold. Then there exists* $\lambda^* > 0$ *such that the boundary value problem* (1.1) *has at least one positive solution for any* $\lambda \in (\lambda^*, +\infty)$.

Proof. By the first limit of (H_4), we have that there exists $N > 0$ such that, for any $t \in [c, d]$ and $u \geq N$, we have

$$f(t, u) \geq \frac{2(\alpha-1)^2 \int_0^1 (1-s)^{\alpha-2} r(s) ds}{\int_c^d (\alpha-2+s)(1-s)^{\alpha-2} ds}. \tag{3.15}$$

Select

$$\lambda^* = \frac{N\Gamma(\alpha-1)}{c\int_0^1 (1-s)^{\alpha-2} r(s) ds}. \tag{3.16}$$

In the rest of the proof, we suppose $\lambda > \lambda^*$.

Let

$$R_1 = \frac{2\lambda(\alpha-1)\int_0^1 (1-s)^{\alpha-2} r(s) ds}{\Gamma(\alpha-1)}. \tag{3.17}$$

Then, for any $u \in \partial P_{R_1}$, we have

$$\begin{aligned} u(t) - \lambda\omega(t) &\geq t\left[\frac{R_1}{\alpha-1} - \lambda\int_0^1 \frac{1}{\Gamma(\alpha-1)}(1-s)^{\alpha-2} r(s) ds\right] \\ &= \frac{\lambda t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} r(s) ds \\ &\geq \frac{\lambda^* t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} r(s) ds = \frac{Nt}{c}, \end{aligned} \tag{3.18}$$

and therefore $u(t) - \lambda\omega(t) \geq N$ on $t \in [c, d]$, $u \in \partial P_{R_1}$. Then,

$$\begin{aligned} Tu(t) &= \lambda\int_0^1 G(t,s)\left[f\left(s, [u(s) - \lambda\omega(s)]^+\right) + r(s)\right] ds \\ &\geq \lambda\int_c^d G(t,s) f\left(s, [u(s) - \lambda\omega(s)]^+\right) ds \\ &\geq \frac{2\lambda(\alpha-1)^2 \int_0^1 (1-s)^{\alpha-2} r(s) ds}{\int_c^d (\alpha-2+s)(1-s)^{\alpha-2} ds}\int_c^d G(t,s) ds \\ &\geq \frac{2t\lambda(\alpha-1)^2 \int_0^1 (1-s)^{\alpha-2} r(s) ds}{\int_c^d (\alpha-2+s)(1-s)^{\alpha-2} ds}\int_c^d \frac{(\alpha-2+s)}{\Gamma(\alpha)}(1-s)^{\alpha-2} ds \\ &= \frac{2\lambda t(\alpha-1)}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} r(s) ds = tR_1, \end{aligned} \tag{3.19}$$

which implies

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial P_{R_1}. \tag{3.20}$$

On the other hand, as $g(t)$ is continuous on $[0,+\infty)$, from the second limit of (H_4), we have

$$\lim_{u\to+\infty}\frac{g^*(u)}{u}=0, \tag{3.21}$$

where $g^*(u)$ is defined by (3.2). In fact, by $\lim_{u\to+\infty}g(u)/u=0$, for any $\epsilon>0$, there exists $N_1>0$ such that for any $u>N_1$ we have $0\le g(u)<\epsilon u$. Let $N=\max\{N_1,g^*(N_1)/\epsilon\}$, for any $u>N$ we have $0\le g^*(u)<\epsilon u+g^*(N_1)<2\epsilon u$. Therefore, $\lim_{u\to+\infty}g^*(u)/u=0$. For

$$\epsilon=\Gamma(\alpha-1)\left[2\lambda\int_0^1(\alpha-2+s)(1-s)^{\alpha-2}z(s)ds\right]^{-1}, \tag{3.22}$$

there exists $X_0>0$ such that when $x\ge X_0$, for any $0\le u\le x$, we have

$$g(u)\le g^*(x)\le\varepsilon x. \tag{3.23}$$

Select

$$R_2\ge\max\left\{X_0,R_1,\frac{2\lambda}{\Gamma(\alpha-1)}\int_0^1(\alpha-2+s)(1-s)^{\alpha-2}r(s)ds\right\}. \tag{3.24}$$

Then, for any $u\in\partial P_{R_2}$, we get

$$\begin{aligned}\|Tu\|&\le\frac{\lambda}{\Gamma(\alpha-1)}\int_0^1(\alpha-2+s)(1-s)^{\alpha-2}\left[z(s)g\left([u(s)-\lambda\omega(s)]^+\right)+r(s)\right]ds\\&\le\frac{\lambda\varepsilon R_2}{\Gamma(\alpha-1)}\int_0^1(\alpha-2+s)(1-s)^{\alpha-2}z(s)ds\\&\quad+\frac{\lambda}{\Gamma(\alpha-1)}\int_0^1(\alpha-2+s)(1-s)^{\alpha-2}r(s)ds\le\frac{R_2}{2}+\frac{R_2}{2}=R_2.\end{aligned} \tag{3.25}$$

Thus,

$$\|Tu\|\le\|u\|,\quad\forall u\in\partial P_{R_2}. \tag{3.26}$$

By Lemma 2.8, T has a fixed point u such that $R_1\le\|u\|\le R_2$. Since $R_1\le\|u\|$, by (3.18), we have $u(t)-\lambda\omega(t)>0, t\in(0,1]$. Let $\overline{u}(t)=u(t)-\lambda\omega(t)$. As $\omega(t)$ is a solution of (2.19) and $u(t)$ is a solution of (2.23), $\overline{u}(t)$ is a positive solution of the singular semipositone boundary value problem (1.1). The proof is completed. □

Corollary 3.3. *The conclusion of Theorem 3.2 is valid if* (H_4) *is replaced by* (H_4^*)*: there exists* $[c,d] \subset (0,1)$ *such that*

$$\liminf_{u\to+\infty} \min_{t\in[c,d]} f(t,u) = +\infty;$$
$$\lim_{u\to+\infty} \frac{g(u)}{u} = 0. \tag{3.27}$$

4. Examples

Example 4.1. Consider the following problem

$$\begin{gathered} {}^cD_{0+}^{5/2}u(t) + \lambda f(t,u) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{gathered} \tag{4.1}$$

where $f(t,u) = u^2/\sqrt{t(1-t)} + \ln t$. Let $z(t) = 1/\sqrt{t(1-t)}$, $r(t) = -\ln t$, $g(u) = u^2$. By direct calculation, we have $\int_0^1 r(t)dt = 1$, $\int_0^1 z(t)dt = \pi$, and

$$\liminf_{u\to+\infty} \min_{t\in[1/4,3/4]} \frac{f(t,u)}{u} = +\infty. \tag{4.2}$$

So all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, BVP (4.1) has at least one positive solution provided λ is sufficiently small.

Example 4.2. Consider the following problem

$$\begin{gathered} {}^cD_{0+}^{9/4}u(t) + \lambda f(t,u) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{gathered} \tag{4.3}$$

where $f(t,u) = \ln(1+u)/\sqrt{t(1-t)} + \ln t$. Let $z(t) = 1/\sqrt{t(1-t)}$, $r(t) = -\ln t$, $g(u) = \ln(1+u)$. By direct calculation, we have $\int_0^1 r(t)dt = 1$, $\int_0^1 z(t)dt = \pi$, and

$$\liminf_{u\to+\infty} \min_{t\in[1/4,3/4]} f(t,u) = +\infty;$$
$$\lim_{u\to+\infty} \frac{g(u)}{u} = 0. \tag{4.4}$$

So all conditions of Theorem 3.2 are satisfied. By Theorem 3.2, BVP (4.3) has at least one positive solution provided λ is sufficiently large.

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