

Research Article

Existence and Attractiveness of Order One Periodic Solution of a Holling I Predator-Prey Model

Huidong Cheng, Tongqian Zhang, and Fang Wang

College of Science, Shandong University of Science and Technology, Qingdao 266510, China

Correspondence should be addressed to Huidong Cheng, chd900517@sdust.edu.cn

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According to the integrated pest management strategies, a Holling type I functional response predator-prey system concerning state-dependent impulsive control is investigated. By using differential equation geometry theory and the method of successor functions, we prove the existence of order one periodic solution, and the attractivity of the order one periodic solution by sequence convergence rules and qualitative analysis. Numerical simulations are carried out to illustrate the feasibility of our main results which show that our method used in this paper is more efficient than the existing ones for proving the existence and attractiveness of order one periodic solution.

1. Introduction

It is one of the significant problems in the world today to prevent plant pests and pesticide pollution and to protect ecological balance for the sustainable development of agriculture and forestry, which is also an important research topic demanding prompt solution. In recent decades, plant pests worldwide are increasingly serious with the damage of the world's natural ecosystems. In agricultural production, pesticides spaying (chemical control) and release of natural enemies (biological control) are the ways commonly used for pest control. But if we implement chemical control as soon as pests appear, many problems are caused: the first is environmental pollution; the second is increase of costs including human and material resources and time; the third is killing natural enemies, such as parasitic wasp; the last is pests' resistance to pesticides, which brings great negative effects [1–3]. The second way, which controls pests with the help of the increasing natural enemies, can avoid problems caused by chemical control and gets more and more attention. So many scholars have been studying and discussing it [4–6].

Considering the effectiveness of the chemical control and nonpollution and limitations of the biological one, people have proposed the method of integrated pest management (IPM), which is a pest management system integrating all appropriate ways and technologies to control economic injury level (EIL) caused by pest populations in view of population dynamics and its relevant environment. In the process of practical application, people usually implement the following two schemes for the integrated pest management: one is to implement control at a fixed time to eradicate pests [7, 8]; the other is to implement measures only when the amount of pests reaches a critical level, which is to make the amount less than certain economic impairment level, not to wipe out pests [9–11].

In consideration of predator-prey capacity, Holling [12] proposes three different predations with functional response based on experiments; the average predator-prey system with Holling response is as follows:

$$\begin{aligned}x'(t) &= xg(x) - y\phi(x), \\y'(t) &= -dy + ey\phi(x),\end{aligned}\tag{1.1}$$

where x represents the densities for prey's density while y is the predator's; $g(x)$ is the unit rate of prey density in lack of predators. $\phi(x)$ is the Holling functional response. Holling type-I functional response is

$$\phi(x) = \begin{cases} cx, & x \leq x_0, \\ cx_0, & x > x_0, \end{cases}\tag{1.2}$$

where C is a constant; when the amount of prey is greater than certain threshold value x_0 , predatory rate is a constant.

Holling type-II and type-III functional responses are as follows, respectively:

$$\phi(x) = \frac{\alpha x}{1 + \omega x}, \quad \phi(x) = \frac{\alpha x^2}{\beta + x^2}.\tag{1.3}$$

Refer to [12] for details.

As the Lotka-Volterra predator-prey system with Holling functional response is more practical, many authors have studied about it [13–15]. The researches mostly focus on Lotka-Volterra predator-prey model with Holling type-II or Holling type-III functional response in contrast to the model with Holling type-I. This paper sets up and adopts new mathematic method to study comprehensive control (releasing natural enemies and spraying pesticide)

adopted when the amount of predators (pests) reaches the economic test (ET), thus the following pest-control model with Holling type-I functional response is set up:

$$\begin{aligned}
 x'(t) &= rx(t) - cx(t)y(t), & x \leq x_0, & x < h, \\
 y'(t) &= -dy(t) + ecx(t)y(t), & x \leq x_0, & x < h, \\
 x'(t) &= rx(t) - cx_0y(t), & x > x_0, & x < h, \\
 y'(t) &= -dy(t) + ecx_0y(t), & x > x_0, & x < h, \\
 \Delta x(t) &= -\alpha x(t), & x &= h, \\
 \Delta y(t) &= -\beta y(t) + q, & x &= h,
 \end{aligned} \tag{1.4}$$

where r, c, d, e are all positive constants, $x(t)$ and $y(t)$ represent the densities of prey (pest) and predator (natural enemy) at time t , respectively; r is the intrinsic growth rate of the prey; d denotes the death rate of the predator; h is the economic threshold ET; $\alpha, \beta \in (0, 1)$ represent the proportion of killed prey and predator by spraying pesticides, respectively, $q > 0$ is the number of natural enemies released at this time. When the amount of prey reaches the threshold h at time t_h , controlling measures are taken and the amount of prey and predator abruptly turns to $(1 - \alpha)h$ and $(1 - \beta)y(t_h) + q$, respectively. $\Delta x = x(t^+) - x(t)$, $\Delta y = y(t^+) - y(t)$, $x(t^+) = \lim_{w \rightarrow 0^+} x(t + w)$, $y(t^+) = \lim_{w \rightarrow 0^+} y(t + w)$.

This paper is organized as follows. In the next section, we present some basic definitions and important lemmas as preliminaries. In Section 3, we prove existence for an order one periodic solution of system (1.4). The sufficient conditions for the attractiveness of order one periodic solutions of system (1.4) are obtained in Section 4. At last, we state conclusion and the main results are carried out to illustrate the feasibility by numerical simulations.

2. Preliminaries

We first consider the model (1.4) without impulse effects

$$\begin{aligned}
 x'(t) &= rx(t) - cx(t)y(t), & x &\leq x_0, \\
 y'(t) &= -dy(t) + ecx(t)y(t), & x &\leq x_0, \\
 x'(t) &= rx(t) - cx_0y(t), & x &> x_0, \\
 y'(t) &= -dy(t) + ecx_0y(t), & x &> x_0.
 \end{aligned} \tag{2.1}$$

We consider the following function:

$$V(x, y) = \int_{x^*}^x \frac{-d + e\phi(s)}{\phi(s)} ds + \int_{y^*}^y \frac{s - y^*}{s} ds, \tag{2.2}$$

and we can easily know that $V(x, y)$ is positive definite in the first quartile and fits for all conditions of Liapunov function.

We can get that

$$V'(x, y) = \frac{exy^*}{\phi(x)} (\phi(x) - \phi(x^*)) \left(\frac{\phi(x^*)}{x^*} - \frac{\phi(x)}{x} \right). \quad (2.2)$$

It is easily proved that $V'(x, y) \equiv 0$ on condition that $x \leq x_0$, so all solutions of model (1.4) which form a set $\{(x, y) / V(x, y) \leq V(x_0, y^*)\}$ are closed trajectory $V(x, y) = C$, where $0 < C < V(x_0, y^*)$.

Since $V'(x, y) > 0$ on condition that $x > x_0$, so the trajectory of system (2.1) passes through closed curve $V(x, y) = C$ when it is out of the curve $V(x, y) = V(x_0, y^*)$.

Therefore, we observe the straight line:

$$L(x, y) = y + x - n, \quad n > 0, \quad x_0 < x \leq h. \quad (2.3)$$

The derivative of $L(x, y)$ along (2.1) is that

$$\begin{aligned} L'(x, y) /_{L=0} &= x' + y' = -dy + ecx_0y + rx - ecx_0 \\ &= -(dn - ecx_0n + cx_0n + cx_0h - dx_0) - (ecx_0 - r - cx_0) \\ &\leq dh - ecx_0^2 + rh + cx_0h - (d - ecx_0 + cx_0)n. \end{aligned} \quad (2.4)$$

We have $L' /_{L=0} < 0$ on condition that $n > (dh - ecx_0^2 + rh + cx_0h) / (d - ecx_0 + cx_0)$. Therefore, we can get the following lemma.

Lemma 2.1. *The system (2.1) possesses the following.*

- (I) *Two steady states $0(0, 0)$ -saddle point, and $R(d/ec, r/c) = R(x^*, y^*)$ -stable centre under the condition $x \leq x_0$ and $d \leq ecx_0$.*
- (II) *The trajectory of system (2.1) goes across the straight line $y + x - n = 0$ from the right to the left under condition $x_0 \leq x \leq h$ and $n > (dh - ecx_0^2 + rh + cx_0h) / (d - ecx_0 + cx_0)$, and intersects with the straight line $x = x_0$.*

Definition 2.2. A triple (X, Π, R^+) is said to be a semi-dynamical system if X is a metric space R^+ is the set of all non-negative real and $\Pi(P, t) : X \times R^+ \rightarrow X$ is a continuous map such that;

- (i) $\Pi(P, 0) = P$ for all $P \in X$;
- (ii) $\Pi(P, t)$ is continuous for t and s ;
- (iii) $\Pi(\Pi(P, t)) = \Pi(P, t + s)$ for all $P \in X$ and $t, s \in R^+$. Sometimes a semi-dynamical system (X, Π, R^+) is denoted by (X, Π) .

Definition 2.3. Assume that

- (i) (X, Π) is a semi-dynamical system;
- (ii) M is a nonempty subset of X ;
- (iii) function $I : M \rightarrow X$ is continuous, and, for any $P \in M$, there exists a $\varepsilon > 0$ such that for any $0 < |t| < \varepsilon$, $\Pi(P, t) \notin M$.

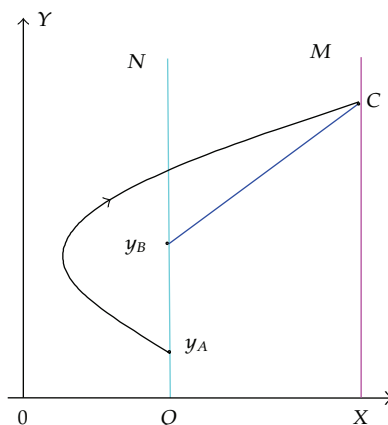


Figure 1

Then, (X, Π, M, I) is called an impulsive semi-dynamical system.

For any P the function $\Pi_P : R^+ \rightarrow X$ defined as $\Pi_P(t) = \Pi(P, t)$ is continuous and we call $\Pi_P(t)$ the trajectory passing through point P . The set $C^+(P) = \{\Pi(P, t) / 0 \leq t < +\infty\}$ is called positive semitrajectory of point P . The set $C^-(P) = \{\Pi(P, t) / -\infty < t \leq 0\}$ is called the negative semi-trajectory of point P .

Definition 2.4. We consider state-dependent impulsive differential equations:

$$\begin{aligned}
 x'(t) &= P(x, y), & (x, y) &\notin M(x, y), \\
 y'(t) &= Q(x, y), & (x, y) &\notin M(x, y), \\
 \Delta x(t) &= \alpha(x, y), & (x, y) &\in M(x, y), \\
 \Delta y(t) &= \beta(x, y), & (x, y) &\in M(x, y),
 \end{aligned} \tag{2.5}$$

where $M(x, y)$ and $N(x, y)$ represent the straight line or curve line on the plane and $M(x, y)$ is called impulsive set. The function I is continuous mapping, and $I(M) = N$, I is called the impulse function. $N(x, y)$ is called the phase set. We define “dynamic system” constituted by the definition of solution of state impulsive differential equation (2.5) as “semi-continuous dynamic system,” which is denoted as (Ω, f, I, M) .

Definition 2.5. Suppose that the impulse set M and the phase set N are both lines, as shown in Figure 1. Define the coordinate in the phase set N as follows: denote the point of intersection Q between N and x -axis as O , then the coordinate of any point in N is defined as the distance between A and Q and is denoted by y_A . Let C denote the point of intersection between the trajectory starting from A and the impulse set M , and let B denote the phase point of C after impulse with coordinate y_B . Then we define B as the successor point of A , and then the successor function of point A is that $f(A) = y_B - y_A$.

Definition 2.6. A trajectory $\tilde{\Pi}(P_0, t)$ is called order one periodic solution with period T if there exists a point $P_0 \in N$ and $T > 0$ such that $P = \Pi(P_0, t) \in M$ and $P^+ = I(P) = P_0$.

We get these lemmas from the continuity of composite function and the property of continuous function.

Lemma 2.7. *Successor function defined in Definition 2.2 is continuous.*

Lemma 2.8. *In system (1.4), if there exist $A \in N$, $B \in N$ satisfying successor function $f(A)f(B) < 0$, then there must exist a point $P(P \in N)$ satisfying $f(P) = 0$: the function between the point of A and the point of B , thus there is an order one periodic solution in system (1.4).*

3. Existence of the Order One Periodic Solution

In this section we shall investigate the existence of an order one periodic solution of system (1.4) by using the successor function defined in this paper. For this goal, we denote that

$$\begin{aligned} M_1 &= \{(x, y) \in R_2^+ \mid x = h, h < x_0, y \geq 0\}, \\ M_2 &= \{(x, y) \in R_2^+ \mid x = h, h > x_0, y \geq 0\}. \end{aligned} \quad (3.1)$$

Phase set N of set M is that

$$N = I(M) = \{(x, y) \in R_2^+ \mid x = (1 - \alpha)h, y \leq q\}. \quad (3.2)$$

Isoclinic line is denoted, respectively, by lines

$$\begin{aligned} L_1 &= \left\{ (x, y) \in R_2^+ \mid y = \frac{r}{c}, 0 \leq x \leq x_0 \right\}, \\ L_2 &= \left\{ (x, y) \in R_2^+ \mid x = \frac{d}{ec}, 0 \leq x \leq x_0, y \geq 0 \right\}, \\ L_3 &= \left\{ (x, y) \in R_2^+ \mid y = \frac{r}{cx_0}x, x \geq x_0, y \geq \frac{r}{c} \right\}. \end{aligned} \quad (3.3)$$

For the convenience, if $P \in \Omega - M$, $F(P)$ is defined as the first point of intersection of $C^+(P)$ and M , that is, there exists a $t_1 \in R_+$ such that $F(P) = \Pi(P, t_1) \in M$, and for $0 < t < t_1$, $\Pi(P, t) \notin M$; if $B \in N$, $R(B)$ is defined as the first point of intersection of $C^-(P)$ and N , that is there exists a $t_2 \in R_+$ such that $R(B) = \Pi(B, -t_2) \in N$, and for $-t < t < 0$, $\Pi(B, t) \notin N$.

For any point P , we denote y_P as its ordinate. If the point $P(h, y_P) \in M$, pulse shall occur at the point P , and the impulsive function transfers the point P into $P^+ \in N$. Without loss of generality, we assume that the initial point of the trajectory lies in phase set N unless otherwise specified.

Due to the practical significance, in this paper we assume the set N always lies in the left side of stable centre R , that is, $(1 - \alpha)h < d/ec$.

In the light of the different position of the set M_1 and the set M_2 , we consider the following three cases.

Case 1 ($0 < h < d/ec$). In this case, sets M_1 and N are both in the left side of stable center R . The line L_1 intersects with N at point $A((1 - \alpha)h, r/c)$, as shown in Figure 2. The trajectory

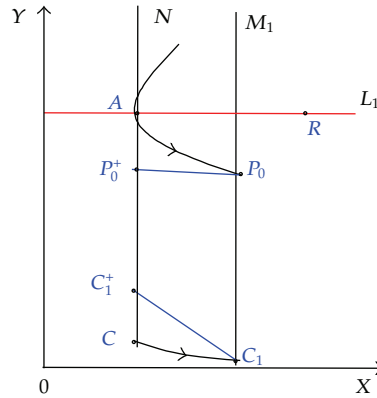


Figure 2

passing through point A which tangents to the set N at point A intersects with the set M_1 at point $P_0(h, y_{P_0})$. Since the point $P_0 \in M_1$, impulse occurs at point P_0 , supposing point P_0 is subject to impulsive effects to point $P_0^+((1-\alpha)h, y_{P_0^+})$, where $y_{P_0^+} = (1-\beta)y_{P_0} + q$, and the position of the point P_0^+ has the following three cases.

Subcase 1.1 $((1-\beta)y_{P_0} + q < r/c)$. In this case, the point P_0^+ lies below the point A , thus the successor function of the point A is $f(A) = (1-\beta)y_{P_0} + q - r/c < 0$.

Take another point $C((1-\alpha)h, \varepsilon) \in N$, where $\varepsilon > 0$ is small enough. Then there must exist a trajectory passing through the point C which intersects M_1 at point $C_1(h, y_{C_1}) \in M_1$. Supposing that the point C_1 is subject to impulsive effects to point $C_1^+((1-\alpha)h, y_{C_1^+})$, $y_{C_1^+} = (1-\beta)y_{C_1} + q > \varepsilon$, so we have $f(C) = y_{C_1^+} - \varepsilon > 0$.

By Lemma 2.8, there exists an order one periodic solution of system (1.4), whose initial point is between the point C and the point A in set N .

Subcase 1.2 $((1-\beta)y_{P_0} + q > r/c)$ (as shown in Figure 3). In this case, the point P_0^+ lies above the point A ; there must exist a trajectory passing through the point P_0^+ which intersects the set M_1 at a point $P_2(h, y_{P_2})$ under P_1 . Denote the coordinates of impulsive point $P_2^+((1-\alpha)h, y_{P_2^+})$ corresponding to the point $P_2(h, y_{P_2})$, then $y_{P_2^+} < y_{P_0^+}$. So we obtain $f(P_0^+) = y_{P_2^+} - y_{P_0^+} < 0$.

Take that another point $B_1((1-\alpha)h, \varepsilon + r/c) \in N$ above A , where $\varepsilon > 0$ is small enough. Then there must exist a trajectory passing through the point B_1 which intersects M_1 at a point $P_1(h, y_{P_1})$. In view of continuous dependence of the solution on initial value and time, we know $y_{P_1} < y_{P_0}$ and the point P_1 is close to P_0^+ enough, then we obtain $f(B_1) = y_{P_1^+} - y_{B_1} > 0$.

From Lemma 2.8, there exists an order one periodic solution of system (1.4), whose initial point is between B_1 and P_0^+ in set N .

Subcase 1.3 $((1-\beta)y_{P_0} + q = r/c)$. P_0^+ coincides with A , and the successor function of A is $f(A) = 0$, so there exists an order one periodic solution of system (1.4) which is just a part of the trajectory passing through the point A .

Now we can summarize the above results as the following theorem.

Theorem 3.1. Assuming that $d < ecx_0, 0 < h \leq d/ec < x_0$, there exists an order one periodic solutions of the system (1.4).

Case 2 $(d/ec < h \leq x_0)$. In this case, the set M_1 is in the right side of R and set N is in the left side of R . Let the line L_1 intersects the sets M_1 and N at points $A(h, r/c)$ and $B((1-\alpha)h, r/c)$,

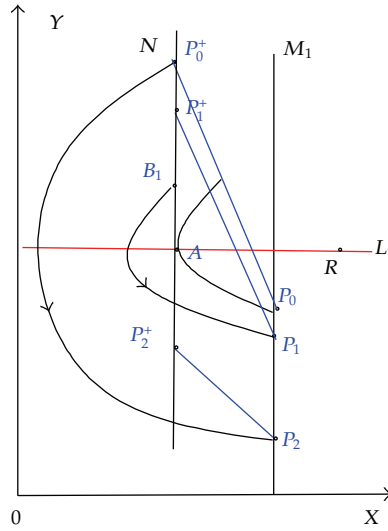


Figure 3

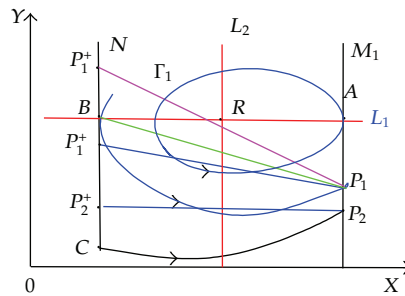


Figure 4

respectively (Figure 4), then there exists a unique closed trajectory Γ_1 of system (1.4) which is contained inside the point R and is tangent to the set M_1 at the point A .

Since Γ_1 is a closed trajectory, we take their minimal value of abscissas δ_{\min} at the trajectory Γ_1 , namely, $\delta_{\min} \leq x$ holds for any abscissas of Γ_1 .

In light of the different positions of the set N , we consider the following three subject cases.

Subcase 2.1 ($0 < (1 - \alpha)h < \delta_{\min} < d/ec$). In this case, there is a trajectory, which is contained inside the point R and is tangent to the set N at point B which intersects M_1 at the point $P_1((1 - \alpha)h, y_{P_1}) \in M_1$. Supposing that point P_1 is subject to impulsive effects to point $P_1^+((1 - \alpha)h, y_{P_1^+}) \in N$, here $y_{P_1^+} = (1 - \beta)y_{P_1} + q$. Like the analysis of Case 1 we can prove there exists an order one periodic solution in system (1.4) in this case.

Now we can summarize the above results as the following theorem.

Theorem 3.2. *If $d < ecx_0$, $0 < (1 - \alpha)h < \delta_{\min} < d/ec < h \leq x_0$, there exists an order one periodic solution in system (1.4).*

Subcase 2.2 ($\delta_{\min} < (1 - \alpha)h < d/ec$). In this case, let the closed trajectory Γ_1 of system (1.4) intersects the set N at two points $A_1((1 - \alpha)h, y_{A_1})$ and $A_2((1 - \alpha)h, y_{A_2})$, as shown in Figure 5. Since $A \in M_1$, impulse occurs at point A . Supposing that point A is subject to impulsive effects to point $P_0^+((1 - \alpha)h, y_{P_0^+})$, here $y_{P_0^+} = (1 - \beta)(r/c) + q$.

If $(1 - \beta)(r/c) + q = y_{A_1}$ or $(1 - \beta)(r/c) + q = y_{A_2}$, P_0^+ coincides with A_1 or P_0^+ coincides with A_2 ; the successor function of A_1 or A_2 is that $f(A_1) = 0$ or that $f(A_2) = 0$, so there exists an order one periodic solution of system (1.4) which is just a part of the trajectory Γ_1 .

If $(1 - \beta)(r/c) + q < y_{A_2}$, the point P_0^+ lies below the point A_2 , like the analysis of Subcase 1.1, and we can prove there exists an order one periodic solution to the system (1.4) in this case.

If $(1 - \beta)(r/c) + q > y_{A_1}$, the point P_0^+ is above the point A_1 . Like the analysis of Subcase 1.2, we can prove there exists an order one periodic solution to the system (1.4) in this case.

Now we can summarize the above results as the following theorem.

Theorem 3.3. *Assuming that $d \leq ecx_0$, $\delta_{\min} < (1 - \alpha)h < d/ec < h < x_0$, if $(1 - \beta)(r/c) + q \geq y_{A_2}$ or $(1 - \beta)(r/c) + q \geq y_{A_1}$, then there exists an order one periodic solution to the system (1.4).*

Subcase 2.3 ($y_{A_2} < (1 - \beta)(r/c) + q < y_{A_1}$). In this case, we note that the point P_0^+ must lie between the point A_1 and the point A_2 (as shown in Figure 6). Taking a point $E_1 \in M_1$ such that E_1 jumps to A_2 after the impulsive effect, we denote $A_2 = E_1^+$. Since $y_{P_0^+} > y_{E_1^+}$, we have $y_A > y_{E_1}$. Let $R(E_1) = E_2^+ \in N$, taking a point $E_2 \in M_1$ such that E_2 jumps to E_2^+ after the impulsive effects, then we have $y_{E_1^+} > y_{E_2^+}$, $y_{E_1} > y_{E_2}$. This process continues until there exists a $E_K^+ \in N$ ($K \in \mathbb{Z}_+$) satisfying $y_{E_K^+} < q$. So we obtain a sequence $\{E_k\}_{k=1,2,\dots,K}$ of set M_1 and a sequence $\{E_k^+\}_{k=1,2,\dots,K}$ of set N satisfying $R(E_{k-1}) = E_k^+ \in N$, $y_{E_{k-1}^+} > y_{E_k^+}$. In the following, we will prove the trajectory of system (1.4) with any initiating point of set N which will ultimately stay in Γ_1 .

From the vector field of system (1.4), we know the trajectory of system (1.4) with any initiating point between the points A_1 and A_2 will be free from impulsive effect and ultimately will stay in Γ_1 .

For any point below A_2 , it must lie between E_k^+ and E_{k-1}^+ , where $k = 2, 3, \dots, K + 1$ and $A_2 = E_1^+$. After k times' impulsive effects, the trajectory with this initiating point will arrive at some point of the set N which must be between A_1 and A_2 , and then ultimately stay in Γ_1 .

The trajectory with any initiating point above A_1 will intersect set N at some point below A_2 with time increasing, so just like the analysis above we obtain that it will ultimately stay in Γ_1 .

Now we can summarize the above results as the following theorem.

Theorem 3.4. *Assuming that $d \leq ecx_0$ and $\delta_{\min} < (1 - \alpha)h < d/ec < h \leq x_0$, if $y_{A_2} < (1 - \beta)(r/c) + q < y_{A_1}$, there is no periodic solution in system (1.4) and the trajectory with any initiating point of set N will stay in Γ_1 .*

Case 3 ($0 < (1 - \alpha)h < d/ec < x_0 < h$). In this case, denote the intersection of the line L_1 and the set N by point $B((1 - \alpha)h, r/c)$, and the intersection of the line L_3 and the set M_2 by point $A(h, rh/cx_0)$ (as shown in Figure 7). Through Lemma 2.8 and qualitative analysis, there exists a unique closed trajectory Γ_2 of system (1.4) which is tangent to the set M_2 at the point A and has minimal value λ_{\min} at the line L_1 . In the light of the different position of the set N , we consider the following two cases.

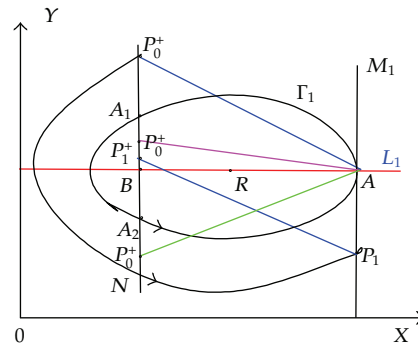


Figure 5

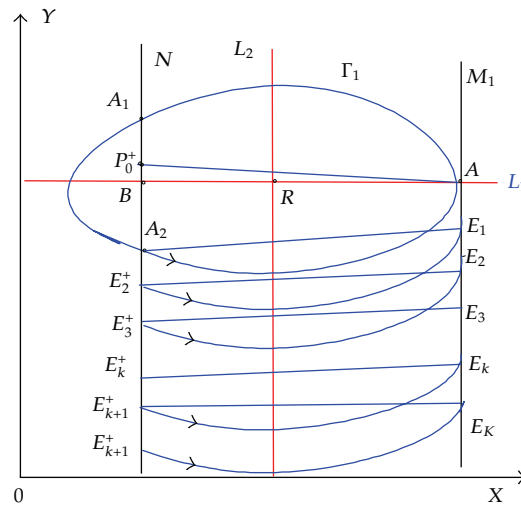


Figure 6

Subcase 3.1 ($0 < (1 - \alpha)h < \lambda_{\min}$). In this case, there exists a unique trajectory of system (1.4) which is tangent to the set N at the point B . Set $F(B) = P_1 \in M_2$, then pulse occurs at point P_1 , and the impulsive function transfers the point P_1 into P_1^+ . Like the analysis of Subcase 2.1, we can prove there exists an order one periodic solution in system (1.4) in this case.

Subcase 3.2 ($\lambda_{\min} < (1 - \alpha)h < x_0 < h$). In this case, let the closed trajectory Γ_2 of system (1.4) intersect with N at two points $A_1((1 - \alpha)h, y_{A_1})$ and $A_2((1 - \alpha)h, y_{A_2})$. Like the analysis of Subcase 2.2 we can prove there exists an order one periodic solution in system (1.4) in this case; like the analysis of Subcase 2.3, we can prove there is no periodic solution in system (1.4) and the trajectory with any initiating point of set N will stay in Γ_2 .

4. Attractiveness of the Order One Periodic Solutions

In this section, under the condition of existence of order one periodic solution to system (1.4) and the initial value of pest population $x(0) \leq h$, we discuss its attractiveness. We focus on Case 1, and by similar method we can obtain similar results about Cases 2 and 3.

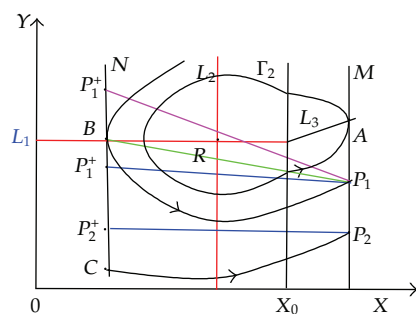


Figure 7

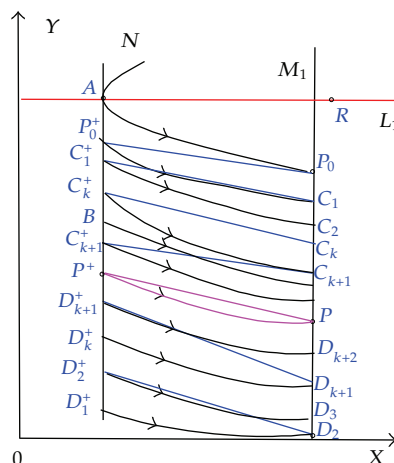


Figure 8

Theorem 4.1. *If system (1.4) satisfies the condition of Theorem 3.1 and $y_{P_0^+} < y_A$, then*

- (I) *there exists an odd number of order one periodic solutions of system (1.4) with initial value between C and A in the set N*
- (II) *if the periodic solution is unique, then the periodic solution is attractive in region $\Omega_1 = \{(x, y) \mid x \geq 0, y \geq 0, x(0) \leq h\}$.*

Proof. (I) According to the Subcase 1.3, $f(A) < 0$ and $f(c) > 0$ (see Figure 2); the successor function $f(x)$ is continuous, so there exists an odd number of root satisfying $f(x) = 0$, then we can get that there exists an odd number of order one periodic solutions of system (1.4) with initial value between C and A in the set N.

(II) By the derivation of Theorem 3.1, we know there exists an order one periodic solution of system (1.4). We assume trajectory $\widehat{P^+P}$ and segment $\overline{PP^+}$ formulate a unique order one periodic solution of system (1.4) with initial point $P^+ \in N$ (Figure 8).

On the one hand, take a point $D_1^+((1-\alpha)h, \varepsilon) \in N$ satisfying $y_{D_1^+} = \varepsilon < q$ and $y_{D_1^+} < y_{P^+}$. The trajectory passes through the point $D_1^+((1-\alpha)h, \varepsilon)$ which intersects set M_1 at point $D_2(h, y_{D_2})$, that is, $F(D_1^+) = D_2 \in M_1$, then we have $y_{D_2} < y_P$, thus $y_{D_2^+} < y_{P^+}$. Since $y_{D_2^+} = (1-\beta)y_{D_2} + q > \varepsilon$, so we obtain $f(D_1^+) = y_{D_2^+} - y_{D_1^+} = y_{D_2^+} - \varepsilon > 0$; set $F(D_2^+) = D_3 \in M_1$, because $y_{D_1^+} < y_{D_2^+} < y_{P^+}$; we know $y_{D_2} < y_{D_3} < y_P$, then we have $y_{D_2^+} < y_{D_3^+} < y_{P^+}$ and

$f(D_2^+) = y_{D_3^+} - y_{D_2^+} > 0$. This process is continuous, then we get a sequence $\{D_k^+\}_{k=1,2,\dots}$ of set N satisfying

$$y_{D_1^+} < y_{D_2^+} < \dots < y_{D_k^+} < \dots < y_{P^+} \quad (4.1)$$

and $f(D_k^+) = y_{D_{k+1}^+} - y_{D_k^+} > 0$. Series $\{y_{D_k^+}\}_{k=1,2,\dots}$ increase monotonously and have upper bound, so that $\lim_{k \rightarrow \infty} y_{D_k^+}$ exists.

Next we will prove $\lim_{k \rightarrow \infty} y_{D_k^+} = y_{P^+}$. Setting $\lim_{k \rightarrow \infty} D_k = D^+$, we will prove $P^+ = D^+$. Otherwise $P^+ \neq D^+$, then there is a trajectory passing through the point D^+ which intersects the set M_1 at point \tilde{D} , then we have $y_{\tilde{D}} < y_P, y_{\tilde{D}^+} < y_{P^+}$. Since $f(D^+) \geq 0$ and $P^+ \neq D^+$, according to the uniqueness of the periodic solution, then we have $f(D^+) = y_{\tilde{D}^+} - y_{D^+} > 0$, thus $y_{D^+} < y_{\tilde{D}^+} < y_{P^+}$ hold. Set $F(\tilde{D}^+) = \tilde{\tilde{D}} \in M_1$. In view of vector field and disjointness of any two trajectories, we know $y_{\tilde{D}} < y_{\tilde{\tilde{D}}} < y_P$ and $y_{\tilde{D}^+} < y_{\tilde{\tilde{D}}^+} < y_{P^+}$, then we have $f(\tilde{\tilde{D}}^+) = y_{\tilde{\tilde{D}}^+} - y_{\tilde{D}^+} > 0$; this contradicts the fact that D^+ is a limit of sequence $\{D_k^+\}_{k=1,2,\dots}$, so we obtain $P^+ = D^+$ and $\lim_{k \rightarrow \infty} y_{D_k^+} = y_{P^+}$.

On the other hand, set $F(P_0^+) = C_1 \in M_1$, then C_1 jumps to $C_1^+ \in N$ under the impulsive effects. Since $y_{P^+} < y_{P_0^+} < y_A$, we have $y_P < y_{C_1} < y_{P_0}$, thus we obtain that $y_{P^+} < y_{C_1^+} < y_{P_0^+}$ and that $f(P_0^+) = y_{C_1^+} - y_{P_0^+} < 0$. Set $F(C_1^+) = C_2 \in M_1$, then C_2 jumps to $C_2^+ \in N$ under the impulsive effects. We have $y_{P^+} < y_{C_2^+} < y_{C_1^+}$, $f(C_1^+) = y_{C_2^+} - y_{C_1^+} < 0$; this process is continuous, and we can obtain a sequence $\{C_k^+\}_{k=1,2,\dots}$ of set N satisfying

$$y_{P_0^+} > y_{C_1^+} > y_{C_2^+} > \dots > y_{C_k^+} > \dots > y_{P^+} \quad (4.2)$$

and $f(C_k^+) = y_{C_{k+1}^+} - y_{C_k^+} < 0$. Series $\{y_{C_k^+}\}_{k=1,2,\dots}$ decrease monotonously and have lower bound, so that $\lim_{k \rightarrow \infty} y_{C_k^+}$ exists. Similarly, we can prove that $\lim_{k \rightarrow \infty} y_{C_k^+} = y_{P^+}$.

Since the trajectory initiating any point of $\Omega_1 = \{(x, y) \mid x \geq 0, y \geq 0, x(0) \leq h\}$ will certainly intersect with set N , next we only need to prove the trajectory initiating any point of N will ultimately tend to be the unique order one periodic solution.

Any point $B \in N$ below A must be in some interval $[y_{C_{k+1}^+}, y_{C_k^+}]_{k=1,2,\dots}, [y_{C_1^+}, y_{P_0^+}], [y_{P_0^+}, y_A], [y_{D_k^+}, y_{D_{k+1}^+}]_{k=1,2,\dots}$. Without loss of generality, we assume the point $B \in [y_{C_{k+1}^+}, y_{C_k^+}]$. The trajectory with initiating point B moves between trajectories $\widehat{C_k^+ C_{k+1}}$ and $\widehat{C_{k+1}^+ C_{k+2}}$ and intersects with M_1 at some point between C_{k+2} and C_{k+1} , and under the impulsive effects it jumps to the point of N which is between $[y_{C_{k+2}^+}, y_{C_{k+1}^+}]$, then trajectory $\tilde{\Pi}(B, t)$ continues to move between trajectories $\widehat{C_{k+1}^+ C_{k+2}}$ and $\widehat{C_{k+2}^+ C_{k+3}}$. This process can be continued unlimitedly. Since $\lim_{k \rightarrow \infty} y_{C_k^+} = y_{P^+}$, the intersection sequence of trajectory $\tilde{\Pi}(B, t)$ and set N will ultimately tend to point P^+ . Similarly, if $B \in [y_{D_k^+}, y_{D_{k+1}^+}]$, we can also get that the intersection sequence of trajectory $\tilde{\Pi}(B, t)$ and set N will ultimately tend to point P^+ . Thus the trajectory initiating any point below A ultimately tends to be the unique order one periodic solution $\widehat{P^+ P P^+}$.

The trajectory with any initiating point above A of N will intersect with set N at some point below A with time increasing, so like the analysis above we obtain the trajectory initiating any point above A will ultimately tend to be the unique order one periodic solution $\widehat{P^+ P P^+}$.

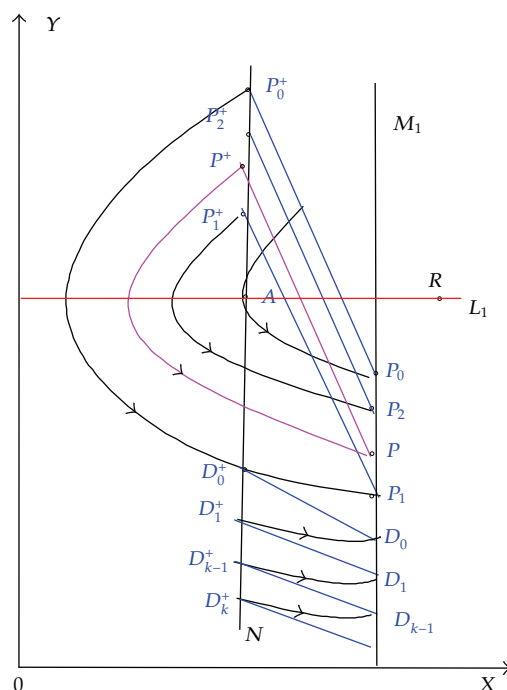


Figure 9

From the above analysis, we know the trajectory initiating any point of N will ultimately tend to be order one periodic solution $\widehat{P^+PP^+}$. Therefore, the unique order one periodic solution is attractive in the region Ω_1 . This completes the proof. \square

Remark 4.2. If system (1.4) satisfies the condition of Theorem 3.1 and that $y_{D_1^+} < y_A < y_{P_0^+}$, then order one periodic solution with initial point between A and P_0^+ is unattractive.

Theorem 4.3. If system (1.4) satisfies the condition of Theorem 3.1 and $y_{P_0^+} > y_{P_1^+} > y_A$, there exists a unique order one periodic solution of system (1.4) which is attractive in the region $\Omega_1 = \{(x, y) / x \geq 0, y \geq 0, x(0) \leq h\}$.

Proof. Through the derivation of Theorem 3.1, we know there exists an order one periodic solution of system (1.4), whose initial point is between A and P_0^+ in set N . We assume trajectory $\widehat{P^+P}$ and segment $\overline{PP^+}$ formulate an order one periodic solution of system (1.4), and $P^+ \in N$ is its initial point satisfying $f(P^+) = 0$ (Figure 9). First we will prove the uniqueness of the order one periodic solution.

We take any two points $C_1(h, y_{C_1}) \in N$, $C_2(h, y_{C_2}) \in N$ satisfying $y_{C_2} > y_{C_1} > y_A$, then we obtain two trajectories, whose initiate points are C_1 and C_2 intersecting with the set M_1 and two points $D_1(h, y_{D_1})$ and $D_2(h, y_{D_2})$, respectively (Figure 10). In view of the vector field of system (1.4) and the disjointness of any two trajectories, we know $y_{D_1} > y_{D_2}$. Suppose the points D_1 and D_2 are subject to impulsive effect to points $D_1^+(h, y_{D_1^+})$ and $D_2^+(h, y_{D_2^+})$ respectively, then we have $y_{D_1^+} > y_{D_2^+}$ and $f(C_1) = y_{D_1^+} - y_{C_1}$, $f(C_2) = y_{D_2^+} - y_{C_2}$, so we get $f(C_1) - f(C_2) < 0$; thus we obtain that the successor function $f(x)$ decreases monotonously on

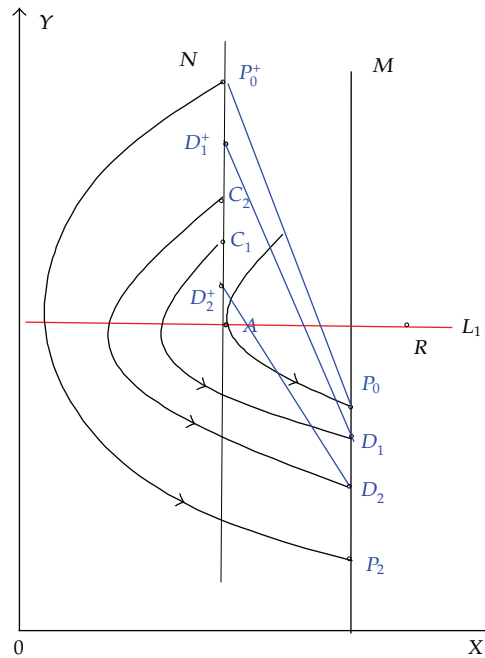


Figure 10

interval $(y_A, y_{P_0^+})$ of N , so there is a unique point $P^+ \in N$ satisfying $f(P^+) = 0$. The trajectory $\widehat{P^+PP^+}$ is a unique order one periodic solution of system (1.4).

Next we prove the attractiveness of the order one periodic solution $\widehat{P^+PP^+}$ in the region Ω_1 .

Denote the first intersection point of the trajectory with initiating point P_0^+ and the impulsive set M_1 by $P_1(h, y_{P_1})$ and that the corresponding consecutive points are $P_2(h, y_{P_2})$, $P_3(h, y_{P_3})$, $P_4(h, y_{P_4})$, and so forth, respectively. Consequently, under the effect of impulsive function I , the corresponding points after pulse are $P_1^+(h, y_{P_1^+})$, $P_2^+(h, y_{P_2^+})$, $P_3^+(h, y_{P_3^+})$, and so forth. In view of $y_{P_0^+} > y_{P_1^+} > y_A$ and disjointness of any two trajectories, we have that

$$y_{P_1^+} < y_{P_3^+} < \cdots < y_{P_{2k-1}^+} < y_{P_{2k+1}^+} < \cdots < y_{P_{2k}^+} < y_{P_{2k-2}^+} < \cdots < y_{P_2^+} < y_{P_0^+} \quad (4.3)$$

so $f(P_{2k-1}^+) = y_{P_{2k}^+} - y_{P_{2k-1}^+} > 0$ and $f(P_{2k}^+) = y_{P_{2k+1}^+} - y_{P_{2k}^+} < 0$ hold. As is proved in Theorem 4.1, we can prove $\lim_{k \rightarrow \infty} y_{P_{2k-1}^+} = \lim_{k \rightarrow \infty} y_{P_{2k}^+} = y_{P^+}$.

Since the trajectory initiating any point of Ω_1 will certainly intersect with set N , next we only need to prove the trajectory initiating any point of N will ultimately tend to be order one periodic solution $\widehat{P^+PP^+}$.

The trajectory with initiating point between D_0^+ and P_0^+ will intersect impulsive set N with time increasing; under the impulsive effects it arrives at a point of N which is between $[y_{P_{2k-1}^+}, y_{P_{2k+1}^+})$ or $[y_{P_{2k}^+}, y_{P_{2k-2}^+})$. Then like the analysis of Theorem 4.1, we know the trajectory initiating any point between D_0^+ and P_0^+ will ultimately tend to be order one periodic solution $\widehat{P^+PP^+}$.

Assume a point D_0 of the set M_1 jumps to D_0^+ under the impulsive effect. Set $R(D_0) = D_1^+ \in N$. Assume a point D_1 of the set M_1 jumps to D_1^+ under the impulsive effect. Set $R(D_1) = D_2^+ \in N$. This process is continuing until there exists a $D_{K_0}^+ \in N$ ($K_0 \in \mathbb{Z}^+$) satisfying $y_{D_{K_0}^+} < q$. So we obtain a sequence $\{D_k\}_{k=0,1,2,\dots,K_0}$ of the set M_1 and a sequence $\{D_k^+\}_{k=0,1,2,\dots,K_0}$ of the set N satisfying $R(D_{k-1}) = D_k^+$, $y_{D_k^+} < y_{D_{k-1}^+}$. For any point of the set N below D_0^+ , it must lie between D_{k+1}^+ and D_k^+ where $k = 1, 2, \dots, K_0$. After $K_0 + 1$ times' impulsive effects, the trajectory from this initiating point will arrive at some point of the set N which must be between D_0^+ and P_0^+ and then will ultimately tend to order one periodic solution $\widehat{P^+PP^+}$. There is no order one periodic solution with the initial point below D_0^+ .

The trajectory with any initiating point above P_0^+ of N will intersect with the set N at some point below U_0^+ with time increasing, so the trajectory with any initiating point above P_0^+ will ultimately tend to be order one periodic solution $\widehat{P^+PP^+}$. There is no order one periodic solution with the initial point above P_0^+ .

From the above analysis, we know there exists a unique order one periodic solution in system (1.4) and the trajectory from any initiating point of N will ultimately tend to be order one periodic solution $\widehat{P^+PP^+}$. Therefore, order one periodic solution $\widehat{P^+PP^+}$ is attractive in the region Ω_1 . This completes the proof. \square

5. Conclusion and Numerical Simulations

In this paper, a Holling I predator-prey model with state-dependent impulsive control model concerning different control methods at different thresholds is proposed to find a new method to study existence and attractive of order one periodic solution of such system. We define semicontinuous dynamical system and successor function and demonstrate the sufficient condition that system (1.4) has order one periodic solution with differential geometry theory and successor function; by means of sequence convergence rules and qualitative analysis, we prove the attractiveness of the order one periodic solution. These results show that the state-dependent impulsive effects contribute significantly to the richness of the dynamics of the model. The methods of the theorems are proved to be new in this paper and these methods are more efficient and easier to operate than the existing research ones that have been applied to the models with impulsive state feedback control [12–15], so they deserve further promotion.

In order to testify the validity of our results, we consider the following example:

$$\begin{aligned}
 \frac{dx}{dt} &= 0.8x - 0.6xy, & x \leq x_0, & x < h, \\
 \frac{dy}{dt} &= -0.2y + 0.3xy, & x \leq x_0, & x < h, \\
 \frac{dx}{dt} &= 0.8x - 0.6x_0y, & x > x_0, & x < h, \\
 \frac{dy}{dt} &= -0.2y + 0.3x_0y, & x > x_0, & x < h, \\
 \Delta x &= -\alpha x, & x &= h, \\
 \Delta y &= -\beta y + q, & x &= h,
 \end{aligned} \tag{5.1}$$

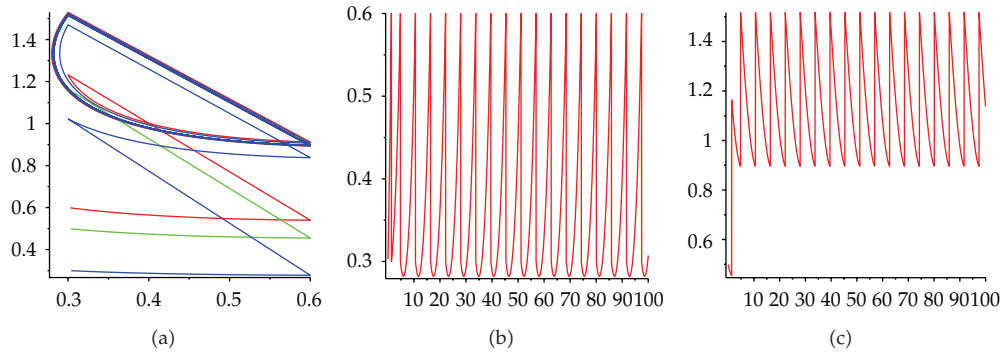


Figure 11: The time series and phase diagram for system (1.4) starting from initial value $(0.3, 0.1)$ (blue), $(0.3, 0.5)$ (green), $(0.3, 0.6)$ (red), $\alpha = 0.5$, $\beta = 0.2$, $q = 0.8$, $h = 0.6$, $x_0 = 1$, $h < x^* < x_0$.

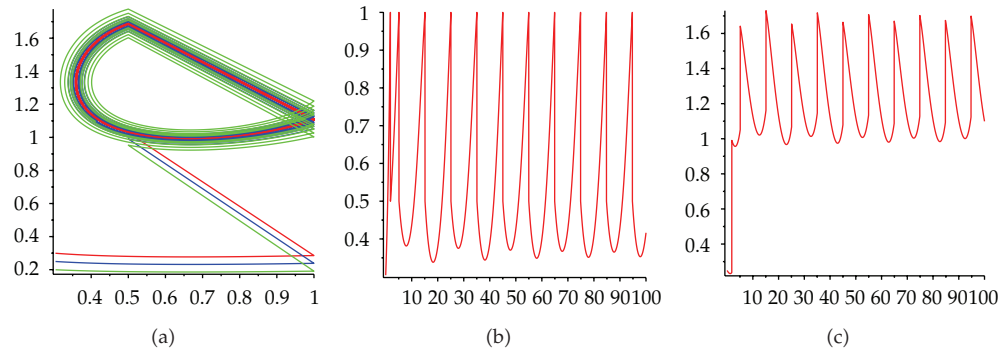


Figure 12: The time series and phase diagram for system (1.4) starting from initial value $(0.3, 0.2)$ (blue), $(0.3, 0.1)$ (green), $(0.3, 0.15)$ (red), $\alpha = 0.5$, $\beta = 0.2$, $q = 0.8$, $h = 1$, $x_0 = 1.5$, $x^* < h < x_0$.

where $\alpha, \beta \in (0, 1)$, $q > 0$, $h > 0$. Now, we consider the impulsive effects on the dynamics of system (5.1).

Example 5.1. Existence and attractiveness of order one periodic solution.

We set $h = 0.6$, $\alpha = 0.5$, $\beta = 0.2$, $q = 0.8$, $h < x^* < x_0$, initiating points are $(0.3, 0.6)$ (red), $(0.3, 0.5)$ (green), and $(0.3, 0.1)$ (blue), respectively. Figure 11 shows that system (5.1) has order one periodic solution when the conditions of Theorems 3.1 and 4.1 hold, and the trajectory from different initiating must ultimately tend to be the order one periodic solution. Therefore order one periodic solution is attractive.

Example 5.2. Existence and attractiveness of positive periodic solutions.

We set $h_1 = 1$, $\alpha = 0.5$, $\beta = 0.2$, $q = 0.8$, $x_0 = 1.5$, $x^* < h < x_0$, initiating points are $(0.3, 0.3)$ (red), $(0.3, 0.2)$ (green), and $(0.3, 0.25)$ (blue), respectively. Figure 12 shows that system (5.1) has order one periodic solution when the conditions of Theorems 3.2 and 4.1 hold, and the trajectory from different initiating must ultimately tend to be the order one periodic solution. Therefore order one periodic solution is attractive.

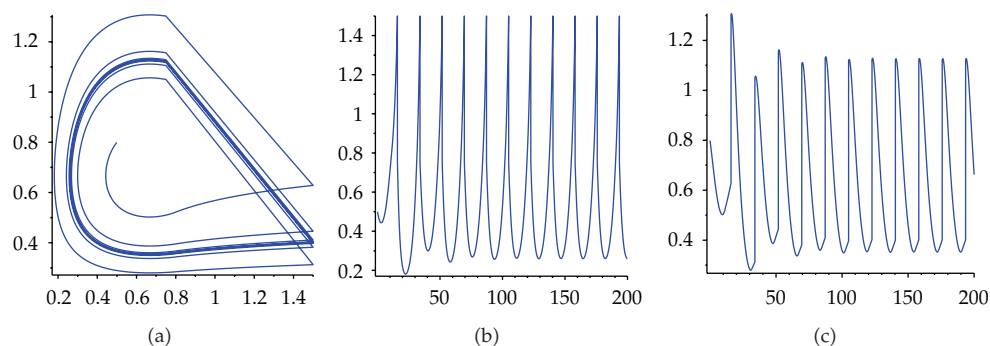


Figure 13: The time series and phase diagram for system (1.4) starting from initial value $(0.4, 0.5)$, $\alpha = 0.5$, $\beta = 0.2$, $q = 0.8$, $h = 1.5$, $x_0 = 0.8$, $x^* < x_0 < h$.

Example 5.3. Existence and attractiveness of positive periodic solutions.

We set $h = 1.5$, $\alpha = 0.5$, $\beta = 0.2$, $q = 0.8$, $x_0 = 0.8$, $x^* < x_0 < h$; initiating points is $(0.4, 0.5)$. Figure 13 shows that results of Case 3 are valid.

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