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Research Article

Blow-Up Analysis for a Quasilinear Degenerate Parabolic Equation with Strongly Nonlinear Source

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We investigate the blow-up properties of the positive solution of the Cauchy problem for a quasilinear degenerate parabolic equation with strongly nonlinear source $u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^l) + u^q$, $(x,t) \in R^N \times (0,T)$, where $N \geq 1$, $p > 2$, and $m, l, q > 1$, and give a secondary critical exponent on the decay asymptotic behavior of an initial value at infinity for the existence and nonexistence of global solutions of the Cauchy problem. Moreover, under some suitable conditions we prove single-point blow-up for a large class of radial decreasing solutions.

1. Introduction

In this paper, we consider the following Cauchy problem to a quasilinear degenerate parabolic equation with strongly nonlinear source

$$\begin{gathered} u_t = \operatorname{div}\left(|\nabla u^m|^{p-2}\nabla u^l\right) + u^q, \quad (x,t) \in R^N \times (0,T), \\ u(x,0) = u_0(x), \quad x \in R^N, \end{gathered} \tag{1.1}$$

where $N \geq 1$, $p > 2$, $m, l, q > 1$, and the initial data $u_0(x)$ is nonnegative bounded and continuous.

Equation (1.1) has been suggested as a mathematical model for a variety of physical problems (see [1]). For instance, it appears in the non-Newtonian fluids and is a nonlinear form of heat equation. Moreover, it can also be used to model the nonlinear heat propagation in a reaction medium (see [2]).

One of the particular features of problem (1.1) is that the equation is degenerate at points where $u = 0$ or $\nabla u = 0$. Hence, there is no classical solution in general and we introduce the following definition of weak solution (see [3, 4]).

Definition 1.1. A nonnegative measurable function $u(x,t)$ defined in $R^N \times (0,T)$ is called a weak solution of the Cauchy problem (1.1) if for every bounded open set Ω with smooth boundary $\partial\Omega$, $u \in C_{\text{loc}}(\Omega \times (0,T))$, u^m, $u^l \in L^p_{\text{loc}}(0,T;W^{1,p}(\Omega))$, and

$$\int_\Omega u\varphi\, dx + \int_{t_0}^t \int_\Omega \left(-u\varphi_t + |\nabla u^m|^{p-2}\nabla u^l \cdot \nabla\varphi\right)dx\, dt = \int_{t_0}^t \int_\Omega u^q \varphi\, dx\, dt + \int_\Omega u(x,t_0)\varphi(x,t_0)dx \tag{1.2}$$

for all $0 \le t_0 \le t \le T$ and all test functions $\varphi \in C_0^1(\Omega \times (0,T))$. Moreover,

$$\lim_{t \to t_0} \int_\Omega u(x,t)\eta(x)dx = \int_\Omega u(x,t_0)\eta(x)dx \tag{1.3}$$

for any $\eta(x) \in C_0^1(\Omega)$.

Under some suitable assumptions, the existence, uniqueness and regularity of a weak solution to the Cauchy problem (1.1) and their variants have been extensively investigated by many authors (see [5–7] and the references therein).

The first goal of this paper is to study the blow-up behavior of solution $u(x,t)$ of (1.1) when the initial data $u_0(x)$ has slow decay near $x = \infty$. For instance, in the following case

$$u_0(x) \cong M|x|^{-a} \quad \text{with } M > 0,\ a \ge 0, \tag{1.4}$$

we investigate the existence of global and nonglobal solutions for the Cauchy problem (1.1) in terms of M and a. In recent years, many authors have studied the properties of solutions to the Cauchy problem (1.1) and their variants (see [8–17] and the references therein). In particular, J.-S. Guo and Y. Y. Guo [18] obtained the secondary critical exponent for the following porous medium type equation in high dimensions:

$$\begin{gathered} u_t = \Delta u^m + u^p, \quad (x,t) \in R^N \times (0,T), \\ u(x,0) = u_0(x), \quad x \in R^N, \end{gathered} \tag{1.5}$$

where $p > 1$, $m > 1$ or $\max\{0, 1 - (2/N)\} < m < 1$, $u_0(x)$ is nonnegative bounded and continuous, and proved that for $p > p_m^* = m + (2/N)$, there exists a secondary critical exponent $a^* = 2/(p-m)$ such that the solution $u(x,t)$ of (1.5) blows up in finite time for the initial data $u_0(x)$, which behaves like $|x|^{-a}$ at $x = \infty$ if $a \in (0, a^*)$, and there exists a global solution for the initial data $u_0(x)$, which behaves like $|x|^{-a}$ at $x = \infty$ if $a \in (a^*, N)$. Here, we say that the solution blows up in finite time; it means that there exists $T \in (0, +\infty)$ such that $\|u(\cdot,t)\|_{L^\infty} < \infty$ for all $t \in [0,T)$, but $\lim_{t \to T^-} \sup \|u(.,t)\|_{L^\infty} = \infty$.

Mu et al. [19] studied the secondary critical exponent for the following p-Laplacian equation with slow decay initial values:

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^q, \quad (x,t) \in R^N \times (0,T),$$
$$u(x,0) = u_0(x), \quad x \in R^N, \tag{1.6}$$

where $p > 2$, $q > 1$, and showed that, for $q > q_c^* = p - 1 + (p/N)$, there exists a secondary critical exponent $a_c^* = (p/(q+1-p))$ such that the solution $u(x,t)$ of (1.6) blows up in finite time for the initial data $u_0(x)$ which behaves like $|x|^{-a}$ at $x = \infty$ if $a \in (0, a_c^*)$, and there exists a global solution for the initial data $u_0(x)$, which behaves like $|x|^{-a}$ at $x = \infty$ if $a \in (a_c^*, N)$.

Recently, Mu et al. [20] also investigated the secondary critical exponent for the doubly degenerate parabolic equation with slow decay initial values and obtained similar results.

On the other hand, in this paper, we will also consider single-point blow-up for the Cauchy problem (1.1). It is interesting to study the set of blow-up points and the behavior of the solution $u(x,t)$ at the blow-up point.

In order to investigate single-point blow-up for the Cauchy problem (1.1), we introduce the concept of the blow-up point.

Definition 1.2. A point $x \in \Omega$ is called a blow-up point if there exists a sequence (x_n, t_n) such that $x_n \to x$, $t_n \to T^-$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$, where T is blow-up time.

In recent years, some authors also studied single-point blow-up for the Cauchy problem to nonlinear parabolic equations (see [21, 22] and the references therein) by different methods. In particular, when $p = 2$, $l = 1$ and $N = 1$, the Cauchy problem (1.1) has been investigated by Weissler in [23], and the author obtained that the solution blows up only at a single point. Galaktionov and Posashkov [24] studied the single-point blow-up and gave the upper and lower bound near the blow-up point for the Cauchy problem (1.1) when $p > 2$ and $m = l = 1$. Recently, when $p > 2$ and $m = l$, Mu and Zeng [25] extended Galaktionov's results to the doubly degenerate parabolic equation. For more works about single-point blow-up, we refer to [26, 27], where the parabolic systems have been considered.

Motivated by the above works, based on a modification of the energy methods, comparison principle, and regularization methods used in [15, 19, 21, 24], we investigate the secondary critical exponent and single-point blow-up for the Cauchy problem (1.1). Before stating the results of the secondary critical exponent, we start with some notations as follows.

Let $C_b(R^N)$ be the space of all bounded continuous functions in R^N. For $a \geq 0$, we define

$$\Phi^a = \left\{\phi(x) \in C_b\left(R^N\right) \mid \phi(x) \geq 0,\ \lim_{|x|\to\infty} \sup |x|^a \phi(x) < \infty\right\},$$
$$\Phi_a = \left\{\phi(x) \in C_b\left(R^N\right) \mid \phi(x) \geq 0,\ \lim_{|x|\to\infty} \inf |x|^a \phi(x) > 0\right\}. \tag{1.7}$$

Moreover, we denote

$$q_c^* = l + m(p-2) + \frac{p}{N}, \qquad a_c^* = \frac{p}{q - l - m(p-2)}. \tag{1.8}$$

Our main results of this paper are stated as follows.

Theorem 1.3. *For $N \geq 2$, $p > 2$, $m > 1$, $l > 1$, and $q > q_c^* = l + m(p-2) + (p/N)$, suppose that $u_0(x) \in \Phi_a$ for some $a \in (0, a_c^*)$; then the solution $u(x,t)$ of the Cauchy problem* (1.1) *blows up in finite time.*

Theorem 1.4. *For $N \geq 2$, $p > 2$, $m > 1$, $l > 1$, and $q > q_c^* = l + m(p-2) + (p/N)$, suppose that $u_0(x) = \lambda\varphi(x)$ for some $\lambda > 0$ and $\varphi(x) \in \Phi^a$ for some $a \in (a_c^*, N)$; then there is $\lambda_0 = \lambda_0(\varphi) > 0$ such that the solution $u(x,t)$ of the Cauchy problem* (1.1) *exists globally for all $t > 0$, and if $\lambda < \lambda_0$, one has*

$$\|u(x,t)\|_\infty \leq Ct^{-a\beta}, \quad \forall t > 0, \tag{1.9}$$

where $\beta = 1/(a(l + m(p-2) - 1) + p)$, $C > 0$.

Remark 1.5. When $p > 2$, $N \geq 2$ and $q > q_c^*$, we have $q_c^* > 1$ and $0 < a_c^* < N$.

Remark 1.6. It follows from Theorems 1.3 and 1.4 that the number $a_c^* = p/(q - l - m(p-2))$ gives another cut-off between the blow-up case and the global existence case. Therefore, the number a_c^* is a new secondary critical exponent of the Cauchy problem (1.1). Unfortunately, in the critical case $a = a_c^*$, we do not know whether the solution of (1.1) exists globally or blows up in finite time.

Remark 1.7. When $m = l = 1$ or $m = l > 1$, the results of Theorems 1.3 and 1.4 are consistent with those in [19, 20], respectively.

Remark 1.8. In [28], Afanas'eva and Tedeev also established the Fujita type results for (1.1) with $m = l$. In particular, if $u_0(x) \sim |x|^{-a}$, $0 < a < N$, they obtained that if $q < m(p-1) + (p/a)$, then every nontrivial solution blows up in finite time, and if $q > m(p-1) + (p/a)$, then the solution exists globally for a small initial data $u_0(x)$. We note that when $m = l$ in (1.1), if $q > m(p-1) + (p/N)$ and $0 < a < p/(q - m(p-1))$, then $0 < a < N$ and $q < m(p-1) + (p/a)$, while if $q > m(p-1) + (p/N)$ and $p/(q - m(p-1)) < a < N$, then $q > m(p-1) + (p/a)$. Therefore, the results of Theorems 1.3 and 1.4 coincide with those in [28].

Finally, we also consider single-point blow-up for a large number of radial decreasing solutions of the Cauchy problem (1.1) and give upper bound of the radial solution $u(r,t)$ in a small neighborhood of the point (x,t), where $x = 0$, $t = T$. We assume that the initial data $u_0(x) = u_0(r)$ satisfies the following condition:

(H) $u_0(x) = u_0(r) \geq 0$ for $r > 0$, $u_0(0) > 0$, and $u_0(r) \in C^1(R_+^1)$, $u_0'(0) = 0$, and $u_0'(r) \leq 0$ for $r > 0$, $M_0 = \sup \ u_0(r) < +\infty$, $K_0 = \sup \ |u_0'(r)| < +\infty$.

Theorem 1.9. *Let $N \geq 1$, $p > 2$, $m > 1$, $l > 1$, and $q > l + m(p-2)$, and let condition (H) hold. In addition, assume that the initial function $u_0(x) = u_0(r)$ satisfies*

$$u_0^q(r) \cdot r^N = o(1) \quad \textit{as } r \longrightarrow \infty, \tag{1.10}$$

$$\lambda_0 = \inf_{r>0,\, u_0(r)>0} \left\{ -\frac{\left|(u_0^m)'\right|^{p-2}\left(u_0^l\right)'}{r u_0^q} \right\} \in \left(0, \frac{p-2}{(p-2)(N+1)+2}\right]. \tag{1.11}$$

Let T be the blow-up time; then one has

$$u(r,t) \le Cr^{-p/(q-l-m(p-2))}, \quad (r,t) \in R_+^1 \times (0,T), \tag{1.12}$$

where

$$C = \left[\frac{q-l-m(p-2)}{(lm^{p-2})^{1/(p-1)} p} \lambda_0^{1/(p-1)} \right]^{-((p-1)/(q-l-m(p-2)))} > 0, \tag{1.13}$$

that is, there is single-point blow-up at point $x = 0$.

Remark 1.10. By (1.11), the best upper estimate (1.12) obtained by our method has the following form:

$$u(r,t) \le \left[\frac{q-l-m(p-2)}{p(lm^{p-2})^{1/(p-1)}} \left(\frac{p-2}{(p-2)(N+1)+2} \right)^{1/(p-1)} \right]^{-((p-1)/(q-l-m(p-2)))} \cdot r^{-(p/(q-l-m(p-2)))} \tag{1.14}$$

in $R_+^1 \times (0,T)$. But, we do not give the lower bound estimate of the radial solution $u(r,t)$ in a small neighborhood of the point (x,t), where $x = 0,\ t = T$.

Remark 1.11. When $m = l = 1$ or $m = l > 1$, the results of Theorem 1.9 are consistent with those in [24, 25], respectively. For $1 < q < l + m(p-2)$, in [29], the authors obtained the results of global blow-up to arbitrary compactly supported initial data.

Remark 1.12. From Theorem 1.9, we obtain the same decay exponent as that of Theorem 1.2 in [29] by different methods. Moreover, it is interesting to see that the decay exponent of the upper estimate of Theorem 1.9 is also the same as the secondary critical exponent of Theorems 1.3 and 1.4.

This paper is organized as follows. In Section 2, by using the energy method, we will obtain a blow-up condition and prove Theorem 1.3. In Section 3, using the comparison principle, we can construct a global supersolution to prove Theorem 1.4. Finally, we consider the single-point blow-up under some suitable conditions and prove Theorem 1.9 in Section 4.

2. Blow-Up Case

By using the energy method, we will obtain a blow-up condition corresponding to (1.1). Therefore, we need to select a suitable test function as follows:

$$\psi_\varepsilon(x) = A_\varepsilon e^{-\varepsilon|x|}, \quad A_\varepsilon = \frac{1}{\int_{R^N} e^{-\varepsilon|x|} dx} = \frac{\varepsilon^N}{\int_{\omega_N} \int_0^\infty e^{-r} r^{N-1} dr\, ds}, \quad \nabla\psi_\varepsilon(x) = -\varepsilon\psi_\varepsilon \frac{x}{|x|}. \tag{2.1}$$

Proof of Theorem 1.3. Suppose that $u(x,t)$ is the solution of the Cauchy problem (1.1) and T is the blow-up time. Let

$$E(t) = \frac{1}{s}\int_{R^N} u^s(x,t)\psi_\varepsilon(x)dx, \quad t \in [0,T), \tag{2.2}$$

where $0 < s < 1/p,\ p > 2$, Then, $E(t) \in C[0,T)\bigcap C^1(0,T)$ and

$$\begin{aligned}
E'(t) = &\int_{R^N} u^{s-1}\psi_\varepsilon u_t dx = \int_{R^N} u^{s-1}\psi_\varepsilon \operatorname{div}\Big(|\nabla u^m|^{p-2}\nabla u^l\Big)dx + \int_{R^N} u^{q+s-1}\psi_\varepsilon dx \\
= &-l(s-1)\int_{R^N} u^{s+l-3}\psi_\varepsilon|\nabla u^m|^{p-2}|\nabla u|^2dx + \varepsilon\int_{R^N} u^{s-1}\psi_\varepsilon|\nabla u^m|^{p-2}\nabla u^l\frac{x}{|x|}dx \\
&+\int_{R^N} u^{q+s-1}\psi_\varepsilon dx \\
\geq &\ lm^{p-2}(1-s)\int_{R^N} u^{l+m(p-2)+s-p-1}\psi_\varepsilon|\nabla u|^p dx \\
&-lm^{p-2}\varepsilon\int_{R^N} u^{l+m(p-2)+s-p}\psi_\varepsilon|\nabla u|^{p-1}dx + \int_{R^N} u^{q+s-1}\psi_\varepsilon dx.
\end{aligned} \tag{2.3}$$

Using Young's inequality, we have

$$\begin{aligned}
\varepsilon\int_{R^N} u^{l+m(p-2)+s-p}\psi_\varepsilon|\nabla u|^{p-1}dx \leq &\frac{p-1}{p}\int_{R^N} u^{l+m(p-2)+s-p-1}\psi_\varepsilon|\nabla u|^p dx \\
&+\frac{\varepsilon^p}{p}\int_{R^N} u^{l+m(p-2)+s-1}\psi_\varepsilon dx.
\end{aligned} \tag{2.4}$$

Since $0 < s < (1/p)$, it follows from (2.3) and (2.4) that

$$E'(t) \geq \int_{R^N} u^{q+s-1}\psi_\varepsilon dx - \frac{lm^{p-2}\varepsilon^p}{p}\int_{R^N} u^{l+m(p-2)+s-1}\psi_\varepsilon dx. \tag{2.5}$$

By $q > q_c^* = l + m(p-2) + (p/N) > l + m(p-2) > 1$, $\int_{R^N}\psi_\varepsilon(x)dx = 1$, and Hölder's inequality, we obtain

$$\begin{aligned}
\int_{R^N} u^{l+m(p-2)+s-1}\psi_\varepsilon dx &= \int_{R^N} u^{l+m(p-2)+s-1}\psi_\varepsilon^{(l+m(p-2)+s-1)/(q+s-1)}\psi_\varepsilon^{q-[l+m(p-2)]/(q+s-1)}dx \\
&\leq \left[\int_{R^N} u^{q+s-1}\psi_\varepsilon dx\right]^{(l+m(p-2)+s-1)/(q+s-1)}.
\end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we have

$$\frac{dE}{dt} \geq \left[\int_{R^N} u^{q+s-1}\varphi_\varepsilon dx\right]^{(l+m(p-2)+s-1)/(q+s-1)} \times \left[\left(\int_{R^N} u^{q+s-1}\varphi_\varepsilon dx\right)^{(q-l-m(p-2))/(q+s-1)} - \frac{lm^{p-2}\varepsilon^p}{p}\right]. \tag{2.7}$$

Applying Jensen's inequality, we obtain

$$\left(\int_{R^N} u^{q+s-1}\varphi_\varepsilon dx\right)^{(q-l-m(p-2))/(q+s-1)} \geq \left(\int_{R^N} u^{s}\varphi_\varepsilon dx\right)^{(q-l-m(p-2))/s}. \tag{2.8}$$

Thus, it follows from (2.7) and (2.8) that

$$\frac{dE}{dt} \geq \frac{1}{2}\left(\int_{R^N} u^{s}\varphi_\varepsilon dx\right)^{(q+s-1)/s} = \frac{1}{2}s^{(q+s-1)/s}E^{(q+s-1)/s}(t) \tag{2.9}$$

as long as

$$E(t) \geq \frac{1}{s}\left(\frac{2lm^{p-2}\varepsilon^p}{p}\right)^{s/(q-l-m(p-2))} \quad \forall t \in [0,T). \tag{2.10}$$

Hence, if $E(0)$ satisfies

$$E(0) \geq \frac{1}{s}\left(\frac{2lm^{p-2}\varepsilon^p}{p}\right)^{s/(q-l-m(p-2))} = C_0, \tag{2.11}$$

then $E(t)$ increases and remains below C_0 for all $t \in [0,T)$.

And by (2.9) we have

$$E(t) \geq \left(E^{q-1/s}(0) - C_1 t\right)^{-s/(q-1)}, \quad \text{where } C_1 = \frac{q-1}{2}s^{(q-1)/s} > 0. \tag{2.12}$$

Therefore, from (2.11) and (2.12), we obtain that $u(x,t)$ blows up in finite time $T = (1/C_1)E^{q-1/s}(0)$: and get an estimate on the blow-up time T of the solution $u(x,t)$ as follows:

$$T \leq \frac{2}{q-1}\left(\frac{2lm^{p-2}\varepsilon^p}{p}\right)^{(1-q)/(q-l-m(p-2))}. \tag{2.13}$$

Finally, it remains to verify the blow-up condition (2.11). Since $u_0(x) \in \Phi_a$ for some $a \in (0, a_c^*)$, there exist two positive constants M and $R_0 > 1$ such that $u_0(x) \geq M|x|^{-a}$ for all $|x| \geq R_0$, and we have

$$\begin{aligned} E(0) &= \frac{1}{s}\int_{R^N} u_0^s(x)\psi_\varepsilon(x)dx \\ &> \frac{M^s A_\varepsilon}{s}\int_{|x|\geq R_0} |x|^{-as}e^{-\varepsilon|x|}dx \\ &= \frac{M^s A_\varepsilon}{s}\varepsilon^{-N+as}\int_{|y|\geq \varepsilon R_0} |y|^{as}e^{-|y|}dy. \end{aligned} \tag{2.14}$$

By the definition of A_ε, $0 < a < a_c^*$, we can choose $0 < \varepsilon \leq (1/R_0)$ so small such that (2.11) holds. The proof of Theorem 1.3 is complete. □

3. Global Existence

In this section, we shall prove Theorem 1.4 by constructing a global supersolution. To do this, we introduce the radially symmetric self-similar solution $U_{M,a}(x,t)$ to the following Cauchy problem:

$$u_t = \operatorname{div}\left(|\nabla u^m|^{p-2}\nabla u^l\right), \quad (x,t) \in R^N \times (0, +\infty), \tag{3.1}$$

$$u(x,0) = u_0(x) = M|x|^{-a}, \quad x \in R^N. \tag{3.2}$$

It is well known that the existence and uniqueness of the solution of (3.1) have been well established (see [7]). By symmetric properties of (3.1), the solution $U_{M,a}(x,t)$ is given by the following form

$$U_{M,a}(x,t) = t^{-a\beta} f_M(r), \quad \text{with } r = \frac{|x|}{t^\beta},\ \beta = \frac{1}{a(l+m(p-2)-1)+p}, \tag{3.3}$$

where the positive function f_M is the solution of the problem

$$\begin{gathered} \left(\left|(f_M^m)'\right|^{p-2}\left(f_M^l\right)'\right)' + \frac{N-1}{r}\left|(f_M^m)'\right|^{p-2}\left(f_M^l\right)'(r) + \beta r f_M'(r) + a\beta f_M(r) = 0, \quad r > 0, \\ f_M(r) \geq 0, \quad r \geq 0, \quad f_M'(0) = 0, \quad \lim_{r\to+\infty} r^a f_M(r) = M. \end{gathered} \tag{3.4}$$

We shall prove the existence of solution $f_M(r)$ to (3.4) by the following ordinary differential equation, and furthermore we obtain the nonincreasing property of the solution $f_M(r)$.

Firstly, given a fixed $\eta > 0$, we consider the following Cauchy problem:

$$\left(\left|(g^m)'\right|^{p-2}\left(g^l\right)'\right)' + \frac{N-1}{r}\left|(g^m)'\right|^{p-2}\left(g^l\right)'(r) + \beta r g'(r) + \alpha\beta g(r) = 0, \quad r > 0,$$
$$g(0) = \eta, \qquad g'(0) = 0. \tag{3.5}$$

According to the standard of the Cauchy problem for ODE and the methods used in [7, 30], we can obtain that the solution $g(r)$ of the Cauchy problem (3.5) is positive, and $g(r) \to 0$ as $r \to \infty$; moreover,

$$\lim_{r\to+\infty} r^a g(r) = M \tag{3.6}$$

for some $M = M(\eta) > 0$.

Secondly, we shall prove that there exists a one-to-one correspondence between $M \in (0, +\infty)$ and $\eta \in (0, +\infty)$. Indeed, this can be seen from the following relation:

$$g_\eta(r) = \eta g_1(\eta^\sigma r), \quad \sigma = \frac{1-l-m(p-2)}{p}, \tag{3.7}$$

where $g_1(r)$ is the solution of (3.5) for $\eta = 1$. Then,

$$M(\eta) = \eta^{1-a\sigma} M(1), \quad \text{with } M(1) = \lim_{r\to+\infty} r^a g_1(r). \tag{3.8}$$

Therefore, we can deduce that, for each $M > 0$, there exists a positive, bounded, and global solution $f_M(r)$ satisfying (3.4).

Finally, we shall prove that the solution $g(r)$ is non-increasing, that is, $f_M(r)$ is also non-increasing. To do this, we need the following lemmas.

Lemma 3.1. *Let $g(r)$ be the solution of* (3.5)*; then*

$$\lim_{r\to 0} \frac{N\left|(g^m)'(r)\right|^{p-2}(g^l)'(r)}{r} = -\alpha\beta g(0). \tag{3.9}$$

Proof. Integrating the (3.5) over $(0, \varepsilon)$ with $\varepsilon > 0$, we have

$$\left(\left|(g^m)'\right|^{p-2}\left(g^l\right)'\right)(\varepsilon) + \int_0^\varepsilon \frac{N-1}{r}\left|(g^m)'\right|^{p-2}\left(g^l\right)' dr + \int_0^\varepsilon \beta r g' dr + \int_0^\varepsilon \alpha\beta g\, dr = 0. \tag{3.10}$$

Dividing by ε and taking $\varepsilon \to 0$ in (3.10), we obtain

$$\lim_{\varepsilon\to 0}\left[\frac{\left|(g^m)'(\varepsilon)\right|^{p-2}(g^l)'(\varepsilon)}{\varepsilon} + \frac{N-1}{\varepsilon}\left|(g^m)'(\varepsilon)\right|^{p-2}\left(g^l\right)'(\varepsilon)\right] = -\lim_{\varepsilon\to 0}\alpha\beta g(\varepsilon), \tag{3.11}$$

which implies that (3.9) holds. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *If there exists* $r_0 \in [0, +\infty)$ *such that* $g(r_0) = 0$, *then* $g(r) = 0$ *for all* $r \geq r_0$.

Proof. We shall prove by contradiction. Assuming that Lemma 3.2 does not hold, it is easy to see that there exists $\varepsilon > 0$ such that

$$g(r) > 0, \quad g'(r) > 0 \quad \text{in } (r_0, r_0 + \varepsilon). \tag{3.12}$$

Multiplying (3.5) by r^{N-1} and integrating over (r_0, r) with $r \in (r_0, r_0 + \varepsilon)$, we obtain

$$r^{N-1}\left|(g^m)'\right|^{p-2}\left(g^l\right)' + \beta r^N g(r) = \int_{r_0}^{r} N\beta r^{N-1} g(r)dr - \int_{r_0}^{r} a\beta r^{N-1} g(r)dr. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\beta r^N g(r) \leq \int_{r_0}^{r} N\beta r^{N-1} g(r)dr - \int_{r_0}^{r} a\beta r^{N-1} g(r)dr \leq \frac{(N-a)\beta}{N} g(r)\left(r^N - r_0^N\right), \tag{3.14}$$

equivalently,

$$1 \leq \frac{N-a}{N}\left(1 - \left(\frac{r_0}{r}\right)^N\right). \tag{3.15}$$

Letting $r \to r_0$ in (3.15), we obtain the inequality $1 \leq 0$, which is a contradiction. The proof of Lemma 3.2 is complete. □

Lemma 3.3. *The solution* $g(r)$ *of* (3.5) *is monotone nonincreasing in* $[0, +\infty)$.

Proof. Our method is based on the contradiction argument. Suppose that, for some $r_0 > 0$, $g'(r_0) > 0$, by Lemma 3.1, there exists $r_1 \in (0, r_0)$ such that

$$g'(r_1) = 0\,, \qquad \left(\left|(g^m)'\right|^{p-2}\left(g^l\right)'\right)'(r_1) \geq 0. \tag{3.16}$$

By Lemma 3.2, we have $g(r_1) > 0$. Using the similar argument in Lemma 3.1, we obtain

$$\lim_{r \to r_1} \frac{N\left|(g^m)'(r)\right|^{p-2}(g^l)'(r)}{r - r_1} = -a\beta g(r_1) < 0, \tag{3.17}$$

which is a contradiction with (3.14). The proof of Lemma 3.3 is complete. □

Next, we apply the monotone properties of $f_M(r)$ to obtain the condition on the global existence of the solution to (1.1).

Proof of Theorem 1.4. We prove Theorem 1.4 by the following steps.

Step 1. Since $\varphi(x) \in \Phi^a$, there exists a constant $K > 0$ such that

$$\varphi(x) \le K(1+|x|)^{-a} \qquad \forall x \in R^N. \tag{3.18}$$

Taking $M > K$ and the self-similar solution $U_{M,a}(x,t)$ of (3.1) defined as (3.3), since $\lim_{r\to+\infty} r^a f_M(r) = M > K$, there exists a positive constant R_0 such that

$$r^a f_M(r) > K \quad \text{for } r \ge R_0. \tag{3.19}$$

Setting $\gamma = f_{M(R_0)} = \min\{f_M(r) \mid r \in [0,R_0]\} > 0$, it is easy to verify that $\varphi(x) \le U_{M,a}(x,t)$ for all $x \in R^N$, where $t_0 \in (0,1)$ and $t_0^{-a\beta}\gamma > \|\varphi\|_\infty$.

Let $\lambda > 0$; then $w(x,t) = \lambda U_{M,a}(x, \lambda^{l+m(p-2)-1}t + t_0)$ is the solution of the following problem

$$\begin{gathered} w_t = \operatorname{div}\left(|\nabla w^m|^{p-2}\nabla w^l\right), \quad (x,t) \in R^N \times (0,+\infty), \\ w(x,0) = \lambda U_{M,a}(x,t_0) \ge \lambda\varphi(x), \quad x \in R^N. \end{gathered} \tag{3.20}$$

Taking $\eta = f_M(0)$ and noting that $f_M(r)$ is non-increasing, we have

$$\|w(x,t)\|_\infty = \eta\lambda\left(\lambda^{l+m(p-2)-1}t + t_0\right)^{-a\beta}. \tag{3.21}$$

Step 2. Set $v(x,t) = A(t)w(x,B(t))$, where $A(t)$ and $B(t)$ are solutions of the following problem:

$$\begin{gathered} A'(t) = \eta^{q-1}\lambda^{q-1}\left[\lambda^{l+m(p-2)-1}B(t) + t_0\right]^{-(a(q-1)/(a(l+m(p-2)-1)+p))} A^q(t), \quad t \in (0,+\infty), \\ B'(t) = A^{l+m(p-2)-1}(t) \quad t \in (0,+\infty), \\ A(0) = 1, \qquad B(0) = 0. \end{gathered} \tag{3.22}$$

By a direct calculation, we obtain that $v(x,t)$ satisfies

$$\begin{gathered} v_t \ge \operatorname{div}\left(|\nabla v^m|^{p-2}\nabla v^l\right) + v^q, \quad (x,t) \in R^N \times (0,+\infty), \\ v(x,0) = w(x,0) = \lambda U_{M,a}(x,t_0) \ge \lambda\varphi(x), \quad x \in R^N. \end{gathered} \tag{3.23}$$

Step 3. We shall prove that there exists a positive constant $\lambda_0 = \lambda_0(\varphi)$ such that the problem (3.22) has a global solution $(A(t),B(t))$ with $A(t)$ bounded in $[0,T)$ if $\lambda \in [0,\lambda_0)$. According to the standard theory of ODE, the local existence and uniqueness of solution $(A(t),B(t))$ of (3.22) hold. By (3.22), we have $A'(t) > 0, A(t) > 1$ for $t > 0$; furthermore, the solution is continuous as long as the solution exists and $A(t)$ is finite.

From (3.22), when $A(t)$ exists in $[0,t]$, then $B(t)$ is uniquely defined by

$$B(t) = \int_0^t A^{l+m(p-2)-1}(s)ds. \tag{3.24}$$

Since $p > 2$ and $A(t)$ is increasing, we obtain

$$B(s) = \int_0^s A^{l+m(p-2)-1}(\tau)d\tau \geq A^{l+m(p-2)-1}(0)s = s \quad \forall s \in [0,t]. \tag{3.25}$$

By (3.22), (3.25), and $a > a_c^* = p/(q-l-m(p-2))$, it follows that

$$\begin{aligned} 1 - A^{1-q}(t) &= (q-1)\eta^{q-1}\lambda^{q-1}\int_0^t \left[\lambda^{l+m(p-2)-1}B(s) + t_0\right]^{-(a(q-1)/(a(l+m(p-2)-1)+p))} ds \\ &\leq (q-1)\eta^{q-1}\lambda^{q-1}\int_0^t \left(\lambda^{l+m(p-2)-1}s + t_0\right)^{-(a(q-1)/(a(l+m(p-2)-1)+p))} ds \\ &\leq \frac{(q-1)\eta^{q-1}\lambda^{q-l-m(p-2)}}{((a(q-1))/(a(l+m(p-2)-1)+p)) - 1} t_0^{1-(a(q-1)/(a(l+m(p-2)-1)+p))}. \end{aligned} \tag{3.26}$$

Let $\lambda_0 = \lambda_0(\varphi)$ be a positive constant defined by

$$\frac{(q-1)\left[a(l+m(p-2)-1)+p\right]}{a(q-l-m(p-2))-p}\eta^{q-1}\lambda_0^{q-l-m(p-2)} t_0^{1-(a(q-1)/(a(l+m(p-2)-1)+p))} = \frac{1}{2}. \tag{3.27}$$

Then from (3.26), $q > q_c^* > l + m(p-2) > 1$ and $a > a_c^* = p/(q-l-m(p-2))$, we have $1 \leq A(t) \leq 2^{1/q-1}$ for any $\lambda \in (0, \lambda_0]$, as long as $A(t)$ exists globally.

On the other hand, by (3.22) and (3.25), we have

$$t \leq B(t) \leq 2^{(p-2)/(q-1)} t \quad \forall t \geq 0. \tag{3.28}$$

Therefore, $B(t)$ is also global.

Step 4. For any $\lambda \in (0, \lambda_0]$, where $\lambda_0 = \lambda_0(\varphi)$ is defined as (3.27), the solution $u(x,t)$ of (1.1) with initial value $u_0(x) = \lambda\varphi(x)$ exists globally, and $u(x,t) \leq v(x,t)$ in $R^N \times (0,+\infty)$. Moreover, there exists a positive constant C such that

$$\|u(\cdot,t)\|_\infty \leq \|v(.,t)\|_\infty \leq 2^{1/(q-1)}\eta\lambda\left(\lambda^{l+m(p-2)-1}B(t) + t_0\right)^{-a\beta} \leq Ct^{-a\beta} \quad \forall t > 0. \tag{3.29}$$

The proof of Theorem 1.4 is complete. □

4. Single Point Blow-Up

In this section, under some suitable assumptions, we shall prove that the blow-up set consists of the single point $x = 0$. Moreover, we also give the upper estimate of the solution $u(x,t)$ in a small neighborhood of the point (x,t), where $x = 0$, $t = T$.

First, we suppose that the solution is radially symmetric, that is, depending only on $r = |x|$ at a given time $t > 0$. Therefore, we study the following problem:

$$\begin{gathered} u_t = r^{-(N-1)}\left[r^{N-1}|(u^m)_r|^{p-2}\left(u^l\right)_r\right]_r + u^q, \quad (r,t) \in (0,+\infty) \times (0,+\infty), \\ u(r,0) = u_0(r), \quad r \in (0,+\infty), \\ r^{N-1}|(u^m)_r|^{p-2}\left(u^l\right)_r = 0, \quad (r,t) \in \{0\} \times [0,+\infty). \end{gathered} \tag{4.1}$$

Proof of Theorem 1.9. It is based on the method in [21]. The main idea is to apply the maximum principle to the auxiliary function $\omega_i(r,t)$, which is defined in (4.7), and to show that ω_i is small enough in $(0,i) \times (0,T)$. Then by integrating the obtained inequality and taking limit as $i \to \infty$, one can get upper bound of the solution $u(r,t)$. Therefore, we divide the proof into the following steps.

Step 1. Since problem (1.1) has no classical solution, we will construct the weak solution by means of regularization of the degenerate equation.

Now define a strictly monotone sequence $\{\varepsilon_i\}$, $\varepsilon_i > 0$ for all $i = 1,2,3,\ldots$, such that

$$\varepsilon_i \longrightarrow 0, \quad \text{as } i \longrightarrow +\infty. \tag{4.2}$$

Then, the weak solution $u(r,t)$ is the limit function of the solution of the following regularized problem (see [31]):

$$\begin{gathered} (u_i)_t = r^{-(N-1)}\left[r^{N-1}\left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-2)/2}\left(\left(u_i^l\right)_r + \varepsilon_i\right)\right]_r + u_i^q, \quad (r,t) \in (0,i) \times (0,T), \\ u_i(r,0) = u_0(r), \quad r \in (0,i), \\ u_i(i,t) = u_0(i), \quad t \in (0,T), \\ r^{N-1}\left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-2)/2}\left(\left(u_i^l\right)_r + \varepsilon_i\right) = 0, \quad (r,t) \in \{0\} \times (0,T). \end{gathered} \tag{4.3}$$

By the standard methods used in [1, 32], the uniform estimates for the passage to the limit which do not depend on ε_i are established. Therefore, for any fixed $\varepsilon_i > 0$, we may assume that, for all sufficiently large i, the function $u_i(r,t)$ satisfies the following conditions:

$$|u_i| \le M_1, \quad |(u_i)_r| \le M_2 \quad \text{in } Q_{i,T} = (0,i) \times (0,T), \tag{4.4}$$

where M_1, M_2 do not depend on i, and

$$(u_i(r,t))_r|_{r=0} = 0 \quad \forall t \in (0,T). \tag{4.5}$$

Moreover, by using condition (H) and the maximum principle in [33], we have

$$(u_i(r,t))_r \le 0 \quad \text{in } Q_{i,T}. \tag{4.6}$$

Step 2. Set

$$w_i(r,t) = r^{N-1}\left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-2)/2}\left(u_i^l\right)_r + \lambda_0 r^N u_i^q, \tag{4.7}$$

where λ_0 is given in (1.11).

By a direct calculation, we find that $w_i(r,t)$ satisfies the following parabolic equation:

$$(w_i)_t = a_i(w_i)_{rr} + b_i(w_i)_r + c_i w_i + d_i + e_i, \tag{4.8}$$

where

$$a_i = l u_i^{l-1}\left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-2)/2} + m(p-2)u_i^{m-1}\left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-4)/2}\left(u_i^l\right)_r (u_i^m)_r, \tag{4.9}$$

$$\begin{aligned} b_i = {} & \left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-2)/2}\left[\frac{l(1-N)u_i^{l-1}}{r} + l(l-1)u_i^{l-2}(u_i)_r\right] + (p-2)\left(u_i^l\right)_r (u_i^m)_r \\ & \cdot \left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-4)/2}\left[\frac{m(1-N)u_i^{m-1}}{r} + m(m-1)u_i^{m-2}(u_i)_r\right], \end{aligned} \tag{4.10}$$

$$c_i = q u_i^{q-1}\left[(p-1) - \lambda_0\left((p-2)(N+1)+2\right)\right], \tag{4.11}$$

$$d_i = Aqr^{N-1}\left[1-\lambda_0(N+1)\right]u_i^{q-1}, \quad \text{with } A \equiv -(p-2)\varepsilon_i^2\left(u_i^l\right)_r\left[\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right]^{(p-4)/2}, \tag{4.12}$$

$$\begin{aligned} e_i = {} & (p-2)r^{N-1}\left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-4)/2}\left(u_i^l\right)_r (u_i^m)_r \\ & \times \left[-(m+q-2)mqr\lambda_0 u_i^{q+m-3}((u_i)_r)^2 + \left((m-1)(1-\lambda_0 N) + q(1-\lambda_0(N+1))\right)mu^{q+m-2}(u_i)_r\right] \\ & + r^{N-1}\left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-2)/2}\left[-(l+q-2)lqr\lambda_0 u_i^{q+l-3}((u_i)_r)^2\right. \\ & \left. \qquad + \left((l-1)(1-\lambda_0 N) + q(1-\lambda_0(N+1))\right)lu^{q+l-2}(u_i)_r\right] \\ & + \lambda_0 q\left[\lambda_0\left((p-2)(N+1)+2\right) - (p-2)\right]r^N u_i^{2q-1}. \end{aligned} \tag{4.13}$$

Since $N \ge 1$, $p > 2$, $m > 1$, $l > 1$, and $q > l + m(p-2)$, it follows from (1.11) and (4.6) that $e_i \le 0$.

Now we consider the coefficients c_i and d_i in $Q_{i,T}$. By (4.6) and $p > 2$, we have

$$A = -(p-2)\varepsilon_i^2\left(u_i^l\right)_r\left[\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right]^{(p-4)/2} \ge 0. \tag{4.14}$$

It follows from (4.4) that $|A| = O(\varepsilon_i^{\nu})$ as $i \longrightarrow \infty$, where $\nu = \min\{2, p-1\}$, and we obtain the following estimate:

$$\sup c_i \leq M_3, \quad \sup d_i \leq M_4 i^{N-1}\varepsilon_i^{\nu} \quad \text{in } Q_{i,T}, \tag{4.15}$$

where M_3, M_4 are positive constants, which are independent of i.

Therefore, we have the parabolic differential inequality

$$(w_i)_t \leq a_i(w_i)_{rr} + b_i(w_i)_r + c_i w_i + d_i, \quad \text{in } Q_{i,T}, \tag{4.16}$$

where c_i, d_i satisfy (4.15).

Next, we consider the function $w_i(r,t)$ on the parabolic boundary of $Q_{i,T}$. At first, it is easy to see that $w_i(0,t) = 0$ for all $t \in (0,T)$. By (1.10), we have $w_i(i,t) \leq \lambda_0 i^N u_0^q(i) = o(1)$, as $i \longrightarrow \infty$ for all $t \in (0,T)$. Finally, it follows from (1.11) that

$$\begin{aligned} w_i(r,0) &= r^{N-1}\Big(\big((u_0^m)_r\big)^2 + \varepsilon_i^2\Big)^{(p-2)/2}\Big(u_0^l\Big)_r + \lambda_0 r^N u_0^q \\ &\leq r^{N-1}\Big[\big|(u_0^m)_r\big|^{p-2}\Big(u_0^l\Big)_r + \lambda_0 r u_0^q\Big] \leq 0 \quad \forall r \in [0,i]. \end{aligned} \tag{4.17}$$

Hence, for all sufficiently large i, there exists $\gamma_i = \sup w_i > 0$ on the parabolic boundary of $Q_{i,T}$ and $\gamma_i = o(1)$ as $i \longrightarrow \infty$.

In order to estimate $w_i(r,t)$ in $Q_{i,T}$, we study the following ODE:

$$\begin{gathered} \frac{d\overline{w}_i}{dt} = M_3\overline{w}_i + M_4 i^{N-1}\varepsilon_i^{\nu}, \quad t > 0, \\ \overline{w}_i(0) = \gamma_i, \end{gathered} \tag{4.18}$$

which has the solution

$$\overline{w}_i(t) = \left(\frac{M_4 i^{N-1}\varepsilon_i^{\nu}}{M_3} + \gamma_i\right)e^{M_3 t} - \frac{M_4 i^{N-1}}{M_3}\varepsilon_i^{\nu}, \quad t > 0. \tag{4.19}$$

Taking the sequence ε_i such that

$$i^{N-1}\varepsilon_i^{\nu} \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty, \tag{4.20}$$

it is obvious that

$$\overline{w}_i(t) \leq \overline{w}_i(T) = \alpha_i \quad \forall t \in (0,T), \tag{4.21}$$

where

$$\alpha_i \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty. \tag{4.22}$$

Setting

$$z(r,t) = w_i(r,t) - \overline{w}_i(t), \tag{4.23}$$

we have $z(r,t) \le 0$ on the parabolic boundary, and $z(r,t)$ satisfies the following parabolic inequality:

$$z_t \le a_i z_{rr} + b_i z_r + c_i z - (M_3 - c_i)\overline{w}_i + \left[d_i - M_4 i^{N-1}\varepsilon_i^{\nu}\right], \quad \text{in } Q_{i,T}. \tag{4.24}$$

It follows from (4.15) that

$$z_t \le a_i z_{rr} + b_i z_r + c_i z, \quad \text{in } Q_{i,T}. \tag{4.25}$$

By the maximum principle (Chapter II, [33]), we obtain that $z(r,t) \le 0$ in $Q_{i,T}$, that is,

$$w_i(r,t) \le \overline{w}_i(t) \le \overline{w}_i(T) = \alpha_i \quad \text{in } Q_{i,T}. \tag{4.26}$$

Step 3. For large i and $(u_i(r,t))_r \in [-M_2, 0]$, we have the following estimate

$$\left(\left((u_i^m)_r\right)^2 + \varepsilon_i^2\right)^{(p-2)/2}\left(u_i^l\right)_r \ge \left|(u_i^m)_r\right|^{p-2}\left(u_i^l\right)_r - \delta_i, \tag{4.27}$$

where

$$0 < \delta_i = O(\varepsilon_i^{\nu}) \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty. \tag{4.28}$$

By (4.7) and (4.26), we have

$$\frac{\left|(u_i^m)_r\right|^{p-2}\left(u_i^l\right)_r}{u_i^q} \le \frac{r^{1-N}\alpha_i + \delta_i}{u_i^q} - \lambda_0 r, \tag{4.29}$$

namely,

$$l m^{p-2} u_i^{(l-1)+(m-1)(p-2)-q}(u_i)_r \left|(u_i)_r\right|^{p-2} \le \frac{r^{1-N}\alpha_i + \delta_i}{u_i^q} - \lambda_0 r. \tag{4.30}$$

Letting $F(b) = b|b|^{-((p-2)/(p-1))}$, we deduce that

$$\left(l m^{p-2}\right)^{1/(p-1)} u_i^{((l+m(p-2)-q-p+1)/(p-1))}(u_i)_r \le F\left[\frac{r^{1-N}\alpha_i + \delta_i}{u_i^q} - \lambda_0 r\right]. \tag{4.31}$$

For arbitrary $t_0 \in (0,T)$, $r_0 \in \operatorname{supp} u(r,t_0) \equiv \{r > 0 \mid u(r,t_0) > 0\}$, we denote $\mu_0 = u(x_0,t_0) > 0$. By (4.6) and the uniform convergence $u_i(r,t_0) \to u(r,t_0)$ as $i \to \infty$, $r \in [0,r_0]$, we have $u_i(r,t_0) \geq (\mu_0/2)$, $r \in [0,r_0]$ for all sufficiently large $i \geq i_0$. Therefore, from (4.31) we obtainly for $t = t_0$, $i > i_0$,

$$\left(lm^{p-2}\right)^{1/(p-1)} u_i^{((l+m(p-2)-q-p+1)/(p-1))} (u_i)_r \leq F\left[\frac{2^q\left(r^{1-N}\alpha_i + \delta_i\right)}{\mu_0^q} - \lambda_0 r\right]. \tag{4.32}$$

Integrating the above inequality over interval $[r_1, r_0]$, where $r_1 > 0$, we obtain

$$-\frac{\left(lm^{p-2}\right)^{1/(p-1)}(p-1)}{q-l-m(p-2)} u_i^{-((q-l-m(p-2))/(p-1))}\Big|_{r_1}^{r_0} \leq \int_{r_1}^{r_0} F\left[\frac{2^q\left(r^{1-N}\alpha_i + \delta_i\right)}{\mu_0^q} - \lambda_0 r\right] dr. \tag{4.33}$$

Letting $i \to \infty$, by (4.22), (4.28), and the uniform convergence $u_i(r,t_0) \to u(r,t_0)$ as $i \to \infty$, $r \in [0,r_0]$, we have the following estimate:

$$\begin{aligned}
&-\frac{\left(lm^{p-2}\right)^{1/(p-1)}(p-1)}{q-l-m(p-2)}\left[u^{-((q-l-m(p-2))/p-1)}(r_0,t_0) - u^{-((q-l-m(p-2))/p-1)}(r_1,t_0)\right] \\
&\leq \lim_{i\to\infty} \int_{r_1}^{r_0} F\left[\frac{2^q\left(r^{1-N}\alpha_i + \delta_i\right)}{\mu_0^q} - \lambda_0 r\right] dr \\
&= -\lambda_0^{1/(p-1)} \frac{p-1}{p}\left(r_0^{p/(p-1)} - r_1^{p/(p-1)}\right).
\end{aligned} \tag{4.34}$$

Setting $r_1 \to 0$, from (4.34), we obtain the upper estimate

$$u(r_0,t_0) \leq \left[\frac{q-l-m(p-2)}{p\left(lm^{p-2}\right)^{1/p-1}} \lambda_0^{1/p-1}\right]^{-((p-1)/(q-l-m(p-2)))} r_0^{-(p/(q-l-m(p-2)))}, \tag{4.35}$$

where $(r_0,t_0) \in R_+^1 \times (0,T)$. The proof of Theorem 1.9 is complete. □

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