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*Research Article*

# Kink Waves and Their Evolution of the RLW-Burgers Equation

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This paper considers the bounded travelling waves of the RLW-Burgers equation. We prove that there only exist two types of bounded travelling waves, the monotone kink waves and the oscillatory kink waves. For the oscillatory kink wave, the regularity of its maximum oscillation amplitude changing with parameters is discussed. Exact expressions of the monotone kink waves and approximate expressions of the oscillatory ones are obtained in some special cases. Furthermore, all bounded travelling waves of the RLW-Burgers equation under different parameter conditions are identified and the evolution of them is discussed to explain the corresponding physical phenomena.

## 1. Introduction

The RLW-Burgers equation,

$$u_t + \alpha u_x + \beta u u_x - \mu u_{xx} - \delta u_{xxt} = 0, \tag{1.1}$$

is put forward to describe propagation of surface water waves in a channel [1], where all variables are rescaled with $x$ proportional to the horizontal coordinate along the channel, $t$ proportional to the elapsed time, and $u(x,t)$ proportional to the vertical displacement of the surface of the water from its equilibrium position. In (1.1) constant $\beta$ characterizes the nonlinearity. Constants $\mu$ and $\delta$ are dissipative and dispersive coefficients, respectively.

In particular, when the Burgers-type dissipative term $\mu u_{xx}$ disappears, (1.1) becomes the regularized long-wave (RLW) equation [2]:

$$u_t + \alpha u_x + \beta u u_x - \delta u_{xxt} = 0. \tag{1.2}$$

In 1981, Bona et al. [1] developed a numerical scheme to solve (1.1) and found that the model could give quite a good description of the spatial and temporal development of periodically generated waves. In 1989 Amick et al. [3] discussed large-time behavior of solutions to the initial-value problem of (1.1) and used the methods such as energy estimates, a maximum principle, and a transformation of Cole-Hopf type to obtain sharp rates of temporal decay of certain norms of the solution. Later, travelling wave solutions of (1.1) were considered due to their important roles in understanding the complicated nonlinear wave phenomena and long-time behavior of solution. People paid more attention to some special exact travelling wave solutions of (1.1) because of the nonintegrability of travelling wave system of it. In [4], Zhang and Wang gave an exact solution of (1.1) for $\alpha = 0$, $\beta = 1$ by the method of undetermined coefficient in 1992. Later, Wang [5] gave a kink-shape exact solutions of (1.1) for $\alpha = 1$, $\beta = 12$ by reducing it to the equation of homogeneous form with a function transformation.

Though there have been some profound results about travelling wave solutions of (1.1) which contributed to our understanding of nonlinear physical phenomena and wave propagation, there still exist some unresolved problems from the viewpoint of physics. For instance, are there other types of bounded travelling waves such as solitary waves, periodic waves, and oscillatory travelling waves? If they exist, how do they evolve? How does the oscillatory amplitude of the oscillatory travelling waves vary with dissipative and dispersive parameters? How can we get their exact expressions and plot their wave profiles? To answer these questions, we need to figure out how the travelling wave solutions of (1.1) depending on the parameters. In fact, it has involved bifurcation of travelling wave solutions. In general, three basic types of bounded travelling waves could occur for a PDE, which are periodic waves, kink waves, and solitary waves. Sometimes, they are also called periodic wave trains, fronts, and pulses, respectively. Recall that heteroclinic orbits are trajectories which have two distinct equilibria as their $\alpha$ and $\omega$-limit sets and homoclinic orbits are trajectories whose $\alpha$ and $\omega$-limit sets consist of the same equilibrium. So, the three basic types of bounded travelling waves mentioned above correspond to periodic, heteroclinic, and homoclinic orbits of the travelling wave system of a PDE, respectively, (see [6, 7]). It is just the relationship that make the bifurcation theory of dynamical system become an effective method to investigate bifurcations of travelling waves of PDEs. In recent decade, many efforts have been devoted to bifurcations of travelling waves of PDEs since it is an effective method to investigate bounded travelling waves. In 1997 Peterhof et al. [8] investigated persistence and continuation of exponential dichotomies for solitary wave solutions of semilinear elliptic equations on infinite cylinders so that Lyapunov-Schmidt reduction can be applied near solitary waves. Sánchez-Garduño and Maini [9] considered the existence of one-dimensional travelling wave solutions in nonlinear diffusion degenerate Nagumo equations and employed a dynamical systems approach to prove the bifurcation of a heteroclinic cycle. Later Katzengruber et al. [6] analyzed the bifurcation of travelling waves such as Hopf bifurcation, multiple periodic orbit bifurcation, homoclinic bifurcation and heteroclinic bifurcation in a standard model of electrical conduction in extrinsic semiconductors, which in scaled variables is actually a singular perturbation problem of a 3-dimensional ODE system. In 2002 Constantin and

Strauss [10] constructed periodic travelling waves with vorticity for the classical inviscid water wave problem under the influence of gravity, described by the Euler equation with a free surface over a flat bottom, and used global bifurcation theory to construct a connected set of such solutions, containing flat waves as well as waves that approach flows with stagnation points. In 2003 Huang et al. [11] employed the Hopf bifurcation theorem to established the existence of travelling front solutions and small amplitude travelling wave train solutions for a reaction-diffusion system based on a predator-prey model, which are equivalent to heteroclinic orbits and small amplitude periodic orbits in $\mathbb{R}^4$, respectively. Besides, many results on bifurcations of travelling waves for Camassa-Holm equation, modified dispersive water wave equation, and KdV equation can be found from [12–15].

Motivated by the reasons above, we try to seek all bounded travelling waves of the RLW-Burgers equation and investigate their dynamical behaviors. By some techniques including analyzing the $\omega$-limit set of unstable manifold, investigating the degenerate equilibria at infinity to give global phase portrait, and so forth, we obtain existence and uniqueness of bounded travelling waves of the RLW-Burgers equation. We prove that there only exist two types of bounded travelling waves for the RLW-Burgers equation, a type of monotone kink waves and a type of oscillatory ones. For the oscillatory kink wave, the regularity of its maximum oscillation amplitude changing with parameters is discussed. In addition, exact and approximate expressions for the monotone kink waves and the oscillatory ones are obtained, respectively, by tanh function method in some special cases. By these results, all bounded travelling waves of the RLW-Burgers equation are identified under different parameter conditions. Furthermore, evolution of the two types of bounded travelling waves is discussed to explain the corresponding physical phenomena. It shows that the ratio $\mu/\delta$ and the travelling wave velocity $c$ are critical factors to affect the evolution of them.

## 2. Preliminaries

It is well known that the travelling wave solution has the form $u(x,t) = u(x-ct)$, where $c \neq 0$ is the wave velocity. So, we can make the transformation $\xi = x - ct$ to change (1.1) into its corresponding travelling wave system

$$\delta c u''' - \mu u'' + (\alpha - c)u' + \beta u u' = 0, \tag{2.1}$$

where $'$ denotes $d/d\xi$. Integrating (2.1) once, we get

$$u'' - g u' - e u - f u^2 = 0, \tag{2.2}$$

which has the equivalent form

$$\begin{aligned} u' &= v = P(u,v), \\ v' &= eu + gv + fu^2 = Q(u,v), \end{aligned} \tag{2.3}$$

where $e = (c-a)/\delta c$, $g = \mu/\delta c$, and $f = -\beta/2\delta c$.

In the following discussion, without loss of generality, we only need to consider the case $e > 0$, $g < 0$, and $f < 0$. In fact, if $e < 0$, we can make the transformation $u = U - e/f$, $v = v$ which converts (2.3) into

$$U' = v,$$
$$v' = EU + gv + fU^2, \tag{2.4}$$

where $E = -e > 0$, that is, the case $e > 0$ for system (2.3). If $g > 0$, we can make transformation $v = -V$, $\xi = -\tau$ which converts (2.3) into

$$u' = V,$$
$$V' = eu + GV + fu^2, \tag{2.5}$$

where $G = -g < 0$, that is, the case $g < 0$ for system (2.3). Similarly, if $f > 0$, we can make transformation $u = -U$, $v = -V$ which converts (2.3) into

$$U' = V,$$
$$V' = eU + gV + FU^2, \tag{2.6}$$

where $F = -f < 0$, that is, the case $f < 0$ for system (2.3).

System (2.3) has two equilibria $E_1(0,0)$ and $E_2(-e/f, 0)$ with the Jacobian matrices, respectively,

$$J(E_1) := \begin{pmatrix} 0 & 1 \\ e & g \end{pmatrix}, \qquad J(E_2) := \begin{pmatrix} 0 & 1 \\ -e & g \end{pmatrix}. \tag{2.7}$$

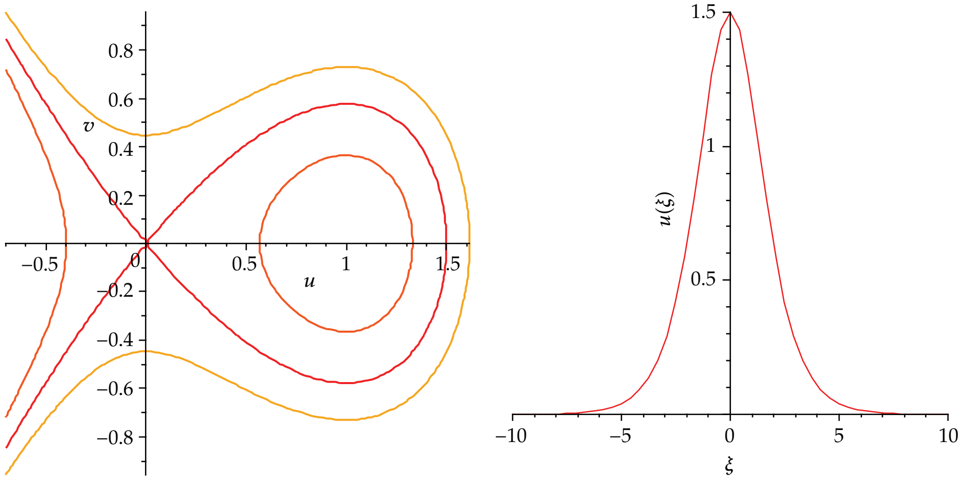
Obviously, $E_1$ is a saddle and $E_2$ is a stable node (resp., focus) for $g^2 - 4e \geq 0$ (resp., $g^2 - 4e < 0$).

As a special case, when $g = 0$, $E_1$ is a saddle and $E_2$ is a center. In fact, in this case system (2.3) is a Hamiltonian system with the first integral

$$H(u,v) := \frac{1}{2}v^2 - \frac{e}{2}u^2 - \frac{f}{3}u^3. \tag{2.8}$$

By the properties of planar Hamiltonian system, we know there is a unique homoclinic orbit $\Upsilon^0$ connecting the saddle $E_1(0,0)$. Taking $e = 1$, $f = -1$, we can give the global phase portrait of system (2.3) in Figure 1(a). The homoclinic orbit $\Upsilon^0$ corresponds to the bell-shape solitary wave of system (1.1) as shown in Figure 1(b).

The homoclinic orbit $\Upsilon^0$ corresponds to the level curve $(1/2)v^2 - (e/2)u^2 - (f/3)u^3 = 0$ which intersects $u$-axis at the point $(u_0, 0)$, where $u_0 = -3e/2f$. Letting $u(0) = u_0$, from




(a) The phase portrait of system (2.3) for $g = 0$, $e = 1$, $f = -1$

(b) The solitary wave of system (1.1) corresponding to homoclinic orbit $\Upsilon^0$ of system (2.3)

**Figure 1:** Homoclinic orbit and solitary wave.

the first equation of system (2.3), we can compute the expression of the bell-shape solitary wave as follows:

$$u(\xi) = \frac{-3e}{2f}\operatorname{sech}^2\left(\frac{\sqrt{e}}{2}\xi\right), \quad \text{for } \xi \in (-\infty, +\infty), \tag{2.9}$$

by two integrals

$$\begin{gathered}\int_u^{u_0} \frac{du}{u\sqrt{e + (2f/3)u}} = \int_\xi^0 d\xi, \quad \text{for } \xi < 0, \\ \int_{u_0}^{u} \frac{du}{-u\sqrt{e + (2f/3)u}} = \int_0^\xi d\xi, \quad \text{for } \xi > 0.\end{gathered} \tag{2.10}$$

## 3. The Existence and Uniqueness of Bounded Travelling Waves

By the Bendixon Theorem, for system (2.3), the expression

$$\frac{\partial P(u,v)}{\partial u} + \frac{\partial Q(u,v)}{\partial v} = g, \tag{3.1}$$

has a fixed sign when $g \neq 0$. It means that system (2.3) has neither closed orbit nor singular closed orbit (homoclinic loop and heteroclinic loop) when $g \neq 0$. So, all bounded travelling waves of system (1.1) can only correspond to heteroclinic orbits connecting the two equilibria $E_1$ and $E_2$. Furthermore, if there exists such a heteroclinic orbit, it is unique (otherwise,

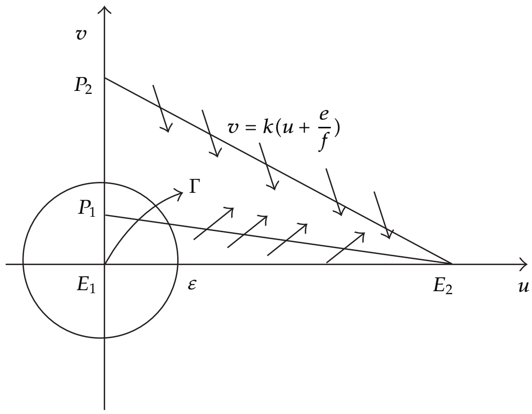



**Figure 2:** The triangle sector.

a heteroclinic loop will arise). Hence, to seek the bounded travelling waves of (1.1) is equivalent to seek the heteroclinic orbits of the system

$$\begin{gathered} u' = v, \\ v' = eu + gv + fu^2, \\ u(-\infty) = 0, \quad u(+\infty) = -\frac{e}{f}. \end{gathered} \tag{3.2}$$

**Theorem 3.1.** *Suppose that $e > 0$, $g < 0$ and $f < 0$. Then system* (1.1) *has either a unique monotone increasing bounded kink wave solution if $g^2 - 4e \geq 0$ or a unique bounded damped oscillatory kink wave solution if $g^2 - 4e < 0$.*

*Proof.* In the case $e > 0$, $g < 0$, $f < 0$, and $g^2 - 4e \geq 0$, the $E_1(0,0)$ is a saddle and $E_2(-e/f, 0)$ is a stable node. Furthermore, by [16], there is an unstable manifold Γ of the saddle $E_1$ in first quadrant, which intersects neither the $u$-axis nor the $v$-axis in the neighborhood $U(0,\varepsilon)$ for $\varepsilon$ small enough. Take a line $L_1 : v = k(u + e/f)$ with the constant $k < 0$, which intersects $v$-axis at point $P_2$. A triangle region is formed by the three lines $u = 0$, $v = 0$, and $L_1$ as shown in Figure 2. If Γ cannot go out of the triangle region, it will tend to $E_2$, since there is no periodic closed orbit and singular closed orbit. It means we need to prove $E_2$ is the $\omega$-limit set of Γ. □

From the vector field defined by (2.3), orbits in first quadrant can only go right when $\xi$ increases. It means that Γ can not intersect the line $u = 0$.

Assume that Γ intersects the boundary $u^2 + v^2 = \varepsilon^2$ of the neighborhood $U(0,\varepsilon)$ at the point $(u_0, v_0)$. Obviously, $u_0 > 0$, $v_0 > 0$. Take a point $P_1(0, v^*)$, $0 < v^* < \varepsilon$. So, on the line segment $\overline{P_1E_2}$: $v = (v^* f/e)u + v^*$, $u \in [0, -e/f]$, we have

$$\frac{dv}{du}\Big|_{\overline{P_1E_2}} = \frac{fu^2 + (e + (gv^* f/e))u + v^* g}{(v^* f/e)u + v^*}. \tag{3.3}$$

The denominator $(v^* f/e)u + v^* > 0$ for $u \in (0, -e/f)$. The numerator have two zeroes $u_1 = -gv^*/e$ and $u_2 = -e/f$. If there exists a $v^*$ which satisfies the relations of both $-gv^*/e < u_0$ and $v^* f u_0/e + v^* \leq v_0$, then $(dv/du)|_{\overline{P_1E_2}} > 0$ for $u \in (u_0, -e/f)$. It means that $\Gamma$ cannot intersect the line segment $\overline{P_1E_2}$. Therefore, it impossibly intersects the line segment $\overline{E_1E_2}$. In fact, we can take $\varepsilon$ so small that $0 < (fu_0/e) + 1 < 1$. So $v^*$ can be chosen by $0 < v^* < \min\,(-eu_0/g, v_0/(1 + fu_0/e))$.

On the line segment $\overline{P_2E_2}$: $v = k(u + e/f)$, $u \in [0, -e/f]$, we have

$$\frac{dv}{du}\Big|_{\overline{P_2E_2}} = \frac{F(u)}{k(u + e/f)} + g, \tag{3.4}$$

where $F(u) = fu^2 + eu$. From the fact $F(-e/f) = 0$,

$$\frac{F(u)}{u + e/f} = \frac{F(u) - F(-e/f)}{[u - (-e/f)]} > F'\left(-\frac{e}{f}\right) = -e. \tag{3.5}$$

So, $(dv/du)|_{\overline{P_2E_2}} = F(u)/k(u + e/f) + g < -e/k + g$. There exists a constant $k$ which satisfies $-e/k + g < k$ since $g^2 - 4e > 0$. Hence, we can choose the constant $k$ in the interval $((g - \sqrt{g^2 - 4e})/2), ((g + \sqrt{g^2 - 4e})/2))$ to make $\Gamma$ not intersect the line segment $\overline{P_2E_2}$.

Now, we can see that $E_2$ is exactly the $\omega$-limit set of $\Gamma$. So, $\Gamma$ is the unique heteroclinic orbit connecting $E_1$ and $E_2$. Moreover, from the proof, we can see $du/d\xi = v > 0$, which means that the bounded kink wave solution corresponding to $\Gamma$ is monotone increasing with respect to $\xi$.

In the case $e > 0$, $g < 0$, $f < 0$, and $g^2 - 4e < 0$, $E_1$ and $E_2$ are a saddle and a stable focus, respectively. We need to discuss (2.3) globally. By the Poincaré transformation $u = 1/y$, $v = x/y$ and $d\tau = d\xi/y$, (2.3) can be changed into

$$\begin{aligned} x' &= f + ey + gxy - x^2 y, \\ y' &= -xy^2, \end{aligned} \tag{3.6}$$

which has no equilibrium in the $(x, y)$-plane.

Then by another Poincaré transformation $u = x/y$, $v = 1/y$, and $d\tau = d\xi/y$, (2.3) can be changed into

$$\begin{aligned} x' &= y + P_2(x, y), \\ y' &= Q_2(x, y), \end{aligned} \tag{3.7}$$

where $P_2(x, y) = -gxy - ex^2 y - fx^3$, $Q_2(x, y) = -gy^2 - exy^2 - fx^2 y$. We only need to consider the equilibrium $(0, 0)$ of system (3.7), which corresponds to the equilibria $E_{+\infty}$ and $E_{-\infty}$ at infinity in $v$-axis, seen in Figure 4. One can check that $(0, 0)$ is a degenerate equilibrium with nilpotent Jacobian matrix. So, we need more precise analysis for it.

Letting $y + p_2(x, y) = 0$, we can obtain that the implicit function

$$\phi(x) = fx^3 + fgx^4 + O\left(x^5\right). \tag{3.8}$$

Then, we have

$$\begin{aligned} \Psi(x) &= Q_2(x, \phi(x)) = -f^2x^5 - 2f^2gx^6 + O\left(x^7\right), \\ \delta(x) &= \frac{\partial P_2(x, \phi(x))}{\partial x} + \frac{\partial Q_2(x, \phi(x))}{\partial y} = -4fx^2 - 3gfx^3 + O\left(x^4\right). \end{aligned} \tag{3.9}$$

By Theorem 7.2 and its corollary in [17], we know that $k = 2m + 1 = 5$, $m = 2$, $a_k = -f^2 < 0$, $n = 2$, $b_n = -4f > 0$, $\lambda = (-4f)^2 + 4(m + 1)(-f^2) = 4f^2 > 0$, which means that the degenerate equilibrium $(0, 0)$ is an unstable degenerate node. Correspondingly, $E_{+\infty}$ is an unstable degenerate node, whereas $E_{-\infty}$ is a stable degenerate node.

Further, we need to judge the behaviors of orbits in $(u, v)$-plane. From system (2.3), two curves $v = 0$ and $eu + gv + fu^2 = 0$ divide $(u, v)$-plane to five regions. In each region, we need to judge the signs of $du/d\xi$ and $dv/d\xi$, which determine the behaviors of orbits. We show our results in Figure 3.

Next, we prove the existence of a saddle-focus heteroclinic orbit. In fact, from [16], there exist four invariant manifolds near the saddle $E_1(0, 0)$, which are, respectively, the unstable manifold $\Gamma_1^+$ in the first quadrant, the stable manifold $\Gamma_2^-$ in the second quadrant, the unstable manifold $\Gamma_3^+$ in the third quadrant and the stable manifold $\Gamma_4^-$ in the fourth quadrant. Since there is no closed orbits and singular closed orbits in whole $(u, v)$-plane, $E_{+\infty}$ is the common $\alpha$-limit set of $\Gamma_2^-$ and $\Gamma_4^-$, that is, $\Gamma_2^-$ and $\Gamma_4^-$ will tend to $E_{+\infty}$ when $\xi \to -\infty$. From the fact that $du/d\xi > 0$ and $dv/d\xi < 0$ in the second quadrant, one can see the unstable manifold $\Gamma_2^-$ can not go out of the second quadrant for $\xi \in (-\infty, +\infty)$. Further, noting that the signs of $du/d\xi$ and $dv/d\xi$ in regions I, III, IV, and V, when $\xi \to -\infty$, one can check that $\Gamma_4^-$ will cross the $u$-axis from the right hand of the focus $E_2(-e/f, 0)$ to tend to $E_{+\infty}$. So, $\Gamma_2^- \cup \Gamma_4^- \cup E_{+\infty} \cup E_1$ forms a boundary of a closed region which contains the unstable manifold $\Gamma_1^+$ and the focus $E_2$. By using the result that there is no closed orbits and singular closed orbits in whole $(u, v)$-plane again, we can come to the conclusion that the $\omega$-limit set of $\Gamma_1^+$ is the stable focus $E_2$. Thus, we prove the existence of saddle-focus heteroclinic orbit.

In addition, from Figure 3, one can check that the unstable manifold $\Gamma_3^+$ can not go out of the third quadrant for $\xi \in (-\infty, +\infty)$ and therefore tends to its $\omega$-limit set $E_{-\infty}$ when $\xi \to +\infty$. Now we are in a position to give the rough global phase portrait of system (2.3) in Figure 4.

In fact, the saddle-focus heteroclinic orbit shown by us corresponds to the oscillatory kink wave. These points on the right (left) hand side of focus $E_2$, where the saddle-focus heteroclinic orbit intersects the $u$-axis, correspond to the peaks (valleys) of the oscillatory kink wave. Let $(u_n, 0)$ be the point where the saddle-focus heteroclinic orbit intersects the $u$-axis for the $n$th time. Obviously, $u_1 > u_2 > u_3 > \cdots$, since $E_2$ is a stable focus. It means that the oscillatory kink wave is a damped wave.

**Theorem 3.2.** *For the oscillatory kink wave in Theorem 3.1, the maximal oscillation amplitude of it is increasing with respect to the parameter $g$.*

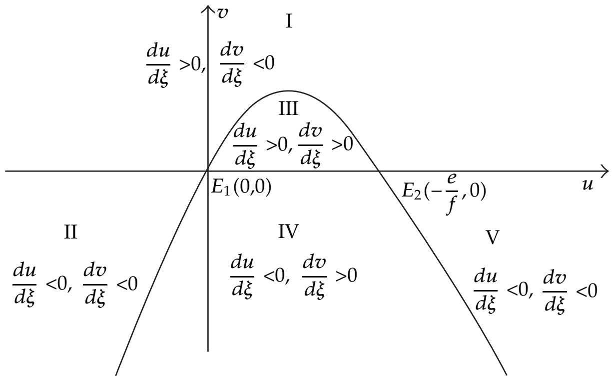



**Figure 3:** Signs of $du/d\xi$ and $dv/d\xi$ in different regions.

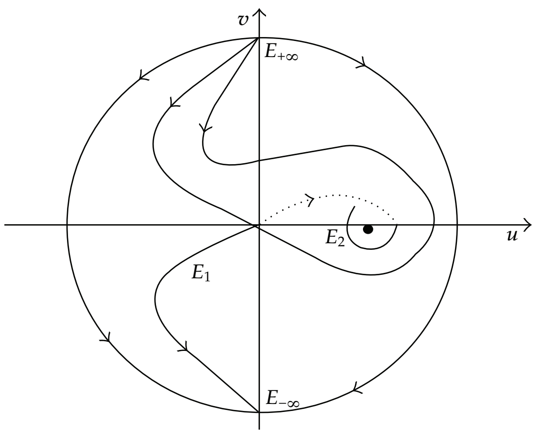



**Figure 4:** Global phase portrait of (2.3) for $e > 0$, $g < 0$, and $g^2 - 4e < 0$.

*Proof.* Let $(u^*, 0)$ be the point at which the saddle-focus heteroclinic orbit firstly intersects the $u$-axis when $\xi = \xi_0$. So, $u^*$ corresponds to the maximal oscillation amplitude of the oscillatory kink wave. Consider $\Gamma^*$, shown by dotted curve in Figure 3, which is the part of the saddle-focus heteroclinic orbit for $\xi \in (-\infty, \xi_0)$. It is equivalent to consider the solution $v(u)$ of the following problem:

$$\begin{gathered} \frac{dv}{du} - \frac{F(u)}{v} = g, \quad u \in (0, u^*), \\ v(0) = 0, \\ v(u) > 0, \end{gathered} \tag{3.10}$$

where $F(u) = fu^2 + eu$.

Assume that $v_1(u)(u \in (0,u_1^*))$ and $v_2(u)(u \in (0,u_2^*))$ satisfy (3.10) for $g = g_1$ and $g = g_2$, respectively, where $g_1 < g_2$. Let $p^* = \min(u_1^*, u_2^*)$. Consider the problem

$$\begin{gathered}\frac{dv_i}{du} - \frac{F(u)}{v_i} = g_i, \quad u \in (0,p^*), \\ v_i(0) = 0, \\ v_i(u) > 0,\end{gathered} \tag{3.11}$$

where $i = 1,2$.

Construct two functions

$$\begin{gathered}M(u) = \exp\left(\int_\delta^u \frac{F(t)}{v_1(t)v_2(t)}dt\right), \\ N(u) = (v_1(u) - v_2(u))M(u),\end{gathered} \tag{3.12}$$

where $0 < u < p^*$ and constant $\delta \in (0,-e/f)$. Noting that

$$(v_1 - v_2)' + \frac{F(u)}{v_1 v_2}(v_1 - v_2) = g_1 - g_2, \tag{3.13}$$

we have $dN/du = (g_1 - g_2)M < 0$ for $u \in (0,p^*)$. Furthermore, $\lim_{u\to 0^+} N(u) = 0$ since $\int_\delta^0 F(t)/v_1(t)v_2(t)dt$ either diverges to $-\infty$ or converges. It means that $N(u) < 0$ for $u \in (0,p^*)$ that is, $v_1(u) < v_2(u)$ for $u \in (0,p^*)$. So, $u_1^* \le u_2^*$. □

## 4. Explicit Expressions of Monotone and Oscillatory Kink Waves

It is difficult to give all exact expressions of the monotone kink waves under the conditions required in Theorem 3.1. But for some special case, for example, $e = 6g^2/25$, it can be found.

Next, we will apply the extended tanh-function method [18] to deal with the problem. Firstly, we guess that the monotone kink wave can be expressed as a finite series of tanh function. Noting that the fact that the Riccati equation:

$$v'(\xi) = b - v(\xi)^2, \quad b > 0 \tag{4.1}$$

has the particular solution $v(\xi) = \sqrt{b}\tanh(\sqrt{b}\xi)$, we suppose that the exact expression of the monotone kink wave has the form

$$u(\xi) = \sum_{i=0}^{n} a_i v(\xi)^i, \tag{4.2}$$

where $v(\xi)$ satisfies (4.1), $n$ and $a_i$ $(i = 0,1\ldots n)$ are constants to be determined later.

Substituting (4.2) into (2.2) and replacing $v'(\xi)$ by $b - v(\xi)^2$ repeatedly, we can obtain a identity with respect to $v(\xi)$. Here, as an example, we calculate the highest nonlinear term $u^2$ and the highest order derivative term $u''$ as follows:

$$\begin{aligned}
u^2 &= \left(\sum_{i=0}^{n} a_i v^i\right)^2, \\
u'' = (u')' &= \left(\sum_{i=1}^{n} i a_i v^{i-1} v'\right)' = \left(\sum_{i=1}^{n} i a_i v^{i-1}\left(b - v^2\right)\right)' \\
&= \left(\sum_{i=1}^{n} i b a_i v^{i-1} - \sum_{i=1}^{n} i a_i v^{i+1}\right)' \\
&= \sum_{i=2}^{n} i(i-1) b a_i v^{i-2} v' - \sum_{i=1}^{n} i(i+1) a_i v^i v' \\
&= \sum_{i=2}^{n} i(i-1) b a_i v^{i-2}\left(b - v^2\right) - \sum_{i=1}^{n} i(i+1) a_i v^i \left(b - v^2\right).
\end{aligned} \tag{4.3}$$

One can check that the highest degree of $u^2$ is $2n$, whereas the highest degree of $u''$ is $n+2$. In order to determine parameter $n$, we need to balance $u^2$ and $u''$ according to the method in [18, 19]. It requires that the highest degrees of $u^2$ and $u''$ should be equal, that is, $2n = n + 2$. Obviously, it is easy to see that $n = 2$. Thus, the identity mentioned above has the form

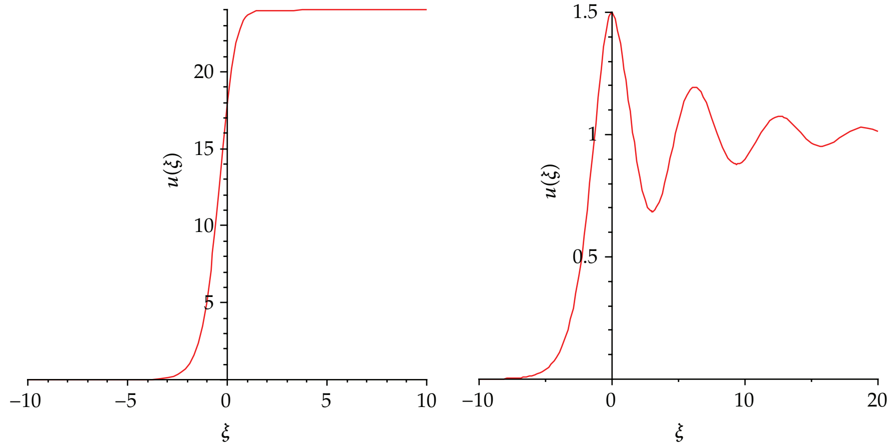
$$\begin{aligned}
&\left(6\ a_2 - f a_2{}^2\right) v^4 + \left(2\ g a_2 + 2\ a_1 - 2\ f a_1\ a_2\right) v^3 \\
&\quad + \left(-8\ a_2\ b + g a_1 - f\left(2\ a_0\ a_2 + a_1{}^2\right) - e\ a_2\right) v^2 \\
&\quad + \left(-2\ a_1\ b - 2\ g a_2\ b - e\ a_1 - 2\ f a_0\ a_1\right) v - f a_0{}^2 \\
&\quad + 2\ a_2\ b^2 - g a_1\ b - e\ a_0 \equiv 0.
\end{aligned} \tag{4.4}$$

Letting all coefficients of $v^i$ $(i = 0, 1, 2, 3, 4)$ be zero, we can obtain $a_0 = -9g^2/50f$, $a_1 = 6g/5f$, $a_2 = 6/f$, $b = g^2/100$, and $e = 6g^2/25$. Substituting these parameters back into (4.2) and the particular solution of (4.1), we obtain a solution of (2.2) expressed by

$$u(\xi) = -\frac{3g^2}{50f}\left(2 + 2\tanh\left(\frac{g}{10}\xi\right) + \operatorname{sech}^2\left(\frac{g}{10}\xi\right)\right). \tag{4.5}$$

One can check that $u(\xi) \to 0$ when $\xi \to -\infty$ and $u(\xi) \to -e/f$ when $\xi \to +\infty$. So, by the uniqueness, it is the exact expression of monotone increasing kink wave of (1.1) corresponding to the heteroclinic orbit of system (2.3) connecting the two equilibria $E_1(0,0)$ and $E_2(-e/f, 0)$. Taking $g = -10$, $f = -1$, we can give the picture of the solution in Figure 5(a).

In contrast to the monotone kink wave, it is more difficult to give the exact expression of the oscillatory kink wave solution. Even when $|g|$ is small, we can only give an approximate solution of it. In fact, the saddle-focus heteroclinic orbit is generated by the unstable manifold




(a) The monotone increasing kink wave (b) The damped oscillatory kink wave

**Figure 5:** The monotone and oscillatory kink wave.

of saddle $E_1(0,0)$ when the homoclinic orbit discussed in Section 2 breaks. So, when $|g|$ is small enough, enlightened by the expression of homoclinic orbit in Section 2, we can assume that the approximate expression of the oscillatory kink wave solution is of the form

$$u(\xi) = \begin{cases} \dfrac{-3e}{2f}\operatorname{sech}^2\left(\dfrac{\sqrt{e}}{2}\xi\right), & \xi \in (-\infty, 0], \\ -\dfrac{e}{f} - \dfrac{e}{2f}\exp(b\xi)\ \cos a\xi, & b < 0,\ \xi \in (0, +\infty). \end{cases} \tag{4.6}$$

Next, we only need to determine the coefficients $a$ and $b$ in (4.6). Substituting the second expression of (4.6) into (2.3), we get

$$-\frac{e^2(\exp(b\xi))^2\cos(a\xi)^2}{4f} + \frac{e(-b^2 + a^2 + gb - e)}{2f}\cos a\xi + \frac{ea(2\,b - g)}{2f}\sin a\xi \equiv 0. \tag{4.7}$$

Neglecting the first small term, we can solve $a = \sqrt{4e - g^2}/2$, $b = g/2$. Taking $e = 1$, $g = -0.3$, $f = -1$, we can give picture of the approximate solution in Figure 5(b).

## 5. Results

From the transformation made in Section 2 and the proofs in Section 3, we can obtain complete results about bounded travelling waves of (1.1) under different parameter conditions as listed in Table 1. Furthermore, from these results and Theorem 3.2, we can see, for the oscillatory kink wave solution, the maximum oscillation amplitude increases with respect to $g$ for $g < 0$ but decreases for $g > 0$. It means that the maximum oscillation amplitude decreases with respect to $|g|$.

So, taking the case $e > 0$, $g < 0$, $f < 0$, for example, we can exhibit the evolution of bounded travelling waves of (1.1) as follows: when $g = 0$, there is a bell-shape solitary wave

**Table 1:** Bounded travelling waves of (1.1) under different parameter conditions.

| Probability of parameters | | | | Type of traveling wave |
|---|---|---|---|---|
| $e > 0$ | $f > 0$ | $g^2 - 4e \geq 0$ | $g > 0$ | I |
| | | | $g < 0$ | II |
| | | $g^2 - 4e < 0$ | $g > 0$ | III |
| | | | $g < 0$ | IV |
| | $f < 0$ | $g^2 - 4e \geq 0$ | $g > 0$ | II |
| | | | $g < 0$ | I |
| | | $g^2 - 4e < 0$ | $g > 0$ | III |
| | | | $g < 0$ | IV |
| $e < 0$ | $f > 0$ | $g^2 + 4e \geq 0$ | $g > 0$ | I |
| | | | $g < 0$ | II |
| | | $g^2 + 4e < 0$ | $g > 0$ | III |
| | | | $g < 0$ | IV |
| | $f < 0$ | $g^2 + 4e \geq 0$ | $g > 0$ | II |
| | | | $g < 0$ | I |
| | | $g^2 + 4e < 0$ | $g > 0$ | III |
| | | | $g < 0$ | IV |

I: monotone increasing kink wave; II: monotone decreasing kink wave; III: increasing oscillatory kink wave; IV: damped oscillatory kink wave.

for system (1.1). Then the solitary wave evolves to an oscillatory kink wave when $-\sqrt{4e} < g < 0$. The oscillation amplitude of it becomes smaller and smaller when $g$ approaches to $-\sqrt{4e}$. Until $g = -\sqrt{4e}$, the oscillation disappears thoroughly and a monotone kink wave appears instead.

From the variable transformation $g = \mu/\delta c$ made in Section 2 and the discussion above, one can see that the ratio $\mu/\delta$ and the wave velocity $c$ play important roles in the evolution of travelling wave of system (1.1). When $\mu/\delta = 0$, that is, without the dissipative term, the RLW-Burgers equation has a solitary wave because of the balance between nonlinearity and dispersion. Once $\mu$ varies from 0 to nonzero, the solitary wave evolves an oscillatory kink wave. Either the ratio $|\mu/\delta|$ increasing or $|c|$ decreasing will lead to $|g|$, the absolute value of coefficient of damping terms in the travelling wave system (2.2), increasing. So, the oscillation amplitude of the oscillatory kink wave will decrease until it evolves to a monotone kink wave with the oscillation disappearing thoroughly.

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