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Research Article

Common Solutions of Generalized Mixed Equilibrium Problems, Variational Inclusions, and Common Fixed Points for Nonexpansive Semigroups and Strictly Pseudocontractive Mappings

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We introduce a new iterative scheme by shrinking projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of common solutions of variational inclusion problems with set-valued maximal monotone mappings and inverse-strongly monotone mappings, the set of solutions of fixed points for nonexpansive semigroups, and the set of common fixed points for an infinite family of strictly pseudocontractive mappings in a real Hilbert space. We prove that the sequence converges strongly to a common element of the above four sets under some mind conditions. Furthermore, by using the above result, an iterative algorithm for solution of an optimization problem was obtained. Our results improve and extend the corresponding results of Martinez-Yanes and Xu (2006), Shehu (2011), Zhang et al. (2008), and many authors.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let 2^H denote the family of all subsets of H, and let C be a closed-convex subset of H. Recall that a mapping $T : C \to C$ is said to be a *k-strict pseudocontraction* [1] if there exists $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C, \tag{1.1}$$

where I denotes the identity operator on C. When $k = 0, T : C \to C$ is said to be *nonexpansive* [2] if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

And when $k = 1, T : C \to C$ is said to be *pseudocontraction* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

Clearly, the class of k-strict pseudocontraction falls into the one between classes of nonexpansive mappings and pseudocontraction mapping. We denote the set of fixed points of T by $F(T)$.

A family $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$,

(ii) $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$,

(iii) $\|S(s)x - S(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$,

(iv) for all $x \in C, s \mapsto S(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S} = \{S(s) : s \geq 0\}$, that is, $F(S) = \bigcap_{s \geq 0} F(S(s))$. It is known that $F(\mathcal{S})$ is closed and convex.

Let $A : H \to H$ be a single-valued nonlinear mapping, and let $M : H \to 2^H$ be a set-valued mapping. We consider the following *variational inclusion problem*, which is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{1.4}$$

where θ is the zero vector in H. The set of solutions of problem (1.4) is denoted by $I(A, M)$.

Let the set-valued mapping $M : H \to 2^H$ be a maximal monotone. We define the *resolvent operator* $J_{M,\lambda}$ associate with M and λ as follows:

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H, \tag{1.5}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive, and 1-inverse-strongly monotone ([3, 4]).

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, let $A : C \to H$ be a mapping, and let $\varphi : C \to \mathbb{R}$ be a real-valued function. The *generalized mixed equilibrium problem* is for finding $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by GMEP(F, φ, A), that is,

$$\mathrm{GMEP}(F, \varphi, A) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in C\}. \tag{1.7}$$

If $A \equiv 0$, then the problem (1.6) is reduced into the mixed equilibrium problem for finding $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.8}$$

The set of solutions of (1.8) is denoted by MEP(F, φ). The (generalized) mixed equilibrium problems include fixed-point problems, variational inequality problems, optimization problems, Nash equilibrium problems, noncooperative games, economics, and the equilibrium problem as special cases ([5–15]). In the last two decades, many papers have appeared in the literature on the existence of solutions of equilibrium problems; see, for example, [9] and references therein. Some solution methods have been proposed to solve the mixed equilibrium problems; see, for example, ([7–10, 12–20]) and references therein.

In 2006, Martinez-Yanes and Xu [21] introduced the following iterative:

$$\begin{gathered}
x_0 \in C, \\
y_n = \alpha_n x_0 + (1 - \alpha_n) T x_n, \\
C_n = \left\{z \in C : \left\|y_n - z\right\|^2 \leq \|x_n - z\|^2 + \alpha_n \left(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle\right)\right\}, \\
Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{gathered} \tag{1.9}$$

where T is a nonexpansive mapping in a Hilbert space H, and P_C is metric projection of H onto a closed and convex subset C of H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

In 2008, Zhang et al. [4] introduced an iterative scheme for finding a common element of the set of solutions to the variational inclusion problem with a multivalued maximal monotone mapping and an inverse-strongly monotone mapping and the set of fixed points of nonexpansive mapping in Hilbert spaces. The following iterative scheme $x_0 = x \in H$ and

$$\begin{gathered}
y_n = J_{M,\lambda}(x_n - \lambda A x_n), \\
x_{n+1} = \alpha_n x + (1 - \alpha_n) S y_n,
\end{gathered} \tag{1.10}$$

for all $n \geq 0$. They proved the strong convergence theorem under some mind conditions.

Recently, Shehu [19] introduced a new iterative scheme by hybrid method for finding a common element of the set of common fixed points of infinite family of k-strictly pseudocontractive mappings, the set of common solutions to a system of generalized mixed equilibrium problems, and the set of solutions to a variational inequality problem in Hilbert spaces. Starting with an arbitrary $x_0 \in C$, $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, and $x_1 = P_{C_1} x_0$ define sequence $\{x_n\}, \{w_n\}, \{u_n\}, \{z_n\}$, and $\{y_{n,i}\}$ as follows:

$$\begin{gathered}
z_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\
y_n = T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n), \\
w_n = P_C(u_n - s_n D u_n), \\
y_{n,i} = \alpha_{n,i} w_n + (1-\alpha_{n,i}) T_i w_n, \quad n \geq 1, \\
C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\| \leq \|x_n - z\| \right\}, \quad n \geq 1, \\
C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1,
\end{gathered} \tag{1.11}$$

where T_i is a k_i-strictly pseudocontractive mapping and for some $0 \leq k_i < 1$, A, B is α, β-inverse-strongly monotone mapping of C into H. He proved that if the sequence $\{\alpha_{n,i}\}, \{r_n\}, \{s_n\}$, and $\{\lambda_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ generated by (1.11) converges strongly to $P_{\Omega} x_0$.

In this paper, motivated by the above results, we present a new general iterative scheme for finding a common element of the set of solutions for a system of generalized mixed equilibrium problems, the set of common solutions of variational inclusion problems with set-valued maximal monotone mappings and inverse-strongly monotone mappings, the set of solutions of fixed points for nonexpansive semigroup mappings, and the set of common fixed points for an infinite family of strictly pseudocontractive mappings in a real Hilbert space. Then, we prove strong convergence theorem under some mind conditions. Furthermore, by using the above result, an iterative algorithm for solution of an optimization problem was obtained. The results presented in this paper extend and improve the results of Martinez-Yanes and Xu [21], Shehu [19], Zhang et al. [4], and many authors.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, and let C be a closed-convex subset of H. When $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x, and $x_n \to x$ means that $\{x_n\}$ converges strongly to x. In a real Hilbert space H, we have

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y \rangle, \tag{2.1}$$

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda) \|y\|^2 - \lambda(1-\lambda) \|x-y\|^2, \quad \forall x, y \in H, \tag{2.2}$$

and $\lambda \in \mathbb{R}$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.3}$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.4}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{2.5}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H,\ y \in C. \tag{2.6}$$

Recall that a mapping A of H into itself is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall, y \in H. \tag{2.7}$$

It is obvious that any α-inverse-strongly monotone mapping A is $(1/\alpha)$-Lipschitz monotone and continuous mapping.

In order to prove our main results, we need the following Lemmas.

Lemma 2.1 (see [22]). *Let $V : C \to H$ be a k-strict pseudocontraction, then*

(1) *the fixed-point set $F(V)$ of V is closed convex, so that the projection $P_{F(V)}$ is well defined;*

(2) *define a mapping $T : C \to H$ by*

$$Tx = tx + (1 - t)Vx, \quad \forall x \in C. \tag{2.8}$$

If $t \in [k, 1)$, then T is a nonexpansive mapping such that $F(V) = F(T)$.

A family of mappings $\{V_i : C \to H\}_{i=1}^{\infty}$ is called *a family of uniformly k-strict pseudocontractions* if there exists a constant $k \in [0, 1)$ such that

$$\|V_i x - V_i y\|^2 \leq \|x - y\|^2 + k\|(I - V_i)x - (I - V_i)y\|^2, \quad \forall x, y \in C,\ \forall i \geq 1. \tag{2.9}$$

Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly k-strict pseudocontractions. Let $\{T_i : C \to C\}_{i=1}^{\infty}$ be the sequence of nonexpansive mappings defined by (2.8), that is,

$$T_i x = tx + (1-t)V_i x, \quad \forall x \in C, \forall i \geq 1, \ t \in [k, 1). \tag{2.10}$$

Let $\{T_i\}$ be a sequence of nonexpansive mappings of C into itself defined by (2.10), and let $\{\mu_i\}$ be a sequence of nonnegative numbers in $[0,1]$. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \mu_n T_n U_{n,n+1} + (1-\mu_n)I,\\
U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1})I,\\
&\ \vdots\\
U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k)I,\\
U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1})I,\\
&\ \vdots\\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2)I,\\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1)I.
\end{aligned} \tag{2.11}$$

Such a mapping W_n is nonexpansive from C to C and it is called the W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\mu_1, \mu_2, \ldots, \mu_n$.

For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (2.11), then we can have the following crucial conclusions concerning W_n. You can find them in [23]. Now, we only need the following similar version in Hilbert spaces.

Lemma 2.2 (see [23]). *Let C be a nonempty closed-convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$, then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$, $\forall n \geq 1$,*

(2) *for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists,*

(3) *a mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \quad \forall x \in C \tag{2.12}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$, and it is called the W-mapping generated by $T_1, T_2, \ldots$ and $\mu_1, \mu_2, \ldots$.

Lemma 2.3 (see [24]). *Let C be a nonempty closed-convex subset of a Hilbert space H, let $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and let $\{\mu_i\}$ be a real* *sequence such that* $0 < \mu_i \leq b < 1,\ \forall i \geq 1$. *If* D *is any bounded subset of* C, *then*

$$\lim_{n \to \infty} \sup_{x \in D} \|Wx - W_n x\| = 0. \tag{2.13}$$

Lemma 2.4 (see [25]). *Each Hilbert space* H *satisfies Opials condition, that is, for any sequence* $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$, *the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{2.14}$$

holds for each $y \in H$ *with* $y \neq x$.

Lemma 2.5 (see [3]). *Let* $M : H \to 2^H$ *be a maximal monotone mapping, and let* $A : H \to H$ *be a monotone mapping, then the mapping* $S = M + A : H \to 2^H$ *is a maximal monotone mapping.*

Remark 2.6. Lemma 2.5 implies that $I(A, M)$ is closed and convex if $M : H \to 2^H$ is a maximal monotone mapping and $A : H \to H$ is a monotone mapping.

Lemma 2.7 (see [4]). *Let* $u \in H$ *be a solution of variational inclusion* (1.4) *if and only if* $u = J_{M,\lambda}(u - \lambda A u), \forall \lambda > 0$, *that is,*

$$I(A, M) = F(J_{M,\lambda}(I - \lambda A)), \quad \forall \lambda > 0. \tag{2.15}$$

Lemma 2.8 (see [20]). *Let* C *be a nonempty bounded closed-convex subset of a Hilbert space* H, *and let* $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on* C, *then for any* $h \geq 0$,

$$\lim_{t \to \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t S(s) x \, ds - S(h) \left(\frac{1}{t} \int_0^t S(s) x \, ds \right) \right\| = 0. \tag{2.16}$$

Lemma 2.9 (see [26]). *Let* C *be a nonempty bounded closed-convex subset of* H, *let* $\{x_n\}$ *be a sequence in* C, *and let* $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on* C. *If the following conditions are satisfied:*

(1) $x_n \rightharpoonup z$,

(2) $\limsup_{s \to \infty} \limsup_{n \to \infty} \|S(s) x_n - x_n\| = 0$, *then* $z \in F(S)$.

For solving the generalized mixed equilibrium problem for $F : C \times C \to \mathbb{R}$, *one gives the following assumptions for the bifunction* F, φ *and the set* C:

(A1) $F(x, x) = 0$ *for all* $x \in C$,

(A2) F *is monotone, that is,* $F(x, y) + F(y, x) \leq 0$ *for all* $x, y \in C$,

(A3) *for each* $x, y, z \in C, \lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$,

(A4) *for each* $x \in C, y \mapsto F(x, y)$ *is convex and lower semicontinuous,*

(A5) *for each* $y \in C, x \mapsto F(x, y)$ *is weakly upper semicontinuous,* (B1) *for each* $x \in H$ *and* $r > 0$, *there exist a bounded subset* $D_x \subseteq C$ *and* $y_x \in C$ *such that for any* $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z), \tag{2.17}$$

(B2) C *is a bounded set,*

then one has the following lemma.

Lemma 2.10 (see [18]). *Let* C *be a nonempty closed-convex subset of* H. *Let* $F : C \times C \to \mathbb{R}$ *be a bifunction that satisfies (A1)–(A5), and let* $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r^{(F,\varphi)}(x) = \left\{z \in C : F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z),\ \forall y \in C\right\}, \tag{2.18}$$

for all $z \in H$, *then the following hold:*

(1) *for each* $x \in H, T_r^{(F,\varphi)}(x) \neq \emptyset$,

(2) $T_r^{(F,\varphi)}$ *is single valued,*

(3) $T_r^{(F,\varphi)}$ *is firmly nonexpansive, that is, for any* $x, y \in H$, $\|T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y\|^2 \leq \langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y\rangle$,

(4) $F(T_r^{(F,\varphi)}) = MEP(F, \varphi)$,

(5) $MEP(F, \varphi)$ *is closed and convex.*

3. Main Result

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions for a system of generalized mixed equilibrium problems, the set of common solutions of variational inclusion problems with set-valued maximal monotone mappings and inverse-strongly monotone mappings, the set of solutions of fixed points for nonexpansive semigroup mappings, and the set of common fixed points for an infinite family of strictly pseudocontractive mappings in a real Hilbert space.

Theorem 3.1. *Let* C *be a nonempty closed-convex subset of a real Hilbert Space* H. *Let* F_1, F_2 *be bifunctions of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5), and let* $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ *be proper lower semicontinuous and convex functions with assumption (B1) or (B2). Let* A, B, E_1, E_2 *be* $\alpha, \beta, \eta_1, \eta_2$*-inverse-strongly monotone mappings of* C *into* H, *respectively, and let* $M_1, M_2 : H \to 2^H$ *be maximal monotone mappings. Let* $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on* C, *and let* $\{t_n\}$ *be a positive real divergent sequence. Let* $\{V_i : C \to C\}_{i=1}^{\infty}$ *be a countable family of uniformly* k*-strict pseudocontractions, let* $\{T_i : C \to C\}_{i=1}^{\infty}$ *be a countable family of nonexpansive mappings defined by* $T_i x = tx + (1-t)V_i x$, $\forall x \in C, \forall i \geq 1$, $t \in [k, 1)$, *and let* W_n *be the* W*-mapping defined by* (2.11) *and* W *a mapping defined by* (2.12) *with* $F(W) \neq \emptyset$. *Suppose that*

$\Theta := F(S) \cap F(W) \cap GMEP(F_1, \varphi_1, A) \cap GMEP(F_2, \varphi_2, B) \cap I(E_1, M_1) \cap I(E_2, M_2) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C$, $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$, *and*

$$\begin{gathered}
t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\
u_n = T_{q_n}^{(F_2,\varphi_2)}(t_n - q_n B t_n), \\
v_n = J_{M_1,\lambda_1}(u_n - \lambda_1 E_1 u_n), \\
w_n = J_{M_2,\lambda_2}(v_n - \lambda_2 E_2 v_n), \\
y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) \frac{1}{t_n} \int_0^{t_n} S(s) W_n w_n ds, \\
C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \leq \| x_n - z \|^2 + \alpha_{n,i} \left(\| x_0 \|^2 + 2\langle x_n - x_0, z \rangle \right) \right\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} = P_{C_{n+1}} x_0,
\end{gathered} \tag{3.1}$$

for every $n \geq 0$, *where* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1)$, $\{r_n\}, \{q_n\} \subset (0, \infty)$, $\lambda_1 \in (0, 2\eta_1)$, *and* $\lambda_2 \in (0, 2\eta_2)$ *satisfy the following conditions:*

(i) $0 < a \leq r_n \leq b < 2\alpha$,

(ii) $0 < c \leq q_n \leq d < 2\beta$,

(iii) $\lim_{n \to \infty} \alpha_{n,i} = 0$,

(iv) $0 < e \leq \lambda_1 \leq f < 2\eta_1$,

(v) $0 < g \leq \lambda_2 \leq j < 2\eta_2$,

then $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

Proof. First, we show that $I - \lambda_1 E_1$ and $I - \lambda_2 E_2$ are nonexpansive. Indeed, for all $x, y \in C$ and $\lambda_1 \in (0, 2\eta_1)$, we obtain

$$\begin{aligned}
\left\| (I - \lambda_1 E_1)x - (I - \lambda_1 E_1)y \right\|^2 &= \left\| x - y - \lambda_1 (E_1 x - E_1 y) \right\|^2 \\
&= \left\| x - y \right\|^2 - 2\lambda_1 \langle x - y, E_1 x - E_1 y \rangle + \lambda_1^2 \left\| E_1 x - E_1 y \right\|^2 \\
&\leq \left\| x - y \right\|^2 - 2\lambda_1 \eta_1 \left\| E_1 x - E_1 y \right\|^2 + \lambda_1^2 \left\| E_1 x - E_1 y \right\|^2 \\
&\leq \left\| x - y \right\|^2 + \lambda_1 (\lambda_1 - 2\eta_1) \left\| E_1 x - E_1 y \right\|^2 \\
&\leq \left\| x - y \right\|^2,
\end{aligned} \tag{3.2}$$

which implies that the mapping $I-\lambda_1 E_1$ is nonexpansive, so is $I-\lambda_2 E_2$. Let $p \in \Theta$. We observe that

$$\begin{aligned}
\left\|w_n-p\right\|^2 &= \left\|J_{M_2,\lambda_2}(v_n-\lambda_2 E_2 v_n)-J_{M_2,\lambda_2}(p-\lambda_2 E_2 p)\right\|^2 \\
&\le \left\|(v_n-\lambda_2 E_2 v_n)-(p-\lambda_2 E_2 p)\right\|^2 \\
&\le \left\|v_n-p\right\|^2 \\
&= \left\|J_{M_1,\lambda_1}(u_n-\lambda_1 E_1 u_n)-J_{M_1,\lambda_1}(p-\lambda_1 E_1 p)\right\|^2 \\
&\le \left\|(u_n-\lambda_1 E_1 u_n)-(p-\lambda_1 E_1 p)\right\|^2 \\
&= \left\|u_n-p\right\|^2.
\end{aligned} \tag{3.3}$$

Since both $I-r_n A$ and $I-q_n B$ are nonexpansive for each $n \ge 1$, let $p \in \Theta$, then $p = T_{r_n}^{F_1,\varphi_1}(p-r_n Ap)$ and $p = T_{q_n}^{F_2,\varphi_2}(p-q_n Bp)$; by conditions (i) and (ii), we have

$$\begin{aligned}
\left\|u_n-p\right\|^2 &= \left\|T_{q_n}^{(F_2,\varphi_2)}(I-q_n B)t_n - T_{q_n}^{(F_2,\varphi_2)}(I-q_n B)p\right\|^2 \\
&\le \left\|(I-q_n B)t_n-(I-q_n B)p\right\|^2 \\
&\le \left\|t_n-p\right\|^2 + q_n(q_n-2\beta)\left\|Bt_n-Bp\right\|^2 \\
&\le \left\|t_n-p\right\|^2, \\
\left\|t_n-p\right\|^2 &= \left\|T_{r_n}^{(F_1,\varphi_1)}(I-r_n A)x_n - T_{r_n}^{(F_1,\varphi_1)}(I-r_n A)p\right\|^2 \\
&\le \left\|(I-r_n A)x_n-(I-r_n A)p\right\|^2 \\
&\le \left\|x_n-p\right\|^2 + r_n(r_n-2\alpha)\left\|Ax_n-Ap\right\|^2 \\
&\le \left\|x_n-p\right\|^2.
\end{aligned} \tag{3.4}$$

Therefore, we get

$$\left\|u_n-p\right\| \le \left\|x_n-p\right\|. \tag{3.5}$$

Next, we will divide the proof into five steps.

Step 1. We show that $\{x_n\}$ is well defined. Let $n=1$, then $C_{1,i}=C$ is closed and convex for each $i \ge 1$. Suppose that $C_{n,i}$ is closed convex for some $n > 1$, then, from the definition of $C_{n+1,i}$, we know that $C_{n+1,i}$ is closed convex for the same $n \ge 1$. Hence, $C_{n,i}$ is closed convex for $n \geq 1$ and for each $i \geq 1$. This implies that C_n is closed convex for $n \geq 1$. Furthermore, we show that $\Theta \subset C_n$. For $n = 1, \Theta \subset C = C_{1,i}$. For $n \geq 2$, let $p \in \Theta$, then

$$\begin{aligned}
\left\|y_{n,i} - p\right\|^2 &= \left\|\alpha_{n,i}(x_0 - p) + (1-\alpha_{n,i})(\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - p)\right\|^2 \\
&\leq \alpha_{n,i}\left\|x_0 - p\right\|^2 + (1-\alpha_{n,i})\left\|\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - p\right\|^2 \\
&\leq \alpha_{n,i}\left\|x_0 - p\right\|^2 + (1-\alpha_{n,i})\left\|w_n - p\right\|^2 \\
&= \left\|w_n - p\right\|^2 + \alpha_{n,i}\left(\left\|x_0 - p\right\|^2 - \left\|w_n - p\right\|^2\right) \\
&\leq \left\|x_n - p\right\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle x_n - x_0, p\rangle\right),
\end{aligned} \tag{3.6}$$

which shows that $p \in C_{n,i}$, $\forall n \geq 2$, $\forall i \geq 1$. Thus, $\Theta \subset C_{n,i}$, $\forall n \geq 1$, $\forall i \geq 1$. Hence, it follows that $\emptyset \neq \Theta \subset C_n$, $\forall n \geq 1$. This implies that $\{x_n\}$ is well defined.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|y_{n,i} - x_n\| = 0$, for $i \geq 1$. Since $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, $\forall n \geq 1$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 1. \tag{3.7}$$

Also, as $\Theta \subset C_n$ by (2.1), it follows that

$$\|x_n - x_0\| \leq \|z - x_0\|, \quad z \in \Theta, \ \forall n \geq 1. \tag{3.8}$$

Form (3.7) and (3.8), we have that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Hence, $\{x_n\}$ is bounded and so are $\{y_{n,i}\}, \forall i \geq 1$, $\{w_n\}$, $\{v_n\}$, $\{u_n\}$, $\{t_n\}$, $\{Ax_n\}$, $\{Bt_n\}$, $\{E_1u_n\}$, $\{E_2v_n\}$, $\{W_nw_n\}$, and $\{(1/t_n)\int_0^{t_n} S(s)W_nw_n ds\}$. For $m > n \geq 1$, we have that $x_m = P_{C_m}x_0 \in C_m \subset C_n$. By (2.5), we obtain

$$\|x_m - x_n\|^2 \leq \|x_n - x_0\|^2 - \|x_m - x_0\|^2. \tag{3.9}$$

Letting $m, n \to \infty$ and taking the limit in (3.9), we have $\|x_m - x_n\| \to 0$, which shows that $\{x_n\}$ is Cauchy. In particular,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Since $\{x_n\}$ is Cauchy, we assume that $x_n \longrightarrow z \in C$. Since $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1}$, then

$$\left\|y_{n,i} - x_{n+1}\right\|^2 \leq \|x_n - x_{n+1}\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle x_n - x_0, x_{n+1}\rangle\right) \longrightarrow 0, \tag{3.11}$$

and it follows that

$$\left\|y_{n,i} - x_n\right\| \leq \left\|y_{n,i} - x_{n+1}\right\| + \|x_{n+1} - x_n\|. \tag{3.12}$$

Therefore,

$$\lim_{n\to\infty}\left\|y_{n,i} - x_n\right\| = 0, \quad \forall i \geq 1. \tag{3.13}$$

Step 3. We claim that the following statements hold:

(1) $\lim_{n\to\infty}\|u_n - t_n\| = 0$,

(2) $\lim_{n\to\infty}\|t_n - x_n\| = 0$,

(3) $\lim_{n\to\infty}\|w_n - v_n\| = 0$,

(4) $\lim_{n\to\infty}\|v_n - u_n\| = 0$.

For $p \in \Theta$, from (3.4), and (3.6), we obtain

$$\begin{aligned}
\left\|y_{n,i} - p\right\|^2 &\leq \alpha_{n,i}\left\|x_0 - p\right\|^2 + (1 - \alpha_{n,i})\left\|w_n - p\right\|^2 \\
&\leq \alpha_{n,i}\left\|x_0 - p\right\|^2 + (1 - \alpha_{n,i})\left\|u_n - p\right\|^2 \\
&\leq \alpha_{n,i}\left\|x_0 - p\right\|^2 + (1 - \alpha_{n,i})\left(\left\|t_n - p\right\|^2 + q_n(q_n - 2\beta)\left\|Bt_n - Bp\right\|^2\right) \\
&\leq \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|x_n - p\right\|^2 + (1 - \alpha_{n,i})q_n(q_n - 2\beta)\left\|Bt_n - Bp\right\|^2.
\end{aligned} \tag{3.14}$$

Since $0 < c \leq q_n \leq d < 2\beta$, we have

$$\begin{aligned}
(1 - \alpha_{n,i})q_n(2\beta - q_n)\left\|Bt_n - Bp\right\|^2 &\leq \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|x_n - p\right\|^2 - \left\|y_{n,i} - p\right\|^2 \\
&\leq \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|x_n - y_{n,i}\right\|\left(\left\|x_n - p\right\| + \left\|y_{n,i} - p\right\|\right).
\end{aligned} \tag{3.15}$$

Hence, by condition (iii) and (3.13), we have

$$\lim_{n\to\infty}\left\|Bt_n - Bp\right\| = 0. \tag{3.16}$$

From (3.6), we have

$$\begin{aligned}\|y_{n,i}-p\|^2 &\le \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|w_n-p\|^2\\ &\le \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|u_n-p\|^2.\end{aligned}\tag{3.17}$$

On the other hand,

$$\begin{aligned}\|u_n-p\|^2 &= \left\|T_{q_n}^{(F_2,\varphi_2)}(t_n-q_nBt_n)-T_{q_n}^{(F_2,\varphi_2)}(p-q_nBp)\right\|^2\\ &\le \langle (t_n-q_nBt_n)-(p-q_nBp), u_n-p\rangle\\ &= \frac{1}{2}\Big\{\|(t_n-q_nBt_n)-(p-q_nBp)\|^2+\|u_n-p\|^2\\ &\qquad -\|(t_n-q_nBt_n)-(p-q_nBp)-(u_n-p)\|^2\Big\}\\ &\le \frac{1}{2}\Big\{\|t_n-p\|^2+\|u_n-p\|^2-\|(t_n-q_nBt_n)-(p-q_nBp)-(u_n-p)\|^2\Big\}\\ &\le \frac{1}{2}\Big\{\|x_n-p\|^2+\|u_n-p\|^2-\|u_n-t_n\|^2+2q_n\langle t_n-u_n, Bt_n-Bp\rangle-s_n^2\|Bt_n-Bp\|^2\Big\},\end{aligned}\tag{3.18}$$

and hence,

$$\|u_n-p\|^2 \le \|x_n-p\|^2-\|u_n-t_n\|^2+2q_n\|t_n-u_n\|\|Bt_n-Bp\|.\tag{3.19}$$

Putting (3.19) into (3.17), for $i \ge 1$, we have

$$\begin{aligned}\|y_{n,i}-p\|^2 &\le \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\Big(\|x_n-p\|^2-\|u_n-t_n\|^2+2q_n\|t_n-u_n\|\|Bt_n-Bp\|\Big)\\ &\le \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-(1-\alpha_{n,i})\|u_n-t_n\|^2+2q_n\|t_n-u_n\|\|Bt_n-Bp\|.\end{aligned}\tag{3.20}$$

It follows that

$$\begin{aligned}(1-\alpha_{n,i})\|u_n-t_n\|^2 &\le \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-\|y_{n,i}-p\|^2+2q_n\|t_n-u_n\|\|Bt_n-Bp\|\\ &\le \alpha_{n,i}\|x_0-p\|^2+\|x_n-y_{n,i}\|(\|x_n-p\|+\|y_{n,i}-p\|)\\ &\quad +2q_n\|t_n-u_n\|\|Bt_n-Bp\|.\end{aligned}\tag{3.21}$$

Therefore, from condition (iii), (3.13), and (3.16), we have

$$\lim_{n\to\infty}\|u_n - t_n\| = 0. \tag{3.22}$$

Furthermore, from (3.4), and (3.6), we get

$$\begin{aligned}
\|y_{n,i} - p\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|w_n - p\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|u_n - p\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|t_n - p\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\left\{\|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2\right\} \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 + (1-\alpha_{n,i})r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.23}$$

Since $0 < a \le r_n \le b < 2\alpha$, we have

$$\begin{aligned}
(1-\alpha_{n,i})r_n(2\alpha - r_n)\|Ax_n - Ap\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - \|y_{n,i} - p\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - y_{n,i}\|\left(\|x_n - p\| + \|y_{n,i} - p\|\right).
\end{aligned} \tag{3.24}$$

Then, by condition (iii) and (3.13), we obtain that

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{3.25}$$

From (3.6), we have

$$\begin{aligned}
\|y_{n,i} - p\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|w_n - p\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|u_n - p\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|t_n - p\|^2.
\end{aligned} \tag{3.26}$$

On the other hand, we note that

$$\begin{aligned}
\|t_n - p\|^2 &\le \left\| T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n) - T_{r_n}^{(F_1,\varphi_1)}(p - r_n Ap) \right\|^2 \\
&\le \langle (x_n - r_n A x_n) - (p - r_n Ap), t_n - p \rangle \\
&= \frac{1}{2}\Big\{ \|(x_n - r_n A x_n) - (p - r_n Ap)\|^2 + \|t_n - p\|^2 \\
&\qquad - \|(x_n - r_n A x_n) - (p - r_n Ap) - (t_n - p)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|t_n - p\|^2 - \|(x_n - r_n A x_n) - (p - r_n Ap) - (t_n - p)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|t_n - p\|^2 - \|t_n - x_n\|^2 + 2r_n \langle x_n - t_n, Ax_n - Ap \rangle - r_n^2 \|Ax_n - Ap\|^2 \Big\},
\end{aligned} \tag{3.27}$$

and hence,

$$\|t_n - p\|^2 \le \|x_n - p\|^2 - \|t_n - x_n\|^2 + 2r_n \|x_n - t_n\| \|Ax_n - Ap\|. \tag{3.28}$$

Putting (3.28) into (3.26), we have

$$\begin{aligned}
\|y_{n,i} - p\|^2 &\le \alpha_{n,i} \|x_0 - p\|^2 + (1 - \alpha_{n,i}) \Big(\|x_n - p\|^2 - \|t_n - x_n\|^2 + 2r_n \|x_n - t_n\| \|Ax_n - Ap\| \Big) \\
&\le \alpha_{n,i} \|x_0 - p\|^2 + \|x_n - p\|^2 - (1 - \alpha_{n,i}) \|t_n - x_n\|^2 + 2r_n \|x_n - t_n\| \|Ax_n - Ap\|.
\end{aligned} \tag{3.29}$$

It follows that

$$\begin{aligned}
(1 - \alpha_{n,i}) \|x_n - t_n\|^2 &\le \alpha_{n,i} \|x_0 - p\|^2 + \|x_n - p\|^2 - \|y_{n,i} - p\|^2 + 2r_n \|x_n - z_n\| \|Ax_n - Ap\| \\
&\le \alpha_{n,i} \|x_0 - p\|^2 + \|x_n - y_{n,i}\| (\|x_n - p\| + \|y_{n,i} - p\|) \\
&\quad + 2r_n \|x_n - z_n\| \|Ax_n - Ap\|.
\end{aligned} \tag{3.30}$$

Therefore, by condition (iii), (3.13), and (3.25), we have

$$\lim_{n \to \infty} \|x_n - t_n\| = 0. \tag{3.31}$$

Condition (iii) implies that

$$\left\| y_{n,i} - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\|^2 = \alpha_{n,i}\left\| x_0 - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| \longrightarrow 0. \tag{3.32}$$

It follows that

$$\left\| x_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| \le \left\| x_n - y_{n,i} \right\| + \left\| y_{n,i} + \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds \right\| \longrightarrow 0. \tag{3.33}$$

From (3.6), we have

$$\begin{aligned}
\left\| y_{n,i} - p \right\|^2 &\le \alpha_{n,i}\left\| x_0 - p \right\|^2 + (1-\alpha_{n,i})\left\| w_n - p \right\|^2 \\
&= \alpha_{n,i}\left\| x_0 - p \right\|^2 + (1-\alpha_{n,i})\left\| J_{M_2,\lambda_2}(v_n - \lambda_2 E_2 v_n) - J_{M_2,\lambda_2}(p - \lambda_2 E_2 p) \right\|^2 \\
&\le \alpha_{n,i}\left\| x_0 - p \right\|^2 + (1-\alpha_{n,i})\left\| (v_n - \lambda_2 E_2 v_n) - (p - \lambda_2 E_2 p) \right\|^2 \\
&\le \alpha_{n,i}\left\| x_0 - p \right\|^2 + (1-\alpha_{n,i})\left(\left\| v_n - p \right\|^2 + \lambda_2(\lambda_2 - 2\eta_2)\left\| E_2 v_n - E_2 p \right\|^2 \right) \\
&\le \alpha_{n,i}\left\| x_0 - p \right\|^2 + \left\| x_n - p \right\|^2 + (1-\alpha_{n,i})\lambda_2(\lambda_2 - 2\eta_2)\left\| E_2 v_n - E_2 p \right\|^2.
\end{aligned} \tag{3.34}$$

Since $0 < g \le \lambda_2 \le j < 2\eta_2$, we have

$$\begin{aligned}
(1-\alpha_{n,i})\lambda_2(2\eta_2 - \lambda_2)\left\| E_2 v_n - E_2 p \right\|^2 &\le \alpha_{n,i}\left\| x_0 - p \right\|^2 + \left\| x_n - p \right\|^2 - \left\| y_{n,i} - p \right\|^2 \\
&\le \alpha_{n,i}\left\| x_0 - p \right\|^2 + \left\| x_n - y_{n,i} \right\|\left(\left\| x_n - p \right\| + \left\| y_{n,i} - p \right\| \right).
\end{aligned} \tag{3.35}$$

Then, by condition (iii) and (3.13), we obtain that

$$\lim_{n\to\infty}\left\| E_2 v_n - E_2 p \right\| = 0. \tag{3.36}$$

From (3.6), we have

$$\left\|y_{n,i}-p\right\|^2 \leq \alpha_{n,i}\left\|x_0-p\right\|^2+(1-\alpha_{n,i})\left\|w_n-p\right\|^2. \tag{3.37}$$

On the other hand, we note that

$$\begin{aligned}\left\|w_n-p\right\|^2 &\leq \left\|J_{M_2,\lambda_2}(v_n-\lambda_2E_2v_n)-J_{M_2,\lambda_2}(p-\lambda_2E_2p)\right\|^2\\ &\leq \langle(v_n-\lambda_2E_2v_n)-(p-\lambda_2E_2p),w_n-p\rangle\\ &=\frac{1}{2}\Big\{\left\|(v_n-\lambda_2E_2v_n)-(p-\lambda_2E_2p)\right\|^2+\left\|w_n-p\right\|^2\\ &\qquad-\left\|(v_n-\lambda_2E_2v_n)-(p-\lambda_2E_2p)-(w_n-p)\right\|^2\Big\}\\ &\leq \frac{1}{2}\Big\{\left\|v_n-p\right\|^2+\left\|w_n-p\right\|^2-\left\|(v_n-\lambda_2E_2v_n)-(p-\lambda_2E_2p)-(w_n-p)\right\|^2\Big\}\\ &\leq \frac{1}{2}\Big\{\left\|x_n-p\right\|^2+\left\|w_n-p\right\|^2-\|w_n-v_n\|^2+2\lambda_2\langle v_n-w_n,E_2v_n-E_2p\rangle\\ &\qquad-\lambda_2^2\left\|E_2v_n-E_2p\right\|^2\Big\},\end{aligned} \tag{3.38}$$

and hence,

$$\left\|w_n-p\right\|^2 \leq \left\|x_n-p\right\|^2-\|w_n-v_n\|^2+2\lambda_2\|v_n-w_n\|\left\|E_2v_n-E_2p\right\|. \tag{3.39}$$

Putting (3.39) into (3.37),

$$\begin{aligned}\left\|y_{n,i}-p\right\|^2 &\leq \alpha_{n,i}\left\|x_0-p\right\|^2+(1-\alpha_{n,i})\Big(\left\|x_n-p\right\|^2-\|w_n-v_n\|^2+2\lambda_2\|v_n-w_n\|\left\|E_2v_n-E_2p\right\|\Big)\\ &\leq \alpha_{n,i}\left\|x_0-p\right\|^2+\left\|x_n-p\right\|^2-(1-\alpha_{n,i})\|w_n-v_n\|^2+2\lambda_2\|v_n-w_n\|\left\|E_2v_n-E_2p\right\|,\end{aligned} \tag{3.40}$$

this implies that

$$\begin{aligned}(1-\alpha_{n,i})\|w_n-v_n\|^2 &\leq \alpha_{n,i}\left\|x_0-p\right\|^2+\left\|x_n-p\right\|^2-\left\|y_{n,i}-p\right\|^2+2\lambda_2\|w_n-v_n\|\left\|E_2v_n-E_2p\right\|\\ &\leq \alpha_{n,i}\left\|x_0-p\right\|^2+\left\|x_n-y_{n,i}\right\|\left(\left\|x_n-p\right\|+\left\|y_{n,i}-p\right\|\right)\\ &\qquad+2\lambda_2\|w_n-v_n\|\left\|E_2v_n-E_2p\right\|.\end{aligned} \tag{3.41}$$

Therefore, by condition (iii), (3.13), and (3.36), we have

$$\lim_{n\to\infty}\|w_n - v_n\| = 0. \tag{3.42}$$

Furthermore, from (3.6), we have

$$\begin{aligned}
\|y_{n,i} - p\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|w_n - p\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|v_n - p\|^2 \\
&= \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|J_{M_1,\lambda_1}(u_n - \lambda_1 E_1 u_n) - J_{M_1,\lambda_1}(p - \lambda_1 E_1 p)\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|(u_n - \lambda_1 E_1 u_n) - (p - \lambda_1 E_1 p)\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\left(\|u_n - p\|^2 + \lambda_1(\lambda_1 - 2\eta_1)\|E_1 u_n - E_1 p\|^2\right) \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 + (1-\alpha_{n,i})\lambda_1(\lambda_1 - 2\eta_1)\|E_1 u_n - E_1 p\|^2.
\end{aligned} \tag{3.43}$$

Since $0 < e \le \lambda_1 \le f < 2\eta_1$, we have

$$\begin{aligned}
&(1-\alpha_{n,i})\lambda_1(2\eta_1 - \lambda_1)\|E_1 u_n - E_1 p\|^2 \\
&\quad\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - \|y_{n,i} - p\|^2 \\
&\quad\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - y_{n,i}\|(\|x_n - p\| + \|y_{n,i} - p\|).
\end{aligned} \tag{3.44}$$

Then, by condition (iii) and (3.13), we obtain that

$$\lim_{n\to\infty}\|E_1 u_n - E_1 p\| = 0. \tag{3.45}$$

From (3.6), we have

$$\begin{aligned}
\|y_{n,i} - p\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|w_n - p\|^2 \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|v_n - p\|^2.
\end{aligned} \tag{3.46}$$

On the other hand, we note that

$$\begin{aligned}
\|v_n - p\|^2 &\le \|J_{M_1,\lambda_1}(u_n - \lambda_1 E_1 u_n) - J_{M_1,\lambda_1}(p - \lambda_1 E_1 p)\|^2 \\
&\le \langle (u_n - \lambda_1 E_1 u_n) - (p - \lambda_1 E_1 p), v_n - p \rangle \\
&= \frac{1}{2}\Big\{ \|(u_n - \lambda_1 E_1 u_n) - (p - \lambda_1 E_1 p)\|^2 + \|v_n - p\|^2 \\
&\qquad - \|(u_n - \lambda_1 E_1 u_n) - (p - \lambda_1 E_1 p) - (v_n - p)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|u_n - p\|^2 + \|v_n - p\|^2 - \|(u_n - \lambda_1 E_1 u_n) - (p - \lambda_1 E_1 p) - (v_n - p)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|v_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_1 \langle u_n - v_n, E_1 u_n - E_1 p \rangle \\
&\qquad - \lambda_1^2 \|E_1 u_n - E_1 p\|^2 \Big\},
\end{aligned} \tag{3.47}$$

and hence,

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_1 \|u_n - v_n\| \|E_1 u_n - E_1 p\|. \tag{3.48}$$

Putting (3.48) into (3.46),

$$\begin{aligned}
\|y_{n,i} - p\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + (1 - \alpha_{n,i})\Big(\|x_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_1 \|u_n - v_n\| \|E_1 u_n - E_1 p\| \Big) \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - (1 - \alpha_{n,i})\|v_n - u_n\|^2 + 2\lambda_1 \|u_n - v_n\| \|E_1 u_n - E_1 p\|,
\end{aligned} \tag{3.49}$$

this implies that

$$\begin{aligned}
(1 - \alpha_{n,i})\|v_n - u_n\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - \|y_{n,i} - p\|^2 + 2\lambda_1 \|v_n - u_n\| \|E_1 u_n - E_1 p\| \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - y_{n,i}\| \left(\|x_n - p\| + \|y_{n,i} - p\| \right) \\
&\quad + 2\lambda_1 \|v_n - u_n\| \|E_1 u_n - E_1 p\|.
\end{aligned} \tag{3.50}$$

Therefore, by condition (iii), (3.13), and (3.45), we have

$$\lim_{n \to \infty} \|v_n - u_n\| = 0. \tag{3.51}$$

Step 4. We show that $z \in \Theta := F(\mathcal{S}) \cap F(W) \cap \mathrm{GMEP}(F_1, \varphi_1, A) \cap \mathrm{GMEP}(F_2, \varphi_2, B) \cap I(E_1, M_1) \cap I(E_2, M_2)$. Since $\{w_{n_i}\}$ is bounded, there exists a subsequence $\{w_{n_{i_j}}\}$ of $\{w_{n_i}\}$ which converges weakly to $z \in C$. Without loss of generality, we can assume that $w_{n_i} \rightharpoonup z$.

(1) First, we prove that $z \in F(\mathcal{S})$. From (3.22), (3.31), (3.33), (3.42), and (3.51), we get

$$\lim_{n\to\infty}\left\|w_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right\| = 0. \tag{3.52}$$

Since $\{W_n w_n\}$ is bounded and from Lemma 2.8 for all $s \geq 0$, we have

$$\lim_{n\to\infty}\left\|\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - S(s)\left(\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right)\right\| = 0, \tag{3.53}$$

and since

$$\begin{aligned}
\|w_n - S(s)w_n\| \leq{}& \left\|w_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right\| \\
&+ \left\|\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - S(s)\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right\| \\
&+ \left\|S(s)\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - S(s)w_n\right\| \\
\leq{}& 2\left\|w_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right\| \\
&+ \left\|\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds - S(s)\frac{1}{t_n}\int_0^{t_n} S(s)W_n w_n ds\right\|.
\end{aligned} \tag{3.54}$$

It follows from (3.52) and (3.53) that

$$\lim_{n\to\infty}\|w_n - S(s)w_n\| = 0. \tag{3.55}$$

Indeed, from Lemma 2.9 and (3.55), we get $z \in F(\mathcal{S})$, that is, $z = S(s)z, \forall s \geq 0$.

(2) Next, we show that $z \in F(W) = \bigcap_{n=1}^{\infty} F(W_n)$, where $F(W_n) = \bigcap_{i=1}^{n} F(T_i), \forall n \geq 1$, and $F(W_{n+1}) \subset F(W_n)$. Assume that $z \notin F(W)$, then there exists a positive integer m such that $z \notin F(T_m)$, and so $z \notin \bigcap_{i=1}^{m} F(T_i)$. Hence, for any $n \geq m, z \notin \bigcap_{i=1}^{n} F(T_i) = F(W_n)$, that is, $z \neq W_n z$. This together with $z = S(s)z, \forall s \geq 0$ shows that $z = S(s)z \neq S(s)W_n z, \forall s \geq 0$; therefore, we have $z \neq (1/t_n)\int_0^{t_n} S(s)W_n z ds, \forall n \geq m$. It follows from the Opial's condition

and (3.52) that

$$\begin{aligned}
\liminf_{i\to\infty}\|w_{n_i}-z\| &< \liminf_{i\to\infty}\left\|w_{n_i}-\frac{1}{t_{n_i}}\int_0^{t_{n_i}}S(s)W_{n_i}z\,ds\right\| \\
&\le \liminf_{i\to\infty}\left(\left\|w_{n_i}-\frac{1}{t_{n_i}}\int_0^{t_{n_i}}S(s)W_{n_i}w_{n_i}\,ds\right\|\right. \\
&\qquad \left.+\left\|\frac{1}{t_{n_i}}\int_0^{t_{n_i}}S(s)W_{n_i}w_{n_i}\,ds-\frac{1}{t_{n_i}}\int_0^{t_{n_i}}S(s)W_{n_i}z\,ds\right\|\right) \\
&\le \liminf_{i\to\infty}\|w_{n_i}-z\|,
\end{aligned} \tag{3.56}$$

which is a contradiction. Thus, we get $z\in F(W)$.

(3) Now, we prove that $z\in \mathrm{GMEP}(F_1,\varphi,A)$. Since $t_n=T_{r_n}^{(F_1,\varphi_1)}(x_n-r_nAx_n), n\ge 1$, we have for any $y\in C$ that

$$F_1(t_n,y)+\varphi_1(y)-\varphi_1(t_n)+\langle Ax_n,y-t_n\rangle+\frac{1}{r_n}\langle y-t_n,t_n-x_n\rangle\ge 0,\quad \forall y\in C. \tag{3.57}$$

From (A2), we also have

$$\varphi_1(y)-\varphi_1(t_n)+\langle Ax_n,y-t_n\rangle+\frac{1}{r_n}\langle y-t_n,t_n-x_n\rangle\ge F_1(y,t_n),\quad \forall y\in C. \tag{3.58}$$

For t with $0<t\le 1$ and $y\in C$, let $y_t=ty+(1-t)z$. Since $y\in C$ and $z\in C$, we have $y_t\in C$. Then, we have

$$\begin{aligned}
\langle y_t-t_{n_i},Ay_t\rangle &\ge \langle y_t-t_{n_i},Ay_t\rangle-\varphi_1(y_t)+\varphi_1(t_{n_i})-\langle y_t-t_{n_i},Ax_{n_i}\rangle \\
&\quad -\left\langle y_t-t_{n_i},\frac{t_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle+F_1(y_t,t_{n_i}) \\
&= \langle y_t-t_{n_i},Ay_t-At_{n_i}\rangle+\langle y_t-t_{n_i},At_{n_i}-Ax_{n_i}\rangle-\varphi_1(y_t)+\varphi_1(t_{n_i}) \\
&\quad -\left\langle y_t-t_{n_i},\frac{t_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle+F_1(y_t,t_{n_i}).
\end{aligned} \tag{3.59}$$

Since $\|t_{n_i}-x_{n_i}\|\to 0$, we have $\|At_{n_i}-Ax_{n_i}\|\to 0$. Furthermore, from the inverse-strongly monotonicity of A, we have $\langle y_t-t_{n_i},Ay_t-At_{n_i}\rangle\ge 0$. So, from (A4), (A5), and the weak lower semicontinuity of φ_1, $(t_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $t_{n_i} \rightharpoonup z$, we have at the limit

$$\langle y_t - z, Ay_t \rangle \geq -\varphi_1(y_t) + \varphi_1(z) + F_1(y_t, z), \tag{3.60}$$

as $i \to \infty$. From (A1), (A4), and (3.60), we also get

$$\begin{aligned}
0 &= F_1(y_t, y_t) + \varphi_1(y_t) - \varphi_1(y_t) \\
&\leq tF_1(y_t, y) + (1-t)F_1(y_t, z) + t\varphi_1(y) - (1-t)\varphi_1(z) - \varphi(y_t) \\
&= t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)[F_1(y_t, z) + \varphi_1(z) - \varphi_1(y_t)] \\
&\leq t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)\langle y_t - z, Ay_t \rangle \\
&= t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)t\langle y - z, Ay_t \rangle,
\end{aligned} \tag{3.61}$$

and hence,

$$0 \leq F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t) + (1-t)\langle y - z, Ay_t \rangle. \tag{3.62}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F_1(z, y) + \varphi_1(y) - \varphi_1(z) + \langle\, y - z, Az \rangle \geq 0. \tag{3.63}$$

This implies that $z \in \mathrm{GMEP}(F_1, \varphi, A)$. By following the same arguments, we can show that $z \in \mathrm{GMEP}(F_2, \varphi, B)$.

(4) At last, we show that $z \in I(E_2, M_2)$. Infact, since E_2 is η_2-inverse-strongly monotone, this implies that E_2 is$(1/\eta_2)$-Lipschitz continuous monotone mapping and domain of E_2 equal to H. It follows from Lemma 2.5 that $M_2 + E_2$ is a maximal monotone. Let $(y, g) \in G(M_2 + E_2)$, that is, $g - E_2 y \in M_2(y)$. Since $w_{n_i} = J_{M_2,\lambda_2}(v_{n_i} - \lambda_2 E_2 v_{n_i})$, we have $v_{n_i} - \lambda_2 E_2 v_{n_i} \in (I + \lambda_2 M_2)(w_{n_i})$, that is,

$$\frac{1}{\lambda_2}(v_{n_i} - w_{n_i} - \lambda_2 E_2 v_{n_i}) \in M_2(w_{n_i}). \tag{3.64}$$

Since $M_2 + E_2$ is a maximal monotone, we have

$$\left\langle y - w_{n_i}, g - By - \frac{1}{\lambda_2}(v_{n_i} - w_{n_i} - \lambda_2 E_2 v_{n_i}) \right\rangle \geq 0, \tag{3.65}$$

and so

$$\begin{aligned}
\langle y - w_{n_i}, g\rangle &\geq \left\langle y - w_{n_i}, E_2 y + \frac{1}{\lambda_2}(v_{n_i} - w_{n_i} - \lambda_2 E_2 v_{n_i})\right\rangle \\
&= \left\langle y - w_{n_i}, E_2 y - E_2 w_{n_i} + E_2 w_{n_i} - E_2 v_{n_i} + \frac{1}{\lambda_2}(v_{n_i} - w_{n_i})\right\rangle \\
&\geq 0 + \langle y - w_{n_i}, E_2 w_{n_i} - E_2 v_{n_i}\rangle + \left\langle y - w_{n_i}, \frac{1}{\lambda_2}(v_{n_i} - w_{n_i})\right\rangle.
\end{aligned} \tag{3.66}$$

It follows from $\|v_n - w_n\| \to 0$, $\|E_2 v_n - E_2 w_n\| \to 0$, and $w_{n_i} \rightharpoonup z$ that

$$\lim_{i\to\infty}\langle y - w_{n_i}, g\rangle = \langle y - w, g\rangle \geq 0. \tag{3.67}$$

It follows from the maximal monotonicity of $M_2 + E_2$ that $0 \in (M_2 + E_2)(z)$, that is, $z \in I(E_2, M_2)$. By following the same arguments, we can show that $z \in I(E_1, M_1)$. Hence, by (1)–(4), we have $z \in \Theta$.

Step 5. Noting that $x_n = P_{C_n} x_0$, by (2.5), we have

$$\langle x_0 - x_n, y - x_n\rangle \leq 0, \quad \forall y \in C_n. \tag{3.68}$$

Since $\Theta \subset C_n$ and by the continuity of inner product, we obtain from the above inequality that

$$\langle x_0 - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{3.69}$$

By (2.5) again, we conclude that z = $P_\Theta x_0$. This completes the proof. □

Using Theorem 3.1, we obtain the following corollaries.

Corollary 3.2. *Let C be a nonempty closed-convex subset of a real Hilbert Space H. Let F_1, F_2 be bifunctions of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5), and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions with assumption (B1) or (B2). Let A, B, E_1, E_2 be $\alpha, \beta, \eta_1, \eta_2$-inverse-strongly monotone mappings of C into H, respectively. Let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly k-strict pseudocontractions, let $\{T_i : C \to C\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x, \forall x \in C, \forall i \geq 1, t \in [k, 1)$, and let W_{n} be the W-mapping defined by* (2.11) *and W a mapping defined by* (2.12) *with $F(W) \neq \emptyset$.*

Suppose that $\Theta := F(\mathcal{S}) \cap F(W) \cap GMEP(F_1, \varphi_1, A) \cap GMEP(F_2, \varphi_2, B) \cap VI(C, E_1) \cap VI(C, E_2) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C$, $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$, *and*

$$\begin{gathered}
t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\
u_n = T_{q_n}^{(F_2,\varphi_2)}(t_n - q_n B t_n), \\
v_n = P_C(u_n - \lambda_1 E_1 u_n), \\
w_n = P_C(v_n - \lambda_2 E_2 v_n), \\
y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) \frac{1}{t_n} \int_0^{t_n} S(s) W_n w_n ds, \\
C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i} \left(\|x_0\|^2 + 2\langle x_n - x_0, z \rangle \right) \right\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} = P_{C_{n+1}} x_0,
\end{gathered} \tag{3.70}$$

for every $n \geq 0$, *where* $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1)$, $\{r_n\}, \{q_n\} \subset (0, \infty)$, $\lambda_1 \in (0, 2\eta_1)$, *and* $\lambda_2 \in (0, 2\eta_2)$ *satisfy the following conditions:*

(i) $0 < a \leq r_n \leq b < 2\alpha$,

(ii) $0 < c \leq q_n \leq d < 2\beta$,

(iii) $\lim_{n \to \infty} \alpha_{n,i} = 0$,

(iv) $0 < e \leq \lambda_1 \leq f < 2\eta_1$,

(v) $0 < g \leq \lambda_2 \leq j < 2\eta_2$,

then $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

Proof. From Theorem 3.1, put $M = \partial \delta_C$, then $J_{M,\lambda_1} = P_C$ and $J_{M,\lambda_2} = P_C$. So we have $v_n = P_C(u_n - \lambda_1 E_1 u_n)$ and $w_n = P_C(v_n - \lambda_2 E_2 v_n)$. The conclusion of Corollary 3.2 can be obtained from Theorem 3.1 immediately. □

4. Applications

In this section, we study a kind of multiobjective optimization problem by using the result of this paper. We will give an iterative algorithm of solution for the following optimization problem with nonempty set of solutions:

$$\begin{aligned}
&\min \quad h_1(x) \\
&\min \quad h_2(x) \\
&\qquad x \in C,
\end{aligned} \tag{4.1}$$

where $h(x)$ is a convex and lower semicontinuous functional, and define C as a closed-convex subset of a real Hilbert space H. We denote the set of solutions of (4.1) by $M(h_1)$ and $M(h_2)$. Let $F_i : C \times C \to \mathbb{R}$ be a bifunction defined by $F_i(x,y) = h_i(y) - h_i(x)$. We consider the equilibrium problem, and it is obvious that $EP(F_i) = M(h_i), i = 1,2$. Therefore, from Theorem 3.1, we obtain the following Corollaries.

Corollary 4.1. *Let C be a nonempty closed-convex subset of a real Hilbert Space H. Let $h_1, h_2 : C \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let E_1, E_2 be η_1, η_2-inverse-strongly monotone mappings of C into H, respectively, and let $M_1, M_2 : H \to 2^H$ be maximal monotone mappings. Let $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C, and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly k-strict pseudocontractions, let $\{T_i : C \to C\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x,\ \forall x \in C,\ \forall i \ge 1,\ t \in [k,1)$, and let W_n be the W-mapping defined by* (2.11) *and W a mapping defined by* (2.12) *with $F(W) \neq \emptyset$. Suppose that $\Theta := F(\mathcal{S}) \cap F(W) \cap M(h_1) \cap M(h_2) \cap I(E_1, M_1) \cap I(E_2, M_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C, C_{1,i} = C,\ C_1 = \bigcap_{i=1}^{\infty} C_{1,i},\ x_1 = P_{C_1} x_0$, and*

$$\begin{gathered}
h_1(t) - h_1(t_n) + \frac{1}{r_n}\langle t - t_n, t_n - x_n\rangle \ge 0, \quad \forall t \in C,\\
h_2(u) - h_2(u_n) + \frac{1}{q_n}\langle u - u_n, u_n - t_n\rangle \ge 0, \quad \forall u \in C,\\
v_n = J_{M_1,\lambda_1}(u_n - \lambda_1 E_1 u_n),\\
w_n = J_{M_2,\lambda_2}(v_n - \lambda_2 E_2 v_n),\\
y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) \frac{1}{t_n}\int_0^{t_n} S(s) W_n w_n ds,\\
C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \le \|x_n - z\|^2 + \alpha_{n,i}\Big(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle\Big)\right\},\\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i},\\
x_{n+1} = P_{C_{n+1}} x_0,
\end{gathered} \tag{4.2}$$

for every $n \ge 0$, where $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1)$, $\{r_n\}, \{q_n\} \subset (0,\infty)$, $\lambda_1 \in (0, 2\eta_1)$, and $\lambda_2 \in (0, 2\eta_2)$ satisfy the following conditions:

(i) $\liminf_{n\to\infty} r_n > 0$,

(ii) $\liminf_{n\to\infty} q_n > 0$,

(iii) $\lim_{n\to\infty} \alpha_{n,i} = 0$,

(iv) $0 < e \le \lambda_1 \le f < 2\eta_1$,

(v) $0 < g \le \lambda_2 \le j < 2\eta_2$,

then $\{x_n\}$ converges strongly to $P_\Theta x_0$.

Proof. From Theorem 3.1, put $F_1(t_n, t) = h_1(t) - h_1(t_n), F_2(u_n, u) = h_2(u) - h_2(u_n)$, and $A, B, \varphi_1, \varphi_2 \equiv 0$. The conclusion of Corollary 4.1 can be obtained from Theorem 3.1 immediately. □

Corollary 4.2. *Let C be a nonempty closed-convex subset of a real Hilbert Space H. Let $h_1, h_2 : C \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let E_1, E_2 be η_1, η_2-inverse-strongly monotone mappings of C into H, respectively, and let $M_1, M_2 : H \to 2^H$ be maximal monotone mappings. Let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly k-strict pseudocontractions, let $\{T_i : C \to C\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x$, $\forall x \in C$, $\forall i \geq 1$, $t \in [k, 1)$, and let W_n be the W-mapping defined by* (2.11) *and W a mapping defined by* (2.12) *with $F(W) \neq \emptyset$. Suppose that $\Theta := F(\mathcal{S}) \cap F(W) \cap M(h_1) \cap M(h_2) \cap VI(C, E_1) \cap VI(C, E_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C, C_{1,i} = C, C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, x_1 = P_{C_1} x_0$, and*

$$
\begin{gathered}
h_1(t) - h_1(t_n) + \frac{1}{r_n}\langle t - t_n, t_n - x_n \rangle \geq 0, \quad \forall t \in C, \\
h_2(u) - h_2(u_n) + \frac{1}{q_n}\langle u - u_n, u_n - t_n \rangle \geq 0, \quad \forall u \in C, \\
v_n = P_C(u_n - \lambda_1 E_1 u_n), \\
w_n = P_C(v_n - \lambda_2 E_2 v_n), \\
y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) \frac{1}{t_n} \int_0^{t_n} S(s) W_n w_n \, ds, \\
C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle x_n - x_0, z \rangle \right) \right\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} = P_{C_{n+1}} x_0,
\end{gathered}
\tag{4.3}
$$

for every $n \geq 0$, where $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1)$, $\{r_n\}, \{q_n\} \subset (0, \infty), \lambda_1 \in (0, 2\eta_1)$, and $\lambda_2 \in (0, 2\eta_2)$ satisfy the following conditions:

(i) $\liminf_{n \to \infty} r_n > 0$,

(ii) $\liminf_{n \to \infty} q_n > 0$,

(iii) $\lim_{n \to \infty} \alpha_{n,i} = 0$,

(iv) $0 < e \leq \lambda_1 \leq f < 2\eta_1$,

(v) $0 < g \leq \lambda_2 \leq j < 2\eta_2$,

then $\{x_n\}$ converges strongly to $P_{\Theta} x_0$.

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