

## Research Article

# Oscillatory Solutions of Neutral Equations with Polynomial Nonlinearities

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Existence uniqueness of an oscillatory solution for nonlinear neutral equations by fixed point method is proved.

## 1. Introduction

In [1, 2], we have considered a lossless transmission line terminated by a nonlinear resistive load and parallel connected capacitance (cf. Figure 1). The nonlinear boundary condition is caused by the polynomial type  $V$ - $I$  characteristics of the nonlinear load at the second end of the transmission line (cf. Figure 1).

The voltage and current  $u(x, t)$ ,  $i(x, t)$  of the lossless transmission line can be found by solving the following mixed problem for the hyperbolic partial differential system:

$$C \frac{\partial u(x, t)}{\partial t} + \frac{\partial i(x, t)}{\partial x} = 0, \quad L \frac{\partial i(x, t)}{\partial t} + \frac{\partial u(x, t)}{\partial x} = 0, \quad (1.1)$$

$$E(t) - u(0, t) = R_0 i(0, t), \quad t \geq 0,$$

$$C_0 \frac{du(\Lambda, t)}{dt} = i(\Lambda, t) - f(u(\Lambda, t)), \quad t \geq 0, \quad (1.2)$$

$$u(x, 0) = u_0(x), \quad i(x, 0) = i_0(x), \quad x \in [0, \Lambda], \quad (1.3)$$

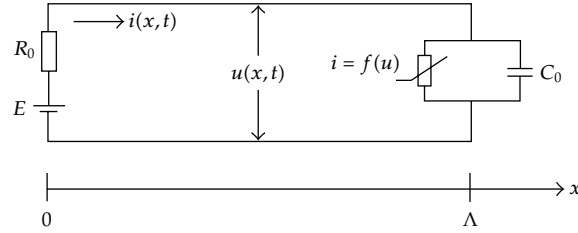


Figure 1

where  $u_0(x)$  and  $i_0(x)$  are prescribed initial functions,  $\Lambda$  is the length of the line,  $C$  is the per-unit length capacitance, and  $L$  is per-unit length inductance (cf. [3–10]). Here, the  $V$ - $I$  characteristic of the nonlinear resistive load is  $i = f(u) = \sum_{n=1}^p r_n u^n$ , where  $r_n$  are real numbers,  $C_0$  is parallel connected capacitance,  $E$  is the source voltage,  $R_0$  is the source resistance, and  $Z_0 = \sqrt{L/C}$  is the line characteristic impedance.

The above formulated mixed problem can be reduced (cf. [1, 2, 11]) to an equivalent initial value problem for a neutral functional differential equation (cf. [12]). Here, we consider the problem of an existence uniqueness of oscillatory solutions of the equation

$$\begin{aligned} \frac{du(t)}{dt} &= \frac{2E}{C_0(Z_0 + R_0)} - \frac{u(t)}{C_0 Z_0} - \frac{1}{C_0} \sum_{n=1}^p r_n [u(t)]^n - \frac{(Z_0 - R_0)u(t-2T)}{Z_0 C_0 (Z_0 + R_0)} \\ &+ \frac{Z_0 - R_0}{C_0(Z_0 + R_0)} \sum_{n=1}^p r_n [u(t-2T)]^n + \frac{Z_0 - R_0}{Z_0 + R_0} \frac{du(t-2T)}{dt}, \quad t \geq T, \\ u(t) &= v_0(t), \quad \frac{du(t)}{dt} = \frac{dv_0(t)}{dt}, \quad t \in [-T, T], \end{aligned} \quad (1.4)$$

where  $(x, t) \in \Pi = \{(x, t) \in R^2 : (x, t) \in [0, \Lambda] \times [0, \infty)\}$ ,  $\kappa = |Z_0 - R_0|/(Z_0 + R_0) < 1$ ,  $u(t) = u(\Lambda, t)$ . In fact, (1.4) is differential difference equation, and the initial function should be prescribed on an interval with length  $2T$ . Let us note that the initial function  $v_0(t)$  can be obtained shifting the initial function  $u_0(x)$  from (1.3) along the characteristics  $x - \nu t = \text{const.}$ , ( $\nu = 1/\sqrt{LC}$ ) on  $[0, T]$  and along the characteristics  $x + \nu t = \text{const.}$  on  $[-T, 0]$  (cf. [1, 2]). So, we obtain an initial function  $v_0(t)$  on  $[-T, T]$ .

Now, we are able to formulate the main problem: to find a solution of (1.4) with advanced prescribed zeros on the interval  $[t_0, \infty)$ ,  $T = t_0$ .

Let  $S_T = \{\tau_k\}_{k=0}^n$ ,  $n \in N$  be the set of zeros of the initial function; that is,  $v_0(\tau_k) = 0$  such that  $\tau_0 = -T$ ,  $\tau_n = T \equiv t_0$ .

Let  $S = \{t_k\}_{k=0}^\infty$  be a strictly increasing sequence of real numbers satisfying the following conditions (C):

- (C1)  $\lim_{k \rightarrow \infty} t_k = \infty$ ,
- (C2)  $0 < l_0 = \inf\{t_{k+1} - t_k : k = 0, 1, 2, \dots\} \leq \sup\{t_{k+1} - t_k : k = 0, 1, 2, \dots\} = T_0 < \infty$ ,
- (C3) for every  $k$  there is  $s < k$  such that  $t_k - T = t_s$  where  $t_s \in S_T \cup S$ .

Introduce the sets:  $C^1[t_0, \infty)$  consisting of all continuous and bounded functions differentiable with bounded derivatives on every interval  $(t_k, t_{k+1})$  (the derivatives at  $t_k$  do

not necessary exist),  $M_S = \{u(\cdot) \in C^1[t_0, \infty) : u(t_k) = 0 \ (k = 0, 1, 2, \dots)\}$ ,  $M_{SU} = \{u(\cdot) \in M_S : |u(t)| \leq U_0 e^{\mu(t-t_k)}, \ t \in [t_k, t_{k+1}]\}$ , where  $U_0, \mu$  are positive constants prescribed below.

We assume that  $|v_0(t)| \leq U_0 e^{\mu(t-\tau_k)}, \ t \in [\tau_k, \tau_{k+1}], \ (k = 0, 1, 2, \dots, n-1)$ .

The set  $M_{SU}$  turns out into a complete uniform space with respect to the family of pseudometrics  $\rho_\mu^{(k)}(f, g) = \max\{\rho_k(f, g), \rho_k(\dot{f}, \dot{g})\}, \ (k = 0, 1, 2, \dots)$ , where  $\rho_k(f, g) = \max\{e^{-\mu(t-t_k)}|f(t) - g(t)| : t \in [t_k, t_{k+1}]\}$ ,  $\rho_k(\dot{f}, \dot{g}) = \max\{e^{-\mu(t-t_k)}|\dot{f}(t) - \dot{g}(t)| : t \in [t_k, t_{k+1}]\}$ .

One can verify that  $M_{SU}$  is closed subset of  $C^1[t_0, \infty)$  with respect to the above metric.

*Remark 1.1.* The functions from  $M_S$  are not necessary differentiable at  $t_k \ (k = 0, 1, 2, \dots)$ . That is why we consider a space with a countable family of pseudometrics, and then, we have to apply the fixed point theory from [13].

Define the operator  $B : M_{SU} \rightarrow M_{SU}$  by

$$B(u)(t) := \int_{t_k}^t U(u)(s) ds - \left( \frac{t - t_k}{t_{k+1} - t_k} \right) \int_{t_k}^{t_{k+1}} U(u)(s) ds, \quad t \in [t_k, t_{k+1}], \ (k = 0, 1, 2, \dots), \quad (1.5)$$

where

$$\begin{aligned} U(u)(t) = & \frac{2E}{C_0(Z_0 + R_0)} - \frac{u(t)}{C_0 Z_0} - \frac{1}{C_0} \sum_{n=1}^p r_n [u(t)]^n - \frac{\kappa(K_T u)(t)}{Z_0 C_0} \\ & + \frac{\kappa}{C_0} \sum_{n=1}^p r_n [(K_T u)(t)]^n + \kappa \frac{d(K_T u)(t)}{dt}, \quad t \geq T, \end{aligned} \quad (1.6)$$

and  $(K_T u)(t) = u(t - 2T)$  is M. A. Krasnoselskii operator (cf. [14]).

*Remark 1.2.* The operator  $K_T$  is well defined, because the initial function is defined on the interval  $[-T, T]$ . We notice that  $K_T$  maps  $M_S$  into itself. Indeed, consider the set  $C^1[-T, \infty)$  consisting of all continuous and bounded functions differentiable with bounded derivatives on every interval  $(t_k, t_{k+1})$ . Introduce the set  $M_S^{v_0} = \{u(\cdot) \in C^1[-T, \infty) : u(t) = v_0(t), \ t \in [-T, T]\}$ . Then,  $K_T$  assigns to every function  $u(\cdot) \in M_S$  the function  $\tilde{u}(\cdot) \in M_S^{v_0}$  translated to the right on the interval  $[T, \infty)$ . So, the function  $(K_T u)(t)$  coincides with  $v_0(t)$  on  $[t_0, t_0 + 2T]$ . Besides  $t_k - 2T = t_s$ , and then

$$(K_T u)(t_k) = \begin{cases} u(t_k - 2T) = v_0(t_s) = 0, & t_k \in [T, 3T], \\ u(t_k - 2T) = u(t_n) = 0, & t \in (3T, \infty), \end{cases} \quad (1.7)$$

that is,  $(K_T u)(\cdot) \in M_S$ .

## 2. Main Results

**Lemma 2.1.** *If  $E \leq U_0$ , problem (1.4) has a solution  $u(\cdot) \in M_{SU}$  iff the operator  $B$  has a fixed point in  $M_{SU}$ , that is,*

$$u(t) = B(u)(t). \quad (2.1)$$

*Proof.* Let  $u(\cdot) \in M_{SU}$  be a solution of (1.4). Then, integrating (1.4) on the interval  $[t_k, t] \subset [t_k, t_{k+1}]$  ( $k = 0, 1, 2, \dots$ ), we obtain  $u(t) - u(t_k) = \int_{t_k}^t U(u)(s)ds \Leftrightarrow u(t) = \int_{t_k}^t U(u)(s)ds$ , and then,

$$u(t) = \int_{t_k}^t U(u)(s)ds \implies 0 = u(t_{k+1}) = \int_{t_k}^{t_{k+1}} U(u)(s)ds \implies \int_{t_k}^{t_{k+1}} U(u)(s)ds = 0. \quad (2.2)$$

Therefore,  $u(t)$  satisfies

$$u(t) = \int_{t_k}^t U(u)(s)ds \iff u(t) = \int_{t_k}^t U(u)(s)ds - \left( \frac{t - t_k}{t_{k+1} - t_k} \right) \int_{t_k}^{t_{k+1}} U(u)(s)ds, \quad (2.3)$$

that is,  $u(\cdot)$  is a fixed point of  $B$ .

Conversely, let  $u(\cdot) \in M_{SU}$  be a solution of  $u = B(u)$ ; that is,

$$u(t) = \int_{t_k}^t U(u)(s)ds - \left( \frac{t - t_k}{t_{k+1} - t_k} \right) \int_{t_k}^{t_{k+1}} U(u)(s)ds. \quad (2.4)$$

Then, introducing  $\mu_0 = \mu T_0$ , we obtain

$$\begin{aligned} & \left| \int_{t_k}^{t_{k+1}} U(u)(s)ds \right| \\ & \leq \frac{2E}{C_0(Z_0 + R_0)} \int_{t_k}^{t_{k+1}} e^{\mu(t-t_k)} dt + \frac{1}{C_0 Z_0} \int_{t_k}^{t_{k+1}} |u(t)| dt \\ & \quad + \frac{1}{C_0} \sum_{n=1}^p |r_n| \int_{t_k}^{t_{k+1}} |u(t)|^n dt + \frac{\kappa}{Z_0 C_0} \int_{t_k}^{t_{k+1}} |u(t - 2T)| dt \\ & \quad + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| \int_{t_k}^{t_{k+1}} |u(t - 2T)|^n dt + \kappa \left| \int_{t_k}^{t_{k+1}} \dot{u}(t - 2T) dt \right| \\ & \leq \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^p |r_n| U_0^n \int_{t_k}^{t_{k+1}} e^{n\mu(t-t_k)} dt \\ & \quad + \frac{\kappa U_0 e^{-2\mu T}}{Z_0 C_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| U_0^n e^{-2n\mu T} \\ & \quad \times \int_{t_k}^{t_{k+1}} e^{n\mu(t-t_k)} dt + \kappa |u(t_{k+1} - 2T) - u(t_k - 2T)| \\ & \leq \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} \frac{e^{\mu T_0} - 1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu T_0} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^p |r_n| U_0^n \frac{e^{n\mu T_0} - 1}{n\mu} \\ & \quad + \frac{U_0 \kappa e^{-2\mu T}}{C_0 Z_0} \frac{e^{\mu T_0} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| U_0^n e^{-2n\mu T} \frac{e^{n\mu T_0} - 1}{n\mu} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{e^{\mu_0} - 1}{\mu C_0} \left( \frac{2U_0 e^{-\mu T}}{Z_0 + R_0} + \frac{U_0(1 + \kappa e^{-2\mu T})}{Z_0} \right) + \frac{1}{\mu C_0} \sum_{n=1}^p \frac{|r_n| U_0^n (1 + \kappa e^{-2n\mu T}) (e^{n\mu_0} - 1)}{n} \\
&\equiv M(\mu).
\end{aligned} \tag{2.5}$$

Let us assume that  $|\int_{t_k}^{t_{k+1}} U(u)(t)dt| = \beta > 0$ . We have just obtained that  $\beta \leq M(\mu)$ . Then, for sufficiently large  $\mu > 0$  (and sufficiently small  $T_0 > 0$ ), one can reach the inequality  $M(\mu) < \beta$ . Consequently,  $\int_{t_k}^{t_{k+1}} U(u)(t)dt = 0$ . It follows that  $u(t) = \int_{t_k}^t U(u)(s)ds$  and, after a differentiation, we obtain (1.4).

Lemma 2.1 is thus proved.  $\square$

**Theorem 2.2.** Let  $S_T = \{\tau_k\}_{k=0}^n$ ,  $n \in \mathbb{N}$  be the set of zeros of the initial function; that is,  $v_0(\tau_k) = 0$  and  $v_0(\cdot) \in C^1[-T, T]$ . If  $E \leq U_0$ ,  $|v_0(t)| \leq U_0 e^{\mu(t-\tau_k)}$ ,  $t \in [\tau_k, \tau_{k+1}]$ ,  $v_0(t_0) = 0$ , then, there exists a unique oscillatory solution of the initial value problem (1.4), belonging to  $M_{SU}$ .

*Proof.* We show that  $B$  maps  $M_{SU}$  into itself; that is,  $u \in M_{SU} \Rightarrow B(u) \in M_{SU}$ .

Indeed, for every  $u(\cdot) \in M_{SU}$ , the function  $B(u)(t)$  is continuous on  $[t_0, \infty)$  and differentiable on every  $(t_k, t_{k+1})$ . We have also  $B(u)(t_k) = 0$  and  $B(u)(t_{k+1}) = 0$ .

We show that  $|B(u)(t)| \leq U_0 e^{\mu(t-t_k)}$ ,  $t \in [t_k, t_{k+1}]$ . (The last inequalities imply that  $B(u)(t)$  is bounded because  $e^{\mu(t-t_k)} \leq e^{\mu T_0}$ ,  $t \in [T, \infty)$ .)

We notice that  $|(t - t_k)/(t_{k+1} - t_k)| \leq 1$ ,  $t \in [t_k, t_{k+1}]$ . For sufficiently large  $\mu$ , we obtain for  $t \in [t_k, t_{k+1}]$

$$|B(u)(t)| \leq \left| \int_{t_k}^t U(u)(s)ds \right| + \left| \int_{t_k}^{t_{k+1}} U(u)(s)ds \right| \equiv B_1 + B_2. \tag{2.6}$$

We have

$$\begin{aligned}
B_1 &\leq \left[ \frac{2}{C_0(Z_0 + R_0)} \int_{t_k}^t |E(s - T)|ds + \frac{1}{C_0 Z_0} \int_{t_k}^t |u(s)|ds + \frac{1}{C_0} \sum_{n=1}^p |r_n| \int_{t_k}^t |u(s)|^n ds \right. \\
&\quad \left. + \frac{\kappa}{Z_0 C_0} \int_{t_k}^t |u(s - 2T)|ds + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| \int_{t_k}^t |u(s - 2T)|^n ds \right] + \kappa \left| \int_{t_k}^t u(s - 2T)ds \right| \\
&\leq \left[ \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} \frac{e^{\mu(t-t_k)} - 1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu(t-t_k)} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^p |r_n| U_0^n \int_{t_k}^t e^{n\mu(s-t_k)} ds \right. \\
&\quad \left. + \frac{\kappa U_0 e^{-2\mu T}}{Z_0 C_0} \frac{e^{\mu(t-t_k)} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| U_0^n e^{-2n\mu T} \int_{t_k}^t e^{n\mu(s-t_k)} ds \right] + \kappa |u(t - 2T)| \\
&\leq e^{\mu(t-t_k)} U_0 \left[ \frac{1}{\mu C_0} \left( \frac{2e^{-\mu T}}{Z_0 + R_0} + \frac{1 + \kappa e^{-2\mu T}}{Z_0} + \sum_{n=1}^p \frac{|r_n| U_0^{n-1} (e^{(n-1)\mu T_0} - 1)(1 + \kappa e^{-2n\mu T})}{n} \right) \right. \\
&\quad \left. + \kappa e^{-2\mu T} \right],
\end{aligned}$$

$$\begin{aligned}
B_2 &\leq \left[ \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^p |r_n| U_0^n \int_{t_k}^{t_{k+1}} e^{n\mu(s-T)} ds \right. \\
&\quad \left. + \frac{\kappa U_0 e^{-2\mu T}}{Z_0 C_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| U_0^n e^{-2n\mu T} \int_{t_k}^{t_{k+1}} e^{n\mu(s-T)} ds \right] \\
&\quad + \kappa |u(t_{k+1} - 2T) - u(t_k - 2T)| \\
&\leq \left[ \frac{2U_0 e^{-\mu T}}{C_0(Z_0 + R_0)} \frac{e^{\mu T_0} - 1}{\mu} + \frac{U_0}{C_0 Z_0} \frac{e^{\mu T_0} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^p |r_n| U_0^n \frac{e^{n\mu T_0} - 1}{n\mu} \right. \\
&\quad \left. + \frac{\kappa U_0 e^{-2\mu T}}{C_0 Z_0} \frac{e^{\mu T_0} - 1}{\mu} + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| U_0^n e^{-2n\mu T} \frac{e^{n\mu T_0} - 1}{n\mu} \right] \\
&\leq e^{\mu(t-t_k)} \frac{U_0}{\mu C_0} \left( \frac{2e^{-\mu T} (e^{\mu T_0} - 1)}{Z_0 + R_0} + \frac{(e^{\mu T_0} - 1)(1 + \kappa e^{-2\mu T})}{Z_0} \right. \\
&\quad \left. + \sum_{n=1}^p \frac{|r_n| U_0^{n-1} (1 + \kappa e^{-2n\mu T}) (e^{n\mu T_0} - 1)}{n} \right).
\end{aligned} \tag{2.7}$$

Therefore, for sufficiently large  $\mu > 0$ , we obtain

$$\begin{aligned}
|(Bu)(t)| &\leq e^{\mu(t-t_k)} U_0 \left[ \frac{1}{\mu C_0} \left( \frac{2e^{-\mu T}}{Z_0 + R_0} + \frac{1 + \kappa e^{-2\mu T}}{Z_0} + \sum_{n=1}^p \frac{|r_n| U_0^{n-1} (e^{(n-1)\mu T_0} - 1)(1 + \kappa e^{-2n\mu T})}{n} \right) \right. \\
&\quad \left. + \kappa e^{-2\mu T} \right] \\
&\quad + e^{\mu(t-t_k)} U_0 \frac{1}{\mu C_0} \left( \frac{2e^{-\mu T} (e^{\mu T_0} - 1)}{Z_0 + R_0} + \frac{(e^{\mu T_0} - 1)(1 + \kappa e^{-2\mu T})}{Z_0} \right. \\
&\quad \left. + \sum_{n=1}^p |r_n| U_0^{n-1} \frac{e^{n\mu T_0} - 1}{n} (1 + \kappa e^{-2n\mu T}) \right) \\
&\leq e^{\mu(t-t_k)} U_0 \left[ \frac{1}{\mu C_0} \left( \frac{2e^{-\mu T} e^{\mu T_0}}{Z_0 + R_0} + \frac{e^{\mu T_0} (1 + \kappa e^{-2\mu T})}{Z_0} \right) \right. \\
&\quad \left. + \sum_{n=1}^p \frac{|r_n| U_0^{n-1} (e^{n\mu T_0} + e^{(n-1)\mu T_0} - 2)(1 + \kappa e^{-2n\mu T})}{n} \right) + \kappa e^{-2\mu T} \Big] \\
&\leq e^{\mu(t-t_k)} U_0.
\end{aligned} \tag{2.8}$$

Consequently, the operator  $B$  maps  $M_{SU}$  into itself.

We show that  $B$  is a contractive operator. Indeed,

$$\begin{aligned} |B(u)(t) - B(\bar{u})(t)| &\leq \left| \int_{t_k}^t [U(u)(s) - U(\bar{u})(s)] ds \right| + \left| \int_{t_k}^{t_{k+1}} [U(u)(s) - U(\bar{u})(s)] ds \right| \\ &\equiv B_1 + B_2, \quad t \in [t_k, t_{k+1}]. \end{aligned} \quad (2.9)$$

We have

$$\begin{aligned} B_1 &\leq \left[ \frac{1}{C_0 Z_0} \int_{t_k}^t |u(s) - \bar{u}(s)| ds + \frac{1}{C_0} \sum_{n=1}^p |r_n| \int_{t_k}^t |u^n(s) - \bar{u}^n(s)| ds \right. \\ &\quad \left. + \frac{\kappa}{Z_0 C_0} \int_{t_k}^t |u(s-2T) - \bar{u}(s-2T)| ds + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| \int_{t_k}^t |u^n(s-2T) - \bar{u}^n(s-2T)| ds \right] \\ &\quad + \kappa \left| \int_{t_k}^t (\dot{u}(s-2T) - \dot{\bar{u}}(s-2T)) ds \right| \\ &\leq \left[ \frac{\rho_k(u, \bar{u})}{C_0 Z_0} \frac{e^{\mu(t-t_k)} - 1}{\mu} + \frac{1}{C_0} \sum_{n=1}^p n |r_n| \operatorname{ess\,sup} \{ |u^{n-1}(s)| : s \in [t_k, t_{k+1}] \} \int_{t_k}^t |u(s) - \bar{u}(s)| ds \right. \\ &\quad + \frac{\kappa}{Z_0 C_0} \rho_k(u, \bar{u}) e^{-2\mu T} \frac{e^{\mu(t-t_k)} - 1}{\mu} \\ &\quad + \frac{\kappa}{C_0} \sum_{n=1}^p n |r_n| \operatorname{ess\,sup} \{ |u^{n-1}(s-2T)| : s \in [t_k, t_{k+1}] \} \int_{t_k}^t |u(s-2T) - \bar{u}(s-2T)| ds \left. \right] \\ &\quad + \kappa \rho_k(\dot{u}, \dot{\bar{u}}) e^{-2\mu T} \frac{e^{\mu(t-t_k)} - 1}{\mu} \\ &\leq e^{\mu(t-t_k)} \left[ \frac{\rho_k(u, \bar{u})}{\mu C_0 Z_0} + \frac{\rho_k(u, \bar{u})}{\mu C_0} \sum_{n=1}^p n |r_n| U_0^{n-1} e^{(n-1)\mu(t_{k+1}-t_k)} + \frac{\kappa \rho_k(u, \bar{u}) e^{-2\mu T}}{\mu Z_0 C_0} \right. \\ &\quad \left. + \frac{\kappa \rho_k(u, \bar{u}) e^{-2\mu T}}{\mu C_0} \sum_{n=1}^p n |r_n| U_0^{n-1} e^{-2(n-1)\mu T} e^{(n-1)\mu(t_{k+1}-t_k)} \right] + e^{\mu(t-t_k)} \frac{\kappa \rho_k(\dot{u}, \dot{\bar{u}}) e^{-2\mu T}}{\mu} \\ &\leq e^{\mu(t-t_k)} \rho_k(\dot{u}, \dot{\bar{u}}) \left[ \frac{1}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^p n |r_n| U_0^{n-1} (1 + \kappa e^{-2n\mu T}) e^{(n-1)\mu T_0} \right) + \frac{\kappa e^{-2\mu T}}{\mu} \right] \\ &\leq e^{\mu(t-t_k)} \rho_\mu^{(k)}(u, \bar{u}) \left[ \frac{1}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^p n |r_n| U_0^{n-1} (1 + \kappa e^{-2n\mu T}) e^{(n-1)\mu T_0} \right) + \frac{\kappa e^{-2\mu T}}{\mu} \right], \\ B_2 &\leq \left[ \frac{1}{C_0 Z_0} \int_{t_k}^{t_{k+1}} |u(s) - \bar{u}(s)| ds + \frac{1}{C_0} \sum_{n=1}^p |r_n| \int_{t_k}^{t_{k+1}} |u^n(s) - \bar{u}^n(s)| ds \right. \\ &\quad \left. + \frac{\kappa}{Z_0 C_0} \int_{t_k}^{t_{k+1}} |u(s-2T) - \bar{u}(s-2T)| ds + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| \int_{t_k}^{t_{k+1}} |u^n(s-2T) - \bar{u}^n(s-2T)| ds \right] \end{aligned}$$

$$\begin{aligned}
& + \kappa \left| \int_{t_k}^{t_{k+1}} (\dot{u}(s-2T) - \dot{\bar{u}}(s-2T)) ds \right| \\
& \leq \left[ \frac{\rho_k(u, \bar{u})}{C_0 Z_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} \right. \\
& \quad + \frac{1}{C_0} \sum_{n=1}^p |r_n| n. \operatorname{ess\,sup} \left\{ |u^{n-1}(s)| : s \in [t_k, t_{k+1}] \right\} \int_{t_k}^{t_{k+1}} |u(s) - \bar{u}(s)| ds \\
& \quad + \frac{\kappa}{Z_0 C_0} \rho_k(u, \bar{u}) e^{-2\mu T} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} \\
& \quad \left. + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| n. \operatorname{ess\,sup} \left\{ |u^{n-1}(s-2T)| : s \in [t_k, t_{k+1}] \right\} \int_{t_k}^{t_{k+1}} |u(s-2T) - \bar{u}(s-2T)| ds \right] \\
& \leq \left[ \frac{\rho_k(u, \bar{u})}{C_0 Z_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} + \frac{\rho_k(u, \bar{u})}{C_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu(t_{k+1}-t_k)} \right. \\
& \quad + \frac{\kappa \rho_k(u, \bar{u}) e^{-2\mu T}}{Z_0 C_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} \\
& \quad \left. + \frac{\kappa \rho_k(u, \bar{u}) e^{-2\mu T}}{C_0} \frac{e^{\mu(t_{k+1}-t_k)} - 1}{\mu} \sum_{n=1}^p n |r_n| U_0^{n-1} e^{(n-1)(\mu T_0 - 2\mu T)} \right] \\
& \leq \rho_k(\dot{u}, \ddot{u}) \frac{e^{\mu T_0} - 1}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} (1 + \kappa e^{-2n\mu T}) \right) \\
& \leq \rho_u^{(k)}(u, \bar{u}) \frac{e^{\mu T_0} - 1}{\mu^2 C_0} \left( \frac{1 + \kappa e^{-2\mu T}}{Z_0} + \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} (1 + \kappa e^{-2n\mu T}) \right).
\end{aligned} \tag{2.10}$$

Consequently,

$$\begin{aligned}
& |B(u)(t) - B(\bar{u})(t)| \\
& \leq e^{\mu(t-t_k)} \rho_\mu^{(k)}(u, \bar{u}) \left[ \frac{1}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^p n |r_n| U_0^{n-1} (1 + \kappa e^{-2n\mu T}) e^{(n-1)\mu T_0} \right) + \frac{\kappa e^{-2\mu T}}{\mu} \right] \\
& \quad + \rho_u^{(k)}(u, \bar{u}) \frac{e^{\mu T_0} - 1}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} (1 + \kappa e^{-2n\mu T}) \right) \\
& \leq \rho_u^{(k)}(u, \bar{u}) \left[ \frac{e^{\mu T_0}}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} (1 + \kappa e^{-2n\mu T}) \right) + \frac{\kappa e^{-2\mu T}}{\mu} \right] \\
& \equiv e^{\mu(t-t_k)} K_U \rho_\mu^{(k)}(u, \bar{u}).
\end{aligned} \tag{2.11}$$



Therefore,  $\rho_k(Bu, B\bar{u}) \leq K_U \rho_\mu^{(k)}(u, \bar{u})$ .

It remains to estimate the derivative of  $B$ .

We have

$$\begin{aligned} |\dot{B}(u)(t) - \dot{B}(\bar{u})(t)| &\leq |U(u)(s) - U(\bar{u})(s)| \\ &\quad + \frac{1}{t_{k+1} - t_k} \left| \int_{t_k}^{t_{k+1}} [U(u)(s) - U(\bar{u})(s)] ds \right| \equiv \dot{B}_1 + \dot{B}_2. \end{aligned} \quad (2.12)$$

We have

$$\begin{aligned} \dot{B}_1 &\leq \frac{1}{C_0 Z_0} |u(t) - \bar{u}(t)| + \frac{1}{C_0} \sum_{n=1}^p |r_n| |u^n(t) - \bar{u}^n(t)| + \frac{\kappa}{C_0 Z_0} |u(t - 2T) - \bar{u}(t - 2T)| \\ &\quad + \frac{\kappa}{C_0} \sum_{n=1}^p |r_n| |u^n(t - 2T) - \bar{u}^n(t - 2T)| + \kappa |\dot{u}(t - 2T) - \dot{\bar{u}}(t - 2T)| \\ &\leq \frac{e^{\mu(t-t_k)} \rho_k(u, \bar{u})}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^p |r_n| n \operatorname{ess\,sup} \{ u^{n-1}(t) : t \in [t_k, t_{k+1}] \} |u(t) - \bar{u}(t)| \\ &\quad + \frac{\kappa e^{\mu(t-t_k)} \rho_k(u, \bar{u}) e^{-2\mu T}}{C_0 Z_0} \\ &\quad + \frac{\kappa}{C_0} \sum_{n=1}^p n |r_n| \operatorname{ess\,sup} \{ u^{n-1}(t - 2T) : t \in [t_k, t_{k+1}] \} |u(t - 2T) - \bar{u}(t - 2T)| \\ &\quad + \kappa |\dot{u}(t - 2T) - \dot{\bar{u}}(t - 2T)| \\ &\leq \frac{e^{\mu(t-t_k)} \rho_k(\dot{u}, \dot{\bar{u}})}{\mu C_0 Z_0} + \frac{e^{\mu(t-t_k)} \rho_k(\dot{u}, \dot{\bar{u}})}{\mu C_0} \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu(t_{k+1}-t_k)} + e^{\mu(t-t_k)} \frac{\kappa \rho_k(\dot{u}, \dot{\bar{u}}) e^{-2\mu T}}{\mu C_0 Z_0} \\ &\quad + \frac{e^{\mu(t-t_k)} \rho_k(\dot{u}, \dot{\bar{u}}) \kappa e^{-2\mu T}}{\mu C_0} \sum_{n=1}^p n |r_n| U_0^{n-1} e^{(n-1)\mu(t_{k+1}-t_k)} + e^{\mu(t-t_k)} \kappa \rho_k(\dot{u}, \dot{\bar{u}}) e^{-2\mu T} \\ &\leq e^{\mu(t-t_k)} \rho_\mu^{(k)}(u, \bar{u}) \left[ \frac{1 + \kappa e^{-2\mu T}}{\mu C_0} \left( \frac{1}{Z_0} + \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} \right) + \kappa e^{-2\mu T} \right], \\ \dot{B}_2 &\leq \frac{1}{t_{k+1} - t_k} \left| \int_{t_k}^{t_{k+1}} (U(u)(s) - U(\bar{u})(s)) ds \right| \leq \frac{1}{l_0} \left| \int_{t_k}^{t_{k+1}} (U(u)(s) - U(\bar{u})(s)) ds \right| \\ &\leq \rho_u^{(k)}(u, \bar{u}) \frac{e^{\mu T_0} - 1}{\mu^2 C_0 l_0} \left( \frac{1 + \kappa e^{-2\mu T}}{Z_0} + \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} (1 + \kappa e^{-2n\mu T}) \right). \end{aligned} \quad (2.13)$$

Therefore,

$$\begin{aligned}
 & |\dot{B}(u)(t) - \dot{B}(\bar{u})(t)| \\
 & \leq e^{\mu(t-t_k)} \rho_\mu^{(k)}(u, \bar{u}) \left[ \frac{1 + \kappa e^{-2\mu T}}{\mu C_0} \left( \frac{1}{Z_0} + \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} \right) + \kappa e^{-2\mu T} \right] \\
 & \quad + \rho_u^{(k)}(u, \bar{u}) \frac{e^{\mu T_0} - 1}{\mu^2 C_0 l_0} \left( \frac{1 + \kappa e^{-2\mu T}}{Z_0} + \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} (1 + \kappa e^{-2n\mu T}) \right) \\
 & \leq \rho_\mu^{(k)}(u, \bar{u}) \left[ \frac{(e^{\mu T_0} + \mu \tau_0 - 1)(1 + \kappa e^{-2\mu T})}{\mu^2 C_0 l_0} \left( \frac{1}{Z_0} + \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu T_0} \right) + \kappa e^{-2\mu T} \right] \\
 & \equiv e^{\mu(t-t_k)} \dot{K}_U \rho_\mu^{(k)}(u, \bar{u}).
 \end{aligned} \tag{2.14}$$

It follows  $\rho_k(\dot{B}(u), \dot{B}(\bar{u})) \leq e^{\mu(t-t_k)} \dot{K}_U \rho_\mu^{(k)}(u, \bar{u})$ .

Then  $\rho_\mu^{(k)}(B(u), B(\bar{u})) \leq \max\{K_U, \dot{K}_U\} \rho_\mu^{(k)}(u, \bar{u})$ .  
Consequently,

$$\rho_\mu^{(k)}(Bu, B\bar{u}) \leq \bar{K} \rho_\mu^{(k)}(u, \bar{u}) \quad (k = 0, 1, 2, \dots), \tag{2.15}$$

where  $\bar{K} = \max\{K_U, \dot{K}_U\} < 1$  does not depend on  $u$  and  $k$ .

We have to verify that  $M_{SU}$  is  $j$ -bounded. Indeed, since  $j$  is an identity mapping,

$$\rho_u^{j^n(k)}(u, \bar{u}) \leq \rho_u^{(k)}(u, \bar{u}) < \infty \quad (n = 0, 1, 2, \dots). \tag{2.16}$$

Therefore, in view of the fixed point theorem for contractive mappings in uniform spaces (cf. [13]), the operator  $B$  has a unique fixed point, and it is an oscillatory solution of (1.4).

Theorem 2.2 is thus proved.  $\square$

### 3. Numerical Example

Finally, we summarize all inequalities needed for the applications:

$$\begin{aligned}
 & \frac{1}{\mu C_0} \left( \frac{2e^{-\mu T} e^{\mu T_0}}{Z_0 + R_0} + \frac{e^{\mu T_0} (1 + \kappa e^{-2\mu T})}{Z_0} \right. \\
 & \quad \left. + \sum_{n=1}^p \frac{|r_n| U_0^{n-1} (e^{n\mu T_0} + e^{(n-1)\mu T_0} - 2)(1 + \kappa e^{-2n\mu T})}{n} \right) + \kappa e^{-2\mu T} \leq 1, \\
 & K_U = \frac{e^{\mu_0}}{\mu^2} \left( \frac{1 + \kappa e^{-2\mu T}}{C_0 Z_0} + \frac{1}{C_0} \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu_0} (1 + \kappa e^{-2n\mu T}) \right) + \frac{\kappa e^{-2\mu T}}{\mu} < 1, \\
 & \dot{K}_U = \frac{(e^{\mu_0} + \mu \tau_0 - 1)(1 + \kappa e^{-2\mu T})}{\mu^2 C_0 l_0} \left( \frac{1}{Z_0} + \sum_{n=1}^p |r_n| n U_0^{n-1} e^{(n-1)\mu_0} \right) + \kappa e^{-2\mu T} < 1.
 \end{aligned} \tag{3.1}$$

Consider a line with the following specific parameters:

$$\begin{aligned}\Lambda &= 1 \text{ m}, & L &= 0,2 \mu\text{H/m}, & C &= 80 \text{ pF/m}, \\ \nu &= \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0,2 \cdot 10^{-6} \cdot 80 \cdot 10^{-12}}} = \frac{1}{4 \cdot 10^{-9}} = 2,5 \cdot 10^8, \\ Z_0 &= \sqrt{\frac{L}{C}} = \sqrt{\frac{0,2 \cdot 10^{-6}}{80 \cdot 10^{-12}}} = 50 \Omega, & R_0 &= 45 \Omega, & C_0 &= 8 \text{ pF} = 8 \cdot 10^{-12} \text{ F}.\end{aligned}\quad (3.2)$$

Then,  $T = \Lambda\sqrt{LC} = 4 \cdot 10^{-9} \text{ s}$ ;  $\kappa = (Z_0 - R_0)/(Z_0 + R_0) = 1/19 = 0,0526$ .

Let us check the propagation of millimeter waves  $\lambda_0 = 10^{-3} \text{ m}$ . We have

$$\begin{aligned}f_0 &= \frac{1}{\lambda_0\sqrt{LC}} = \frac{1}{10^{-3} \cdot 4 \cdot 10^{-9}} = 2,5 \cdot 10^{11} \text{ Hz} \\ \Rightarrow T_0 &= \frac{1}{f_0} = \frac{1}{2,5 \cdot 10^{11}} = 4 \cdot 10^{-12} \text{ sec}; & l &= 2 \cdot 10^{-12} \text{ sec}.\end{aligned}\quad (3.3)$$

If we choose  $\mu = (1/4)10^{12}$ , then  $\mu T_0 = \mu_0 = 1$ ,  $\mu\tau_0 = (1/2)$ , and  $T = 4 \cdot 10^{-9} \cdot (1/4) \cdot 10^{12} T_0 = 1000 \cdot T_0$ .

Consequently,  $\mu T = (1/4)10^{12} \cdot 2 \cdot 10^{-8} = (1/2)10^4$ ,  $\mu C_0 = (1/4)10^{12} \cdot 8 \cdot 10^{-12} = 2$ , and  $\mu^2 C_0 = (1/2) \cdot 10^{12}$ .

Since  $e^{-\mu T} = e^{-5000} = 0$ , then the above inequalities (omitting the second one) become

$$\begin{aligned}\frac{e}{100} + \sum_{n=1}^p |r_n| U_0^{n-1} \frac{e^n + e^{n-1} - 2}{2n} &\leq 1, \\ \dot{K}_U &= 2 \left( e - \frac{1}{2} \right) \left( \frac{1}{50} + \sum_{n=1}^p |r_n| n U_0^{n-1} e^{n-1} \right) < 1.\end{aligned}\quad (3.4)$$

If the V-I characteristic of the nonlinear resistive element is  $f(u) = -0,12u + 0,8u^3$ , then  $U_0 \leq 0,41$ ;  $\dot{K}_U = U_0 < 0,06$ . It follows that  $U_0 < 0,06$ .

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