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Research Article

On the Drazin Inverse of the Sum of Two Matrices

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We deduce the explicit expressions for $(P+Q)^D$ and $(PQ)^D$ of two matrices P and Q under the conditions $P^2Q = PQP$ and $Q^2P = QPQ$. Also, we give the upper bound of $\|(P+Q)^D - P^D\|_2$.

1. Introduction

The symbol $\mathbb{C}^{m\times n}$ stands for the set of $m\times n$ complex matrices, and I_n (for short I) stands for the $n\times n$ identity matrix. For $A\in\mathbb{C}^{n\times n}$, its Drazin inverse, denoted by A^D, is defined as the unique matrix satisfying

$$A^{k+1}A^D = A^k, \qquad A^DAA^D = A^D, \qquad AA^D = A^DA, \tag{1.1}$$

where $k = \operatorname{Ind}(A)$ is the index of A. In particular, if $k = 0$, A is invertible and $A^D = A^{-1}$ (see, e.g., [1–3] for details). Recall that for $A\in\mathbb{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$, there exists an $n\times n$ nonsingular matrix X such that

$$A = X\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}X^{-1}, \tag{1.2}$$

where C is a nonsingular matrix and N is nilpotent of index k, and

$$A^D = X\begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix}X^{-1} \tag{1.3}$$

(see [1, 3]). It is well known that if A is nilpotent, then $A^D = 0$. We always write $A^{\pi} := I - AA^D$.

Drazin [2] proved that in associative ring $(A+B)^D = A^D + B^D$ when A, B are Drazin invertible and $AB = BA = 0$. In [4], Hartwig et al. relaxed the condition to $AB = 0$ and put forward the expression for $(A+B)^D$ where $A, B \in \mathbb{C}^{n\times n}$. In recent years, the Drazin inverse of the sum of two matrices or operators has been extensively investigated under different conditions (see, [5–15]). For example, in [7], the conditions are $PQ = \lambda QP$ and $PQ = PQP$, in [9] they are $P^3Q = QP$ and $Q^3P = PQ$, and in [15], they are $PQP = 0$ and $PQ^2 = 0$. These results motivate us to investigate how to explicitly express the Drazin inverse of the sum $P+Q$ under the conditions $P^2Q = PQP$ and $Q^2P = QPQ$, which are implied by the condition $PQ = QP$.

The paper is organized as follows. In Section 2, we will deduce some lemmas. In Section 3, we will present the explicit expressions for $(P+Q)^D$ and $(PQ)^D$ of two matrices P and Q under the conditions $P^2Q = PQP$ and $Q^2P = QPQ$. We also give the upper bound of $\|(P+Q)^D - P^D\|_2$.

2. Some Lemmas

In this section, we will make preparations for discussing the Drazin inverse of the sum of two matrices in next section. To this end, we will introduce some lemmas.

The first lemma is a trivial consequence of [16, Theorem 3.2].

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times m}$, and $C \in \mathbb{C}^{m\times n}$ with $BC = 0$, and define*

$$M = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}. \tag{2.1}$$

Then,

$$M^D = \begin{pmatrix} A^D & 0 \\ C\left(A^D\right)^2 & B^D \end{pmatrix}. \tag{2.2}$$

Lemma 2.2. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $P^2Q = PQP$, then, for any positive integers i, j,*

(i) $P^{i+1}Q = P^iQP = PQP^i$, $P^{2i}Q = P^iQP^i$,

(ii) $P^iQ^i = (PQ)^i$.

Moreover, if $Q^2P = QPQ$, then

$$PQ^jP^i = P^{i+1}Q^j. \tag{2.3}$$

Proof. (i) By induction, we can easily get the results.

(ii) For $i = 1$, it is evident. Assume that, for $i = k$, the equation holds, that is, $P^k Q^k = (PQ)^k$. When $i = k+1$, by (i), we have

$$P^{i+1}Q^{i+1} = PQP^iQ^i = PQ(PQ)^i = (PQ)^{i+1}. \tag{2.4}$$

Hence, by induction, we have $P^iQ^i = (PQ)^i$ for any i.

Assume $Q^2P = QPQ$. By induction on j for (2.3). Obviously, when $j = 1$, it holds by statement (i). Assume that it holds for $j = k$, that is, $PQ^kP^i = P^{i+1}Q^k$. When $j = k+1$,

$$PQ^{k+1}P^i = PQ^{k-1}Q^2PP^{i-1} = PQ^k\left(PQP^{i-1}\right) = PQ^kP^iQ = P^{i+1}Q^kQ = P^{i+1}Q^{k+1}. \tag{2.5}$$

Hence (2.3) holds for any j. □

Lemma 2.3. *Let $P, Q \in \mathbb{C}^{n\times n}$. Suppose that $P^2Q = PQP$ and $Q^2P = QPQ$. Then, for any positive integer m,*

$$(P+Q)^m = \sum_{i=0}^{m-1} C_{m-1}^i\left(P^{m-i}Q^i + Q^{m-i}P^i\right), \tag{2.6}$$

where the binomial coefficient $C_j^i = j!/i!(j-i)!,\ j \ge i$.

Moreover, if P, Q are nilpotent with $P^s = 0$ and $Q^t = 0$, then $P+Q$ is nilpotent and its index is less than $s+t$.

Proof. We will show by induction that (2.6) holds. Trivially, (2.6) holds for $m = 1$. Assume that (2.6) holds for $m = k$, that is,

$$(P+Q)^k = \sum_{i=0}^{k-1} C_{k-1}^i\left(P^{k-i}Q^i + Q^{k-i}P^i\right). \tag{2.7}$$

Then, for $m = k+1$, we have, by Lemma 2.2,

$$\begin{aligned}
(P+Q)^{k+1} &= \sum_{i=0}^{k-1} C_{k-1}^i\left(P^{k-i}Q^i + Q^{k-i}P^i\right)(P+Q) \\
&= \sum_{i=0}^{k-1} C_{k-1}^i\left(P^{k+1-i}Q^i + P^{k-i}Q^{i+1} + Q^{k-i}P^{i+1} + Q^{k+1-i}P^i\right) \\
&= P^{k+1} + \sum_{i=1}^{k-1}\left(C_{k-1}^i + C_{k-1}^{i-1}\right)P^{k+1-i}Q^i + PQ^k \\
&\quad + Q^{k+1} + \sum_{i=1}^{k-1}\left(C_{k-1}^i + C_{k-1}^{i-1}\right)Q^{k+1-i}P^i + QP^k
\end{aligned}$$

$$
\begin{aligned}
&= P^{k+1} + \sum_{i=1}^{k-1} C_k^i P^{k+1-i} Q^i + PQ^k + Q^{k+1} + \sum_{i=1}^{k-1} C_k^i Q^{k+1-i} P^i + QP^k \\
&= \sum_{i=0}^{k} C_k^i P^{k+1-i} Q^i + \sum_{i=0}^{k} C_k^i Q^{k+1-i} P^i.
\end{aligned} \tag{2.8}
$$

Hence (2.6) holds for any $m \geq 1$.

If P, Q are nilpotent with $P^s = 0$ and $Q^t = 0$, then taking $m = s + t - 1$ in (2.6) yields $(P+Q)^{s+t-1} = 0$, that is, $P + Q$ is nilpotent of index less than $s + t$. □

Lemma 2.4 (see [1, Theorem 7.8.4]). *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQ = QP$, then $(PQ)^D = Q^D P^D = P^D Q^D$ and $P^D Q = Q P^D$.*

Lemma 2.5. *Let $P, Q \in \mathbb{C}^{n\times n}$ and P be invertible. If $PQ = QP$, then*

$$(P+Q)^D = \left(I + P^{-1}Q\right)^D P^{-1} = P^{-1}\left(I + P^{-1}Q\right)^D. \tag{2.9}$$

Moreover, if Q is nilpotent of index t, then $P + Q$ is invertible and

$$(P+Q)^{-1} = \sum_{i=0}^{t-1} (-Q)^i P^{-i-1} = \sum_{i=0}^{t-1} P^{-i-1} (-Q)^i. \tag{2.10}$$

Proof. Since $P + Q = P(I + P^{-1}Q) = (I + P^{-1}Q)P$, by Lemma 2.4,

$$(P+Q)^D = \left(I + P^{-1}Q\right)^D P^{-1} = P^{-1}\left(I + P^{-1}Q\right)^D. \tag{2.11}$$

Note that the nilpotency of Q with commuting with P implies that $P^{-1}Q$ is nilpotent of index t. Thus, $I + P^{-1}Q$ is invertible and so is $P + Q$, and

$$(P+Q)^{-1} = \left(I + P^{-1}Q\right)^{-1} P^{-1} = \sum_{i=0}^{t-1} (-Q)^i P^{-i-1} = \sum_{i=0}^{t-1} P^{-i-1} (-Q)^i. \tag{2.12}$$

□

Lemma 2.6. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $Q = Q_1 \oplus Q_2$, where Q_1 is invertible and Q_2 is nilpotent of index t, and let*

$$P = \begin{pmatrix} P_1 & P_3 \\ P_4 & P_2 \end{pmatrix} \tag{2.13}$$

be partitioned conformably with Q. Suppose that $Q^2P = QPQ$ and $P^2Q = PQP$. Then, $P_3 = 0$ and

$$Q_1P_1 = P_1Q_1, \tag{2.14}$$

$$Q_2P_4 = P_2P_4 = 0, \tag{2.15}$$

$$Q_2^2P_2 = Q_2P_2Q_2, \tag{2.16}$$

$$P_i^2Q_i = P_iQ_iP_i, \quad i = 1, 2. \tag{2.17}$$

Moreover, if P is nilpotent of index s, then $P_4P_1^{s-1} = 0$.

Proof. Since $Q^2P = QPQ$, $Q^{2t}P = Q^tPQ^t$ by Lemma 2.2, that is,

$$\begin{pmatrix} Q_1^{2t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1 & P_3 \\ P_4 & P_2 \end{pmatrix} = \begin{pmatrix} Q_1^{t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1 & P_3 \\ P_4 & P_2 \end{pmatrix}\begin{pmatrix} Q_1^{t} & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.18}$$

namely,

$$\begin{pmatrix} Q_1^{2t}P_1 & Q_1^{2t}P_3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Q_1^{t}P_1Q_1^t & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.19}$$

Thus, $P_3 = 0$ because the invertibility of Q_1. So from $Q^2P = QPQ$ and $P^2Q = PQP$, it follows, respectively, that

$$P_1Q_1 = Q_1P_1, \qquad Q_2^2P_4 = Q_2P_4Q_1, \qquad Q_2^2P_2 = Q_2P_2Q_2, \tag{2.20}$$

and that

$$P_2P_4Q_1 = P_2Q_2P_4, \qquad P_i^2Q_i = P_iQ_iP_i, \quad i = 1, 2. \tag{2.21}$$

Since $Q_2^t = 0$,

$$Q_2P_4 = Q_2^2P_4Q_1^{-1} = Q_2^tP_4Q_1^{-t+1} = 0, \tag{2.22}$$

and then $P_2P_4 = P_2Q_2P_4Q_1^{-1} = 0$. From this, we can easily verify

$$P^s = \begin{pmatrix} P_1^s & 0 \\ P_4P_1^{s-1} & P_2^s \end{pmatrix}. \tag{2.23}$$

Therefore, if $P^s = 0$, then $P_4P_1^{s-1} = 0$. □

3. Main Results

In this section, we will give the explicit expressions for $(P+Q)^D$ and $(PQ)^D$, under the conditions $P^2Q = PQP$ and $Q^2P = QPQ$. Now, we begin with the following theorem.

Theorem 3.1. *Let $Q \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(Q) = t$ and $P \in \mathbb{C}^{n\times n}$ be nilpotent with $P^s = 0$. If $P^2Q = PQP$ and $Q^2P = QPQ$, then*

$$\begin{aligned}(P+Q)^D &= \sum_{i=0}^{s-1}\left(Q^D\right)^{i+1}(-P)^i + Q^\pi P\sum_{i=0}^{s-2}(-1)^i(i+1)\left(Q^D\right)^{i+2}P^i \\ &= QQ^D\sum_{i=0}^{s-1}(-P)^i\left(Q^D\right)^{i+1} + Q^\pi PQQ^D\sum_{i=0}^{s-2}(-1)^i(i+1)P^i\left(Q^D\right)^{i+2}.\end{aligned} \tag{3.1}$$

Proof. If $t = 0$, then Q is invertible, and therefore $QP = PQ$. So, by Lemma 2.5, (3.1) holds.

Now assume that $t > 0$ and, without loss of generality, Q can be written as $Q = Q_1 \oplus Q_2$, where Q_1 is invertible and Q_2 is nilpotent of index t. So $Q^D = Q_1^{-1} \oplus 0$. Since $P^2Q = PQP$ and $Q^2P = QPQ$, we can write P, partitioned conformably with Q, by Lemma 2.6, as follows:

$$P = \begin{pmatrix} P_1 & 0 \\ P_4 & P_2 \end{pmatrix}, \tag{3.2}$$

where P_1, P_2 are nilpotent since P is nilpotent. We also write $I = I_1 \oplus I_2$, partitioned conformably with Q.

Since P_1 is nilpotent and Q_1 is invertible, by Lemma 2.5,

$$(P_1+Q_1)^{-1} = \sum_{i=0}^{s-1} Q_1^{-i-1}(-P_1)^i = \sum_{i=0}^{s-1}(-P_1)^i Q_1^{-i-1}. \tag{3.3}$$

Also, the nilpotency of P_2, Q_2 implies $(P_2+Q_2)^D = 0$ by Lemma 2.3.

By (2.15), $(P_2+Q_2)P_4 = 0$. Hence, by Lemma 2.1, the argument above, and (2.14), we have

$$(P+Q)^D = \begin{pmatrix} P_1+Q_1 & 0 \\ P_4 & P_2+Q_2 \end{pmatrix}^D = \begin{pmatrix} Q_1^{-1}\left(I_1+Q_1^{-1}P_1\right)^{-1} & 0 \\ P_4Q_1^{-2}\left(I_1+Q_1^{-1}P_1\right)^{-2} & 0 \end{pmatrix}. \tag{3.4}$$

By (3.3), it is easy to verify that

$$\left(I_1+Q_1^{-1}P_1\right)^{-2} = \sum_{i=0}^{s-1}(-1)^i(i+1)Q_1^{-i}P_1^i = \sum_{i=0}^{s-1}(-1)^i(i+1)P_1^iQ_1^{-i}. \tag{3.5}$$

Since $P^s = 0$, we have $P_4 P_1^{s-1} = 0$ by Lemma 2.6, and, therefore, by (3.5),

$$\begin{aligned}Q^{\pi} P \sum_{i=0}^{s-2} (-1)^i (i+1) \left(Q^D\right)^{i+2} P^i &= \sum_{i=0}^{s-2} (-1)^i (i+1) \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ P_4 & P_2 \end{pmatrix} \begin{pmatrix} Q_1^{-(i+2)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^i & 0 \\ * & P_2^i \end{pmatrix} \\ &= \sum_{i=0}^{s-2} (-1)^i (i+1) \begin{pmatrix} 0 & 0 \\ P_4 Q_1^{-(i+2)} P_1^{\mathrm{i}} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ P_4 Q_1^{-2} \left(I + Q_1^{-1} P_1\right)^{-2} & 0 \end{pmatrix}. \end{aligned} \tag{3.6}$$

Analogous to the argument above, we can see, by Lemma 2.5,

$$\sum_{i=0}^{s-1} \left(Q^D\right)^{i+1} (-P)^i = \begin{pmatrix} Q_1^{-1} \left(I_1 + Q_1^{-1} P_1\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.7}$$

Thus, putting (3.6) and (3.7) into (3.4) yields the first equation of (3.1).
Similar to the discussion of (3.6), we have

$$\begin{aligned} QQ^D \sum_{i=0}^{s-1} (-P)^i \left(Q^D\right)^{i+1} &= \begin{pmatrix} Q_1^{-1} \left(I + Q_1^{-1} P_1\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \\ Q^{\pi} PQQ^D \sum_{i=0}^{s-2} (-1)^i (i+1) P^i \left(Q^D\right)^{i+2} &= \begin{pmatrix} 0 & 0 \\ P_4 Q_1^{-2} \left(I + Q_1^{-1} P_1\right)^{-2} & 0 \end{pmatrix}, \end{aligned} \tag{3.8}$$

and then putting them into (3.4) yields the second equation of (3.1). □

The following theorem is our main result, and Theorem 3.1 and Lemma 2.5 can be regarded as its special cases.

Theorem 3.2. *Let* $P, Q \in \mathbb{C}^{n \times n}$ *with* $\operatorname{Ind}(P) = s \geq 1$ *and* $\operatorname{Ind}(Q) = t$. *If* $P^2 Q = PQP$ *and* $Q^2 P = QPQ$. *Then,*

(i)

$$(PQ)^D = P^D Q^D = PP^D Q^D P^D = PQ^D \left(P^D\right)^2, \tag{3.9}$$

$$Q^2 P^D = QP^D Q, \tag{3.10}$$

$$\left(P^D\right)^2 Q = P^D Q P^D. \tag{3.11}$$

(ii)

$$(P+Q)^D = P^D\left(I+P^DQ\right)^D + P^{\pi}Q\left[P^D\left(I+P^DQ\right)^D\right]^2 + \sum_{i=0}^{s-1}\left(Q^D\right)^{i+1}(-P)^iP^{\pi}$$
$$+ Q^{\pi}P\sum_{i=0}^{s-2}(-1)^i(i+1)\left(Q^D\right)^{i+2}P^iP^{\pi}. \tag{3.12}$$

Proof. If $s=0$, then P is invertible and $PQ=QP$. So, by Lemmas 2.4 and 2.5, (3.9) and (3.12) hold, respectively. Therefore, assume that $s>0$, and, without loss of generality, let $P=P_1\oplus P_2$, where P_1 is invertible and P_2 is nilpotent of index s. From hypotheses, by Lemma 2.6, we can write

$$Q=\begin{pmatrix} Q_1 & 0 \\ Q_4 & Q_2 \end{pmatrix}, \tag{3.13}$$

partitioned conformably with P, and those equations in Lemma 2.6 hold. By Lemma 2.1, therefore, we have

$$Q^D=\begin{pmatrix} Q_1^D & 0 \\ Q_4\left(Q_1^D\right)^2 & Q_2^D \end{pmatrix}, \qquad Q^2=\begin{pmatrix} Q_1^2 & 0 \\ Q_4Q_1 & Q_2^2 \end{pmatrix}. \tag{3.14}$$

(i) By (2.14) and (2.15),

$$Q^2P^D=\begin{pmatrix} Q_1^2P_1^{-1} & 0 \\ Q_4Q_1P_1^{-1} & 0 \end{pmatrix}=\begin{pmatrix} Q_1P_1^{-1}Q_1 & 0 \\ Q_4P_1^{-1}Q_1 & 0 \end{pmatrix}=QP^DQ,$$
$$\left(P^D\right)^2Q=\begin{pmatrix} P_1^{-2}Q_1 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} P_1^{-1}Q_1P_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}=P^DQP^D,$$
$$PQ^D\left(P^D\right)^2=\begin{pmatrix} P_1Q_1^DP_1^{-2} & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} P_1^{-1}Q_1^D & 0 \\ 0 & 0 \end{pmatrix}= P^DQ^D$$
$$=\begin{pmatrix} Q_1^DP_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}=PP^DQ^DP^D, \tag{3.15}$$

$$PQ=\begin{pmatrix} P_1Q_1 & 0 \\ 0 & P_2Q_2 \end{pmatrix}=\begin{pmatrix} Q_1P_1 & 0 \\ 0 & P_2Q_2 \end{pmatrix}. \tag{3.16}$$

By (2.17) and Lemma 2.2, $(P_2Q_2)^s = P_2^sQ_2^s = 0$. By Lemma 2.4 and (3.16), we have

$$(PQ)^D = \begin{pmatrix} (Q_1P_1)^D & 0 \\ 0 & (P_2Q_2)^D \end{pmatrix} = \begin{pmatrix} P_1^{-1}Q_1^D & 0 \\ 0 & 0 \end{pmatrix} = P^DQ^D. \tag{3.17}$$

As a result, (3.9) holds.

(ii) By Lemma 2.6, $(P_2 + Q_2)Q_4 = 0$ and then, by Lemma 2.1, we have

$$(P+Q)^D = \begin{pmatrix} P_1+Q_1 & 0 \\ Q_4 & P_2+Q_2 \end{pmatrix}^D = \begin{pmatrix} (P_1+Q_1)^D & 0 \\ Q_4\left[(P_1+Q_1)^D\right]^2 & (P_2+Q_2)^D \end{pmatrix}. \tag{3.18}$$

By Lemma 2.5, we have

$$\begin{aligned} P^D\left(I+P^DQ\right)^D &= \begin{pmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_1+P_1^{-1}Q_1 & 0 \\ 0 & I_2 \end{pmatrix}^D \\ &= \begin{pmatrix} P_1^{-1}(I_1+P_1^{-1}Q_1)^D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (P_1+Q_1)^D & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{3.19}$$

and, therefore,

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ Q_4\left[(P_1+Q_1)^D\right]^2 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} Q_1 & 0 \\ Q_4 & Q_2 \end{pmatrix}\begin{pmatrix} \left[(P_1+Q_1)^D\right]^2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= P^\pi Q\left[P^D\left(I+P^DQ\right)^D\right]^2. \end{aligned} \tag{3.20}$$

By (3.1), we have

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & (P_2+Q_2)^D \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & \sum_{i=0}^{s-1}(Q_2^D)^{i+1}(-P_2)^i + Q_2^\pi P_2\sum_{i=0}^{s-2}(-1)^i(i+1)(Q_2^D)^{i+2}P_2^i \end{pmatrix} \\ &= \sum_{i=0}^{s-1}\left(Q^D\right)^{i+1}(-P)^iP^\pi + Q^\pi P\sum_{i=0}^{s-2}(-1)^i(i+1)\left(Q^D\right)^{i+2}P^iP^\pi. \end{aligned} \tag{3.21}$$

Thus, substituting (3.19), (3.21), and (3.20) in (3.18) yields (3.12). □

Note that $PQ = QP$ implies $P^2Q = PQP$ and $Q^2P = QPQ$.

Corollary 3.3 (see [14, Theorem 2]). *If $P, Q \in \mathbb{C}^{n\times n}$ with $PQ = QP$ and $\operatorname{Ind}(P) = s$, then*

$$(P+Q)^D = \left(I + P^D Q\right)^D P^D + P^\pi \sum_{i=0}^{s-1}\left(Q^D\right)^{i+1}(-P)^i. \tag{3.22}$$

Proof. From (3.19) and Lemma 2.5, we can obtain

$$\left(I + P^D Q\right)^D P^D = P^D\left(I + P^D Q\right)^D. \tag{3.23}$$

□

Since $P^k Q = 0$ for some k, $P^D Q = 0$. Thus, we have the following corollary.

Corollary 3.4. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(P) = s \geq 1$ and $\operatorname{Ind}(Q) = t$. Suppose $P^2 Q = PQP$ and $Q^2 P = QPQ$. If there exist two positive integers k and h such that $P^k Q = 0$ and $Q^h P = 0$, then*

$$(P+Q)^D = P^D + Q^D + Q\left(P^D\right)^2. \tag{3.24}$$

If Q is a perturbation of P, then, we have the following result in which $\|(P+Q)^D - P^D\|_2$ has an upper bound. Before the theorem, let us recall that if $\|A\|_2 < 1$, then $I + A$ is invertible and

$$\begin{aligned} \left\|(I+A)^{-1}\right\|_2 &\leq \frac{1}{1-\|A\|_2}, \\ \left\|I-(I+A)^{-1}\right\|_2 &\leq \frac{\|A\|_2}{1-\|A\|_2}. \end{aligned} \tag{3.25}$$

Theorem 3.5. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(P) = s \geq 1$ and $\operatorname{Ind}(Q) = t$. Suppose $P^2 Q = PQP$ and $Q^2 P = QPQ$. If $\|P^D Q\|_2 < 1$, then*

$$\begin{aligned} \left\|(P+Q)^D - P^D\right\|_2 \leq{}& \frac{\|P^D\|_2\|P^D Q\|_2}{1-\|P^D Q\|_2} + \frac{\|P^\pi\|_2\|Q\|_2\|P^D\|_2^2}{\left(1-\|P^D Q\|_2\right)^2} \\ &+ \frac{\|Q^D\|_2\|P^\pi\|_2\left(1-\|Q^D\|_2^s\|P\|_2^s\right)}{1-\|Q^D\|_2\|P\|_2} + \frac{\|Q^D\|_2^2\|Q^\pi\|_2\|P^\pi\|_2\|P\|_2}{\left(1-\|Q^D\|_2\|P\|_2\right)^2} \\ &\times\left[1 - s\left\|Q^D\right\|_2^{s-1}\|P\|_2^{s-1} + (s-1)\left\|Q^D\right\|_2^{s}\|P\|_2^{s}\right]. \end{aligned} \tag{3.26}$$

Proof. Since $\|P^D Q\|_2 < 1$, $I + P^D Q$ is invertible. Then by (3.12), we have

$$\begin{aligned} (P+Q)^D - P^D ={}& P^D\left[\left(I+P^D Q\right)^{-1} - I\right] + P^\pi Q\left[P^D\left(I+P^D Q\right)^{-1}\right]^2 \\ &+ \sum_{i=0}^{s-1}\left(Q^D\right)^{i+1}(-P)^i P^\pi + Q^\pi P\sum_{i=0}^{s-2}(-1)^i(i+1)\left(Q^D\right)^{i+2}P^i P^\pi. \end{aligned} \tag{3.27}$$

In order to verity (3.26), we need to calculate the 2-norms of the right-hand side of the above equation. By (3.25),

$$\begin{aligned}
\left\| P^D\left[\left(I+P^DQ\right)^{-1}-I\right]\right\|_2 &\le \frac{\|P^D\|_2\|P^DQ\|_2}{1-\|P^DQ\|_2},\\
\left\| P^\pi Q\left[P^D\left(I+P^DQ\right)^{-1}\right]^2\right\|_2 &\le \frac{\|P^\pi\|_2\|Q\|_2\|P^D\|_2^2}{(1-\|P^DQ\|_2)^2},\\
\left\|\sum_{i=0}^{s-1}\left(Q^D\right)^{i+1}(-P)^iP^\pi\right\|_2 &\le \sum_{i=0}^{s-1}\left\|Q^D\right\|_2^{i+1}\|P\|_2^i\|P^\pi\|_2\\
&= \sum_{i=1}^{s}\left\|Q^D\right\|_2^{i}\|P\|_2^{i-1}\|P^\pi\|_2\\
&= \frac{\|Q^D\|_2\|P^\pi\|_2\left(1-\|Q^D\|_2^s\|P\|_2^s\right)}{1-\|Q^D\|_2\|P\|_2},\\
\left\|Q^\pi P\sum_{i=0}^{s-2}(-1)^i(i+1)\left(Q^D\right)^{i+2}P^iP^\pi\right\|_2 &\le \sum_{i=0}^{s-2}(i+1)\left\|Q^D\right\|_2^{i+2}\|P\|_2^{i+1}\|Q^\pi\|_2\|P^\pi\|_2\\
&= \sum_{i=1}^{s-1}i\left\|Q^D\right\|_2^{i+1}\|P\|_2^{i}\|Q^\pi\|_2\|P^\pi\|_2.
\end{aligned} \tag{3.28}$$

Let $q := \|Q^D\|_2\|P\|_2$ and $S := \sum_{i=1}^{s-1} iq^i$. Then,

$$(1-q)S=\sum_{i=1}^{s-1}q^i-(s-1)q^s=\frac{q(1-q^{s-1})}{1-q}-(s-1)q^s=\frac{q-sq^s+(s-1)q^{s+1}}{1-q}. \tag{3.29}$$

Thus

$$\begin{aligned}
&\sum_{i=1}^{s-1}i\left\|Q^D\right\|_2^{i+1}\|P\|_2^{i}\|Q^\pi\|_2\|P^\pi\|_2\\
&\quad=\frac{\|Q^D\|_2^2\|Q^\pi\|_2\|P^\pi\|_2\|P\|_2\left[1-s\left\|Q^D\right\|_2^{s-1}\|P\|_2^{s-1}+(s-1)\left\|Q^D\right\|_2^{s}\|P\|_2^{s}\right]}{(1-\|Q^D\|_2\|P\|_2)^2}.
\end{aligned} \tag{3.30}$$

By the above argument, we can get (3.26). □

Finally, we give an example to illustrate our results.

Example 3.6. Consider the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{3.31}$$

We observe that $P^2Q = PQP$ and $Q^2P = QPQ$, but $PQ \neq QP$. It is obvious that $s = \text{Ind}(P) = 2$, and

$$P^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q^D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \tag{3.32}$$

Since $\|P^DQ\|_2 = (1/3) < 1$, $I + P^DQ$ is invertible and

$$\left(I + P^DQ\right)^{-1} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.33}$$

By (3.12),

$$(P + Q)^D - P^D = \begin{pmatrix} -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \tag{3.34}$$

We can compute $\|(P + Q)^D - P^D\|_2 = 3\sqrt{10}$. On the other hand, it is easy to get that $\|P\|_2 = \|P^D\|_2 = \|P^\pi\|_2 = \|Q^\pi\|_2 = 1$, $\|Q\|_2 = 1/3$, $\|Q^D\|_2 = 3$. By (3.26), we get the upper bound of $\|(P + Q)^D - P^D\|_2$ is $16(1/4)$, it is bigger than and close to the exact norm.

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