Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 829107, 15 pages
doi:10.1155/2011/829107

Research Article

The Existence of Multiple Periodic Solutions of Nonautonomous Delay Differential Equations

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Received 13 March 2011; Accepted 13 May 2011

Academic Editor: Meng Fan

We study the multiplicity of periodic solutions of nonautonomous delay differential equations which are asymptotically linear both at zero and at infinity. By making use of a theorem of Benci, some sufficient conditions are obtained to guarantee the existence of multiple periodic solutions.

1. Introduction

The existence and multiplicity of periodic solutions of delay differential equations have received a great deal of attention. In 1962, Jones [1] firstly investigated the existence of periodic solutions to the following scalar equation:

$$u'(t) = -au(t-1)[1+u(t)]. \tag{1.1}$$

By making use of Browder fixed point theorem, the author showed that there exist periodic solutions of (1.1) for each $a > \pi/2$. Since then, various fixed point theorems have been used to study the existence of periodic solutions of delay differential equations (cf. [2]). As pointed out in [3], by making change of variable $1 + u = e^x$, (1.1) turns into

$$x'(t) = -f(x(t-1)). \tag{1.2}$$

In 1974, Kaplan and Yorke [4] studied the following more general form of (1.2)

$$x'(t) = -f(x(t-1)) - f(x(t-2)) - \cdots - f(x(t-n)). \tag{1.3}$$

They introduced a technique which translates the problem of the existence of periodic solutions of a scalar delay differential equation to that of the existence of critical points of an associated ordinary differential system. Using this method, they proved that (1.3) has a periodic solution with minimal period 4 (resp., 6) when (1.3) has one delay (resp., two delays). In this direction, Fei, Li and He did some excellent work and got some signification results (cf. [5–8]).

Many other approaches, such as coincidence degree theory, the Hopf bifurcation theorem, and the Poincaré-Bendixson theorem, have also been used to study the existence of periodic solutions of delay differential equations (cf. [9, 10]). However, most of those results are concerned with scalar delay equations. In 2005, Guo and Yu [3] studied vector delay differential system (1.2). They built a variational structure for (1.2) on certain suitable spaces. Then they reduced the existence of periodic solutions of (1.2) to that of critical points of an associated variational functional. By making use of pseudoindex theory, they obtained some sufficient conditions to guarantee the existence of multiple periodic solutions.

In spite of so many papers on periodic solutions of delay differential equations, there are a quite few researches on nonautonomous case (see for example [11]). The main goal of this paper is to investigate the following nonautonomous system:

$$x'(t) = -f\left(t, x\left(t - \frac{\pi}{2}\right)\right). \tag{1.4}$$

We assume that

($\mathbf{f}_1$) there exists $F \in C^1([0, \pi/2] \times \mathbb{R}^n, \mathbb{R})$ such that f is the gradient of F with respect to x, and

$$F(t, x) = F(t, -x), \quad F\left(t + \frac{\pi}{2}, x\right) = F(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.5}$$

($\mathbf{f}_2$) $f(t, x) = B_0(t)x + o(|x|)$ as $|x| \to 0$ uniformly for $t \in [0, \pi/2]$,

($\mathbf{f}_3$) $f(t, x) = B_\infty(t)x + o(|x|)$ as $|x| \to \infty$ uniformly for $t \in [0, \pi/2]$,

where B_0, B_∞ are $n \times n$ symmetric continuous $\pi/2$-periodic matrix functions.

Hypothesis ($\mathbf{f}_3$) is known as asymptotically linear condition at infinity. Hypothesis ($\mathbf{f}_2$) is an asymptotically linear condition at zero, which implies that 0 is a trivial solution of (1.4). We are interested in nontrivial periodic solutions of (1.4). Similar to [3], we build a variational structure for (1.4) and convert the existence of periodic solutions to that of critical points of variational functional. Since the asymptotically linear hypothesis at infinity is given by a periodic loop of symmetric matrix, it will be more difficult to deal with more than a constant matrix. However, we can prove the existence of multiple periodic solutions by making use of a multiple critical points theorem of Benci (cf. [12]).

The rest of this paper is organized as follows: in Section 2, we build the variational functional and state some useful lemmas; in Section 3, the main results will be proved.

2. Variational Tools

Denote $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. The space $H = H^{1/2}(S^1, \mathbb{R}^n)$ has been introduced in [3]. The space H can be equipped with inner product as follows:

$$\langle x, y\rangle = (a_0, c_0) + \sum_{j=1}^{\infty}(1+j)\left[(a_j, c_j) + (b_j, d_j)\right], \tag{2.1}$$

where $x = a_0/\sqrt{2\pi} + 1/\sqrt{\pi}\sum_{j=1}^{\infty}(a_j \cos jt + b_j \sin jt)$, $y = c_0/\sqrt{2\pi} + 1/\sqrt{\pi}\sum_{j=1}^{\infty}(c_j \cos jt + d_j \sin jt)$, $a_0, c_0 \in \mathbb{R}^n$, $a_j, c_j, b_j, d_j \in \mathbb{R}^n$, $j \in \mathbb{N}$.

Set

$$E = \{x \in H \mid x(t+\pi) = -x(t),\ \forall t \in \mathbb{R}\}. \tag{2.2}$$

Then E is a closed subspace of H. If $x \in E$, it has Fourier expansion

$$x(t) = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{\infty}\left[a_j \cos(2j-1)t + b_j \sin(2j-1)t\right]. \tag{2.3}$$

Let $x \in L^2(S^1, \mathbb{R}^n)$. If for every $z \in C^{\infty}(S^1, \mathbb{R}^n)$

$$\int_0^{2\pi}(x(t), z'(t))\,dt = -\int_0^{2\pi}(y(t), z(t))\,dt, \tag{2.4}$$

then y is called a weak derivative of x, denoted by $\dot{x}$.

The variational functional defined on H, corresponding to (1.4), is

$$J(x) = \int_0^{2\pi}\left[\frac{1}{2}\left(x\left(t+\frac{\pi}{2}\right), \dot{x}(t)\right) - F(t, x(t))\right]dt. \tag{2.5}$$

Define a linear bounded operator $A : H \to H$ by setting

$$\langle Ax, y\rangle = \int_0^{2\pi}\left(x\left(t+\frac{\pi}{2}\right), \dot{y}(t)\right)dt. \tag{2.6}$$

It is easy to prove that E is an invariant subspace of H with respect to A and A is self-adjoint if it is restricted to E.

Lemma 2.1 (see [3]). *The essential spectrum of the operator A restricted to E is just* $\{2, -2\}$.

Define

$$\varphi(x) = -\int_0^{2\pi}F(t, x(t))\,dt, \quad \forall x \in H. \tag{2.7}$$

Then J can be rewritten as

$$J(x) = \frac{1}{2}\langle Ax, x\rangle + \varphi(x), \quad \forall x \in H. \tag{2.8}$$

Similar to the argument as in [3], we can prove the following two basic lemmas.

Lemma 2.2. *Assume that f satisfies* $(\mathbf{f}_1)$–$(\mathbf{f}_3)$. *Then J is continuous differentiable on H and*

$$\langle J'(x), h\rangle = \int_0^{2\pi}\left[\frac{1}{2}\left(\dot{x}\left(t-\frac{\pi}{2}\right) - \dot{x}\left(t+\frac{\pi}{2}\right), h(t)\right) - (f(t, x(t)), h(t))\right]dt, \quad \forall h \in H. \tag{2.9}$$

Moreover, $\varphi' : H \to H^*$ is a compact mapping defined as follows:

$$\langle \varphi'(x), h\rangle = -\int_0^{2\pi}(f(t, x(t)), h(t))dt, \quad \forall x, h \in H. \tag{2.10}$$

Lemma 2.3. *The existence of 2π-periodic solutions of* (1.4) *belonging to E is equivalent to the existence of critical points of functional J restricted to E.*

Lemma 2.3 implies that we can restrict our discussion on space E. At the end of this section, we recall a useful embedding theorem.

Lemma 2.4 (see [13]). *For every $p \in [1, +\infty)$, H is compactly embedded into the Banach space $L^p(S^1, \mathbb{R}^n)$. In particular, there is an α_p such that*

$$\|x\|_{L^p} \le \alpha_p\|x\|, \quad \forall x \in H. \tag{2.11}$$

Remark 2.5. Here and hereafter, α_p $(p \in [1, \infty))$ denotes the real number satisfying (2.11).

3. Main Results

Let $B(t)$ be an $n\times n$ symmetric continuous $\pi/2$-periodic matrix function. We define a bounded self-adjoint linear operator $B \in L(E)$ by extending the bilinear forms

$$\langle Bx, y\rangle = \int_0^{2\pi}(B(t)x(t), y(t))dt, \quad \forall x, y \in E. \tag{3.1}$$

It is well known that B is compact (cf. [14]).

Denote by B_0, B_∞ the operators defined by (3.1), corresponding to $B_0(t), B_\infty(t)$, respectively. Set

$$\begin{gathered} n_0 = \dim \operatorname{Ker}(A - B_0), \qquad n_\infty = \dim \operatorname{Ker}\,(A - B_\infty), \\ G_i(t, x) = F(t, x) - \frac{1}{2}(B_i(t)x, x), \quad \varphi_i(x) = \int_0^{2\pi} G_i(t, x)dt, \quad i = 0, \infty. \end{gathered} \tag{3.2}$$

Then the functional J defined by (2.5) can be rewritten as

$$J(x) = \frac{1}{2}\langle (A - B_i)x, x\rangle - \varphi_i(x), \quad \forall x \in E,\ i = 0, \infty. \tag{3.3}$$

Lemma 3.1. *Suppose that f satisfies $(\mathbf{f}_1)$–$(\mathbf{f}_3)$. Then*

$$\lim_{\|x\| \to 0} \frac{\|\varphi_0'(x)\|}{\|x\|} = 0, \qquad \lim_{\|x\| \to +\infty} \frac{\|\varphi_\infty'(x)\|}{\|x\|} = 0. \tag{3.4}$$

The proof uses the same arguments of [5].

In order to prove our results, we need an abstract theorem by Benci [12].

Proposition 3.2. *Let $\chi \in C^1(E, \mathbb{R})$ satisfy the following:*

- (J1) *$\chi(x) = 1/2\langle Lx, x\rangle + \omega(x)$, where L is a bounded linear self-adjoint operator and ω' is compact, where ω' denotes the Frechét derivative of ω;*
- (J2) *every sequence $\{x_j\}$ such that $\chi(x_j) \to c < \varphi(0)$ and $\|\chi'(x_j)\| \to 0$ as $j \to +\infty$ has a convergent subsequence;*
- (J3) *$\omega(x) = \omega(-x)$, $x \in E$;*
- (J4) *there are two closed subspaces of E, E^+, and E^-, and some constant c_0, c_∞, ρ with $c_0 < c_\infty < \omega(0)$ and $\rho > 0$ such that*
  - (a) *$\chi(x) > c_0$ for $x \in E^+$,*
  - (b) *$\chi(x) < c_\infty < \omega(0)$ for $u \in E^- \cap S_\rho (S_\rho = \{u \in E \|x\| = \rho\})$.*

Then the number of pairs of nontrivial critical points of χ is greater than or equal to $\dim(E^+ \cap E^-) - \operatorname{codim}(E^- + E^+)$. *Moreover, the corresponding critical values belong to* $[c_0, c_\infty]$.

Definition 3.3. Let $B_1(t)$ and $B_2(t)$ be symmetric matrices function in $\mathbb{R}^n$, continuous and $\pi/2$-periodic in t. A index I of $B_1(t)$ and $B_2(t)$ is defined as follows:

$$\begin{aligned} I(B_1(t), B_2(t)) = {} & \dim\Big(M^+(A - B_1) \bigcap M^-(A - B_2)\Big) \\ & - \dim\Big[\Big(M^-(A - B_1) \oplus M^0(A - B_1)\Big) \bigcap \Big(M^+(A - B_2) \oplus M^0(A - B_2)\Big)\Big], \end{aligned} \tag{3.5}$$

where B_i $(i = 1, 2)$ are the operators, defined by (3.1), corresponding to $B_i(t)$ $(i = 1, 2)$ and $M^+(A - B_i)$ (resp., $M^-(A - B_i(t))$, $M^0(A - B_i)$) denotes the subspace of E on which $A - B_i$ is positive definite (resp., negative definite, null).

Lemma 3.4. *If f satisfies $(\mathbf{f}_1)$–$(\mathbf{f}_3)$, then J, defined by (2.8), satisfies (J1), (J3) and (J4).*

Proof. Hypothesis $(\mathbf{f}_1)$, (2.8), and Lemma 2.2 imply both (J1) and (J3). By definition of φ_0 and Lemma 3.1, we have

$$\varphi_0(x) = -\varphi(0) + o\Big(\|x\|^2\Big), \quad \text{for } \|x\| \longrightarrow 0. \tag{3.6}$$

Since B_0 and B_∞ are compact operators from E to E, it follows from Lemma 2.1 and a well-known theorem (cf. [15]) that the essential spectrum of $A - B_0$ and $A - B_\infty$ is $\{2, -2\}$. Thus 0 is either an isolated eigenvalue of finite multiplicity or it belongs to the resolvent. Hence, we decompose E as follows:

$$E = M^+(A - B_0) \oplus M^0(A - B_0) \oplus M^-(A - B_0) = M^+(A - B_\infty) \oplus M^0(A - B_\infty) \oplus M^-(A - B_\infty). \tag{3.7}$$

Setting $E^+ = M^+(A - B_\infty)$, $E^- = M^-(A - B_0)$, there exists positive constant α, β such that

$$\langle (A - B_0)x, x\rangle \le -\alpha\|x\|^2, \quad \forall x \in E^-, \qquad \langle (A - B_\infty)x, x\rangle \ge \beta\|x\|^2, \quad \forall x \in E^+. \tag{3.8}$$

It follows that, for any $x \in E^-$, it is

$$J(x) \le -\frac{\alpha}{2}\|x\|^2 + \varphi(0) - o\left(\|x\|^2\right), \quad \text{as } \|x\| \longrightarrow 0. \tag{3.9}$$

Then there exist constants $\rho > 0$ and $\gamma > 0$ such that

$$J(x) < -\gamma + \varphi(0), \quad \forall x \in E^- \cap S_\rho. \tag{3.10}$$

Setting $c_\infty = -\gamma/2 + \varphi(0)$, (J4)(b) is satisfied.

By ($\mathbf{f}_3$), there exists $R_0 > 0$ such that

$$G_\infty(t, x) \le \frac{\beta}{4\alpha_2^2}|x|^2, \quad \forall |x| > R_0. \tag{3.11}$$

Since $G_\infty(t, x)$ is continuous with respect to (t, x), denote by $M = \max_{0\le t\le \pi/2, |x|=R_0}\{G_\infty(t, x)\}$. Then M is finite. Thus

$$\left|\varphi_\infty(x)\right| \le \int_0^{2\pi} |G_\infty(t, x)|dt \le \int_0^{2\pi} \left[\frac{\beta}{4\alpha_2^2}|x|^2 + M\right]dt \le \frac{\beta}{4}\|x\|^2 + 2\pi M. \tag{3.12}$$

Then, for every $x \in E^+$,

$$J(x) = \frac{1}{2}\langle (A - B_\infty)x, x\rangle - \varphi_\infty(x) \ge \frac{\beta}{2}\|x\|^2 - \left|\varphi_\infty(x)\right| \ge \frac{\beta}{4}\|x\|^2 - 2\pi M. \tag{3.13}$$

Thus J is bounded from below on E^+. Setting

$$c_0 = \inf_{x\in E^+} J(x) - \omega \tag{3.14}$$

with $\omega > 0$ such that $c_0 < c_\infty$, then (J4)(a) is satisfied. □

Remark 3.5. Supposing that $0 \notin \sigma_e(A - B_\infty)$, any bounded sequence has a convergent subsequence (cf. [12]).

Theorem 3.6. *Suppose that f satisfies* $(\mathbf{f}_1)$–$(\mathbf{f}_3)$, *and $n_0 = n_\infty = 0$, then* (1.4) *has at least $|I(B_\infty, B_0)|$ pairs of nonconstant 2π–periodic solutions if $|I(B_\infty, B_0)| > 0$.*

Proof. Since $n_\infty = 0$, $\dim M^0(A - B_\infty) = 0$. By Proposition 3.2 and Lemma 3.4, we only need to check (J2). Let $\{x_j\}$ be a sequence such that

$$J'(x_j) \to 0, \qquad J(x_j) \longrightarrow c, \tag{3.15}$$

where $c \in \mathbb{R}$, $c < \varphi(0)$. Suppose to the contrary that we can choose $\|x_j\| \to +\infty$ as $j \to +\infty$. Clearly, x_j can be written as $x_j = x_j^+ + x_j^- \in M^+(A - B_\infty) \oplus M^-(A - B_\infty)$. On one hand,

$$\frac{\left|\left\langle J'(x_j), x_j^+ - x_j^- \right\rangle\right|}{\left|\langle x_j, x_j \rangle\right|} \le \frac{\|J'(x_j)\| \|x_j\|}{\|x_j\|^2}, \tag{3.16}$$

then we have

$$0 \le \limsup_{j \to +\infty} \frac{\left|\left\langle J'(x_j), x_j^+ - x_j^- \right\rangle\right|}{\left|\langle x_j, x_j \rangle\right|} \le \limsup_{j \to +\infty} \frac{\|J'(x_j)\| \|x_j\|}{\|x_j\|^2} = 0. \tag{3.17}$$

Thus

$$\limsup_{j \to +\infty} \frac{\left|\left\langle J'(x_j), x_j^+ - x_j^- \right\rangle\right|}{\left|\langle x_j, x_j \rangle\right|} = 0. \tag{3.18}$$

On the other hand,

$$\left\langle J'(x_j), x_j^+ - x_j^- \right\rangle = \left\langle (A - B_\infty) x_j, x_j^+ - x_j^- \right\rangle - \left\langle \varphi'_\infty(t, x_j), x_j^+ - x_j^- \right\rangle. \tag{3.19}$$

Since

$$\frac{\left|\left\langle \varphi'_\infty(x_j), x_j^+ - x_j^- \right\rangle\right|}{\left|\langle x_j, x_j \rangle\right|} \le \frac{\|\varphi'_\infty(x_j)\| \|x_j\|}{\|x_j\|^2} = \frac{\|\varphi'_\infty(x_j)\|}{\|x_j\|}, \tag{3.20}$$

it follows by Lemma 3.1 that

$$\lim_{j \to +\infty} \frac{\left|\left\langle \varphi'_\infty(x_j), x_j^+ - x_j^- \right\rangle\right|}{\left|\langle x_j, x_j \rangle\right|} = 0. \tag{3.21}$$

Using a similar discussion as (3.8), there exists $\beta_0 > 0$ such that $\langle (A - B_\infty)x, x\rangle \le -\beta_0\|x\|^2$ for all $x \in M^-(A - B_\infty)$. Choosing $\beta' = \min(\beta, \beta_0) > 0$, we have

$$\left\langle (A - B_\infty)x_j, x_j^+ - x_j^- \right\rangle = \left\langle (A - B_\infty)x_j^+, x_j^+ \right\rangle - \left\langle (A - B_\infty)x_j^-, x_j^- \right\rangle \ge \beta'\|x\|^2. \tag{3.22}$$

Thus,

$$\begin{aligned}\liminf_{j\to+\infty}\frac{\left|\left\langle J'(x_j), x_j^+ - x_j^- \right\rangle\right|}{\left|\langle x_j, x_j\rangle\right|} &= \liminf_{j\to+\infty}\frac{\left|\left\langle (A - B_\infty)x_j, x_j^+ - x_j^- \right\rangle - \left\langle g'_\infty(x_j), x_j^+ - x_j^- \right\rangle\right|}{\left|\langle x_j, x_j\rangle\right|}\\ &= \liminf_{j\to+\infty}\frac{\left|\left\langle (A - B_\infty)x_j, x_j^+ - x_j^- \right\rangle\right|}{\left|\langle x_j, x_j\rangle\right|} \ge \beta' > 0\end{aligned} \tag{3.23}$$

which contradicts (3.18). This proves (J2). By Lemma 3.1, (1.4) has at least $\dim(E^+ \cap E^-) - \operatorname{codim}(E^- + E^+) = I(B_\infty, B_0)$ pairs of nontrivial solutions if $I(B_\infty, B_0) > 0$. Since the Sobolev space E does not contain $\mathbb{R}^n$ as its subspace, all nontrivial periodic solutions are nonconstant periodic solutions.

If $I(B_\infty, B_0) < 0$, then $I(B_0, B_\infty) = -I(B_\infty, B_0) > 0$. In this case, we replace J by $-J$ and let $E^+ = M^-(A - B_0)$ and $E^- = M^+(A - B_\infty)$. It is easy to see that (J1)–(J4) are satisfied. Similarly, we can show that (1.4) has at least $I(B_0, B_\infty)$ pairs of nonconstant solutions. □

Remark 3.7. When Theorem 3.6 is applied to autonomous delay differential equations, we obtain the same number of periodic solutions as that in [3].

Theorem 3.8. *Suppose f satisfies* ($\mathbf{f}_1$)–($\mathbf{f}_3$) *and*

($\mathbf{f}_4$) *$G'_\infty(t, x)$ is bounded, where G'_∞ denotes the derivative of G_∞ with respect to x,*

$(\mathbf{f}_5)^\pm$ *$G_\infty(t, x) \to \pm\infty$ as $|x| \to +\infty$, uniformly for $t \in [0, \pi/2]$.*

Then (1.4) *has at least $I(B_\infty, B_0)$ pairs of nonconstant 2π-periodic solutions provided $I(B_\infty, B_0) > 0$.*

Proof. By Proposition 3.2 and Lemma 3.4, it suffices to check condition (J2). Let $\{x_j\}$ be a sequence satisfying (3.15). Suppose to the contrary that $\{x_j\}$ is unbounded. Clearly, x_j can be written as $x_j = x_j^+ + x_j^0 + x_j^- \in M^+(A - B_\infty) \oplus M^0(A - B_\infty) \oplus M^-(A - B_\infty)$. Since $J'(x_j) \to 0$, for j large enough, we get

$$\left|\left\langle (A - B_\infty)x_j, x_j^+ \right\rangle - \int_0^{2\pi}\left(G'_\infty(t, x_j), x_j^+\right)dt\right| \le \left\|x_j^+\right\|. \tag{3.24}$$

By ($\mathbf{f}_4$), there exists $c_1 > 0$ such that $|G'_\infty(t, x)| \le c_1$. Then the above inequality and (3.8) imply

$$\beta\left\|x_j^+\right\|^2 \le \left|\left\langle (A - B_\infty)x_j, x_j^+ \right\rangle\right| \le \left\|x_j^+\right\| + c_1\alpha_2\sqrt{2\pi}\left\|x_j^+\right\|. \tag{3.25}$$

This gives a uniform bound for $\{x_j^+\}$. In the same manner, one gets a uniform bound for $\{x_j^-\}$. Since $\{J(x_j)\}$ is convergent, it is bounded and there exist positive constants c_2, c_3, c_4 such that

$$\begin{aligned} c_2 \le J(x_j) &\le -\psi_\infty(x_j) + \frac{1}{2}\left|\langle (A - B_\infty)x_j, x_j\rangle\right| \\ &\le -\psi_\infty\left(x_j^0\right) + \left(\psi_\infty\left(x_j^0\right) - \psi_\infty(x_j)\right) + c_3 \\ &\le -\psi_\infty\left(x_j^0\right) + c_1\int_0^{2\pi}\left|x_j^0 - x_j\right|dt + c_3 \\ &\le -\psi_\infty\left(x_j^0\right) + c_4. \end{aligned} \tag{3.26}$$

Therefore, $\psi_\infty(x_j^0)$ is bounded from above. $(\mathbf{f}_5)^+$ implies that $\|x_j^0\|$ is bounded. Otherwise, since the kernel of $A - B_\infty$ is a finite dimensional space, thus $\psi_\infty(x_j^0) = \int_0^{2\pi} G_\infty(t, x_j^0)dt \to \infty$ as $j \to \infty$, which contradicts to (3.26).

If $(\mathbf{f}_5)^-$ holds, we replace (3.26) by

$$c_2 \ge J(x_j) \ge -\psi_\infty(x_j) - \left|\langle (A - B_\infty)x_j, x_j\rangle\right|. \tag{3.27}$$

Arguing as above, we can get a contradiction and complete our proof. □

Theorem 3.9. *Suppose that f satisfies $(\mathbf{f}_1)$–$(\mathbf{f}_3)$ and*

$(\mathbf{f}_6)$ *there exist constants $r > 0$, $p \in (1,2)$, $a_1 > 0$, and $a_2 > 0$ such that*

$$\begin{gathered} pG_\infty(t,x) \ge \left(x, G_\infty'(t,x)\right) > 0 \quad \text{for } |x| \ge r,\ t \in \left[0, \frac{\pi}{2}\right]; \\ G_\infty(t,x) \ge a_1|x|^p - a_2, \quad \forall x \in \mathbb{R}^n,\ t \in \left[0, \frac{\pi}{2}\right]. \end{gathered} \tag{3.28}$$

Then (1.4) has at least $I(B_\infty, B_0)$ pairs of nonconstant 2π-periodic solutions provided $I(B_\infty, B_0) > 0$.

Proof. Let $\{x_j\}$ be a sequence satisfying (3.15). We want to show that $\{x_j\}$ is a bounded sequence in E. Decompose x_j as $x_j = x_j^+ + x_j^0 + x_j^- \in M^+(A - B_\infty) \oplus M^0(A - B_\infty) \oplus M^-(A - B_\infty)$. Then

$$\left\langle J'(x_j), x_j^+\right\rangle = \left\langle (A - B_\infty)x_j^+, x_j^+\right\rangle - \left\langle \psi_\infty'(x_j), x_j^+\right\rangle \ge \beta\left\|x_j^+\right\|^2 - \|\psi_\infty'(x_j)\| \cdot \left\|x_j^+\right\|. \tag{3.29}$$

Combining the above inequality with (3.15) and Lemma 3.1, we have

$$\frac{\left\|x_j^+\right\|}{\|x_j\|} \to 0, \quad \text{as } j \to \infty. \tag{3.30}$$

Similarly, we have

$$\frac{\left\|x_j^-\right\|}{\left\|x_j\right\|} \longrightarrow 0, \quad \text{as } j \longrightarrow \infty. \tag{3.31}$$

Then by (3.30) and (3.31), there exists a positive integer j_0 such that for $j \geq j_0$

$$\left\|x_j^0\right\| \geq \left\|x_j^+ + x_j^-\right\|. \tag{3.32}$$

It follows that

$$\left\|x_j\right\| = \left\|x_j^+ + x_j^- + x_j^0\right\| \leq \left\|x_j^+ + x_j^-\right\| + \left\|x_j^0\right\| \leq 2\left\|x_j^0\right\|. \tag{3.33}$$

By $(\mathbf{f}_6)$, there exist positive constants M_1, M_2, M_3, M_4 such that for j large

$$\begin{aligned} M_1 + \frac{1}{2}\left\|x_j\right\| &\geq \frac{1}{2}\langle J'(x_j), x_j\rangle - J(x_j) = \int_0^{2\pi}\left[G_\infty(t, x_j) - \frac{1}{2}(G'_\infty(t, x_j), x_j)\right]dt \\ &\geq \left(1 - \frac{p}{2}\right)\int_0^{2\pi} G_\infty(t, x_j)\,dt - M_2 \\ &\geq M_3\left\|x_j\right\|_{L^p}^p - M_4. \end{aligned} \tag{3.34}$$

Let q be such that $p^{-1} + q^{-1} = 1$. Since $E \subset L^q(S^1, \mathbb{R}^n)$, the embedding being continuous, the dual space E^* of E, contains $L^p(S^1, \mathbb{R}^n)$ with continuous embedding. Therefore, by (3.34)

$$M_5(1 + \left\|x_j\right\|) \geq \left\|x_j\right\|_{E^*}^p. \tag{3.35}$$

Since $\|x_j\|_{E^*} = \sup_{\|w\|_E \leq 1}(x_j, w)_{L^2} = \sup_{\|w\|_E \leq 1}[(x_j^0, w^0)_{L^2} + (x_j^-, w^-)_{L^2} + (x_j^+, w^+)_{L^2}]$, taking $w = x_j^0/\|x_j^0\|_E$, it follows that

$$\left\|x_j\right\|_{E^*} \geq \frac{1}{\left\|x_j^0\right\|_E}\left\|x_j^0\right\|_{L^2}^2. \tag{3.36}$$

Owing to the fact that $M^0(A - B_\infty)$ is a finite dimensional subspace of E, there exist two positive constants c_1 and c_2 such that

$$c_1\|x_j^0\|_E \leq \|x_j^0\|_{L^2} \leq c_2\|x_j^0\|_E. \tag{3.37}$$

Therefore by (3.35), (3.36), and (3.37),

$$M_6(1 + \left\|x_j\right\|) \geq \left\|x_j^0\right\|^p. \tag{3.38}$$

Both (3.33) and (3.38) imply that there exists $M_8 > 0$ such that

$$M_7\left(1 + \left\|x_j^0\right\|\right) \geq \left\|x_j^0\right\|^p \tag{3.39}$$

which yields a bound for $\|x_j^0\|$ and hence x_j via (3.33). Thus (J2) holds. □

Theorem 3.10. *Suppose that f satisfies* $(\mathbf{f}_1)$–$(\mathbf{f}_3)$ *and*

$(\mathbf{f}_7)^{\pm}$ *there exist positive constants $c_1, c_2 > 0$ such that*

$$\pm\left[2G_\infty(t,x) - \left(G'_\infty(t,x), x\right)\right] \geq c_1|x| - c_2 \quad \forall x \in \mathbb{R}^n,\ t \in \left[0, \frac{\pi}{2}\right]. \tag{3.40}$$

Then (1.4) *has at least $I(B_\infty, B_0)$ pairs of nonconstant 2π-periodic solutions provided $I(B_\infty, B_0) > 0$.*

Proof. Let $\{x_j\}$ be a sequence satisfying (3.15). We want to prove that $\{x_j\}$ is bounded in E. Suppose, to the contrary, $\{x_j\}$ is unbounded in E. Decompose x_j as $x_j = x_j^+ + x_j^0 + x_j^- \in M^+(A - B_\infty) \oplus M^0(A - B_\infty) \oplus M^-(A - B_\infty)$. Clearly, (3.30)–(3.33) still hold.

Assume that $(\mathbf{f}_7)^+$ holds. Since $M^0(A - B_\infty)$ is a finite dimensional subspace of E, we have

$$\begin{aligned}
\langle J'(x_j), x_j\rangle - 2J(x_j) &= \int_0^{2\pi} \left[2G_\infty(t,x_j) - \left(G'_\infty(t,x_j), x_j\right)\right] dt \\
&\geq c_1 \int_0^{2\pi} \left|x_j\right| dt - 2\pi c_2 \\
&\geq c_1 \int_0^{2\pi} \left|x_j^0\right| dt - c_1 \int_0^{2\pi} \left(\left|x_j^+\right| + \left|x_j^-\right|\right) dt - 2\pi c_2 \\
&\geq c_3\left\|x_j^0\right\| - c_4\left(\left\|x_j^+\right\| + \left\|x_j^-\right\| + 1\right).
\end{aligned} \tag{3.41}$$

Combining the above inequality with (3.30), (3.31), we have

$$\frac{\left\|x_j^0\right\|}{\left\|x_j\right\|} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{3.42}$$

But this implies the following contradiction:

$$1 = \frac{\left\|x_j\right\|}{\left\|x_j\right\|} = \frac{\left\|x_j^0\right\| + \left\|x_j^-\right\| + \left\|x_j^+\right\|}{\left\|x_j\right\|} \longrightarrow 0 \quad \text{as } j \longrightarrow +\infty, \tag{3.43}$$

therefore, $\{x_j\}$ must be a bounded sequence.

If $(\mathbf{f}_7)^-$ holds, using a similar argument, we can get a contradiction which completes our proof. □

Theorem 3.11. *Suppose that f satisfies* $(\mathbf{f}_1)$–$(\mathbf{f}_3)$ *and*

$(\mathbf{f}_8)^{\pm}$ *there exist constants* $1 \leq \gamma < 2$, $0 < \delta < \gamma/2$, *and* $b_1, b_2, L > 0$ *such that*

$$\left|G'_{\infty}(t,x)\right| \leq b_1|x|^{\delta}, \quad \pm G_{\infty}(t,x) \geq b_2|x|^{\gamma}, \quad \forall |x| \geq L, t \in \left[0, \frac{\pi}{2}\right]. \tag{3.44}$$

Then (1.4) *has at least* $I(B_{\infty}, B_0)$ *pairs of nonconstant* 2π*-periodic solutions provided* $I(B_{\infty}, B_0) > 0$.

Proof. Let $\{x_j\}$ be a sequence satisfying (3.15). Suppose, to the contrary, $\|x_j\| \to +\infty$ as $j \to +\infty$. Decompose x_j as $x_j = x_j^+ + x_j^0 + x_j^- \in M^+(A - B_{\infty}) \oplus M^0(A - B_{\infty}) \oplus M^-(A - B_{\infty})$. First, we show that for j large enough

$$\left\|x_j^+ + x_j^-\right\| \leq b_3 \left\|x_j^0\right\|^{\delta} + \eta, \tag{3.45}$$

where $b_3 > 0$ and $\eta > 0$ are constants independent of j. Since $|x_j| \geq L$ for sufficiently large j, therefore, $|G'_{\infty}(t, x_j)| \leq b_1 |x_j|^{\delta}$, and we have

$$\begin{aligned}
&\left|G'_{\infty}(t,x_j)\right|^2 \leq b_1^2 \left|x_j\right|^{2\delta} + b_4; \\
&\left|\langle \varphi'_{\infty}(x_j), y\rangle\right| \leq \int_0^{2\pi} \left|G'_{\infty}(t,x_j)\right| |y| dt \leq \left(\int_0^{2\pi} \left|G'_{\infty}(t,x_j)\right|^2 dt\right)^{1/2} \|y\|_{L^2} \\
&\qquad \leq \alpha_2 \left[b_1^2 (2\pi)^{1-\delta} \alpha_2^{2\delta} \|x_j\|^{2\delta} + 2\pi b_4\right]^{1/2} \|y\|, \quad \text{for any } y \in E.
\end{aligned} \tag{3.46}$$

This implies that for j large enough

$$\frac{\|\varphi'_{\infty}(x_j)\|}{\|x_j\|^{\delta}} \leq b_5. \tag{3.47}$$

By (3.15), (3.22), (3.33) and (3.47), for j large enough, we have

$$\begin{aligned}
\left|\left\langle J'(x_j), x_j^+ - x_j^-\right\rangle\right| &= \left|\left\langle (A - B_{\infty})x_j, x_j^+ - x_j^-\right\rangle - \left\langle \varphi'_{\infty}(x_j), x_j^+ - x_j^-\right\rangle\right| \\
&\geq \beta' \left\|x_j^+ + x_j^-\right\|^2 - b_5 \|x_j\|^{\delta} \left\|x_j^+ - x_j^-\right\| \\
&\geq \beta' \left\|x_j^+ + x_j^-\right\|^2 - b_5 2^{\delta} \left\|x_j^0\right\|^{\delta} \left\|x_j^+ - x_j^-\right\|.
\end{aligned} \tag{3.48}$$

Therefore, for sufficiently large j,

$$\|J'(x_j)\| \geq \beta' \left\|x_j^+ + x_j^-\right\| - b_5 2^{\delta} \left\|x_j^0\right\|^{\delta}. \tag{3.49}$$

This implies that (3.45) holds, where $b_3 = b_5 2^{\delta}/\beta'$.

By (3.15) and (3.45), for j large enough, there exist positive constants b_6, b_7, b_7', b_8 such that

$$\begin{aligned}\varphi_\infty(x_j) &= \frac{1}{2}\left\langle (A - B_\infty)\left(x_j^+ + x_j^-\right), x_j^+ + x_j^- \right\rangle - J(x_j)\\ &\le b_6\left\|x_j^+ + x_j^-\right\|^2 + b_7' \le b_8\left\|x_j^0\right\|^{2\delta} + b_7.\end{aligned} \tag{3.50}$$

Now, we claim that there exists $b_9 > 0$ such that, for j large enough,

$$\int_0^{2\pi} |x_j|^\gamma dt \ge b_9\left\|x_j^0\right\|^\gamma, \tag{3.51}$$

In fact, for $\gamma > 1$, by (3.45) and the fact that $\delta < 1$, we have

$$\begin{aligned}\int_0^{2\pi}\left(x_j, x_j^0\right)dt &\le \left(\int_0^{2\pi} |x_j|^\gamma dt\right)^{1/\gamma}\left(\int_0^{2\pi}\left|x_j^0\right|^{\gamma/\gamma-1} dt\right)^{\gamma-1/\gamma}\\ &\le b_{10}\left(\int_0^{2\pi} |x_j|^\gamma dt\right)^{1/\gamma}\left\|x_j^0\right\|;\\ \int_0^{2\pi}\left(x_j, x_j^0\right)dt &= \int_0^{2\pi}\left(x_j^0, x_j^0\right)dt + \int_0^{2\pi}\left(x_j^+ + x_j^-, x_j^0\right)dt\\ &\ge \int_0^{2\pi}\left|x_j^0\right|^2 dt - \left\|x_j^+ + x_j^-\right\|_{L^2}\left\|x_j^0\right\|_{L^2}\\ &\ge b_{11}\left\|x_j^0\right\|^2 - b_3\alpha_2^2\left\|x_j^0\right\|^{1+\delta} - \alpha_2^2\eta\left\|x_j^0\right\| \ge b_{12}\left\|x_j^0\right\|^2,\end{aligned} \tag{3.52}$$

for j large enough. This implies (3.51) for $\gamma > 1$.

For $\gamma = 1$, since $M^0(A - B_\infty)$ is a finite dimensional subspace of E, we know that for any j,

$$b_{13}\left\|x_j^0\right\| \le \left\|x_j^0\right\|_\infty \le b_{14}\left\|x_j^0\right\|. \tag{3.53}$$

where $b_{13}, b_{14} > 0$ are constants independent of j. Now we have

$$\int_0^{2\pi}\left(x_j, x_j^0\right)dt \le \int_0^{2\pi} |x_j|\left|x_j^0\right| dt \le \left(\int_0^{2\pi} |x_j| dt\right)\left\|x_j^0\right\|_\infty \le b_{15}\left\|x_j^0\right\|\left(\int_0^{2\pi} |x_j| dt\right). \tag{3.54}$$

Combining (3.52) with (3.54), we get (3.51) for $\gamma = 1$.

On the other hand, by $(\mathbf{f}_8)^+$

$$\varphi_\infty(x_j) = \int_0^{2\pi} G_\infty(t, x_j)dt \ge \int_0^{2\pi} b_2|x_j|^\gamma dt - 2\pi b_{16} \ge b_{17}\left\|x_j^0\right\|^\gamma - 2\pi b_{16}. \tag{3.55}$$

Since that $\gamma > 2\delta$, we get a contradiction from (3.50) and (3.55). Therefore, $\{x_j\}$ is bounded.

If $(\mathbf{f}_8)^-$ holds, using a similar argument as above, we get a contradiction and completes our proof. □

Theorem 3.12. *Suppose f satisfies $(\mathbf{f}_1)$–$(\mathbf{f}_3)$ and*

$(\mathbf{f}_9)^\pm$ *there exist positive constants $1 \le \gamma < 2$, $0 < \delta < \gamma/2$, and b_1, b_2, L such that*

$$\left|G'_\infty(t,x)\right| \le b_1|x|^\delta, \pm\langle G'_\infty(t,x), x\rangle \ge b_2|x|^\gamma, \quad \forall |x| \ge L,\ t \in \left[0, \frac{\pi}{2}\right]. \tag{3.56}$$

Then (1.4) *has at least $I(B_\infty, B_0)$ pairs of nonconstant 2π-periodic solutions provided $I(B_\infty, B_0) > 0$.*

Proof. If $(\mathbf{f}_9)^+$ holds, for j large enough, by (3.45) and (3.51), we have

$$\begin{aligned}\int_0^{2\pi} (G'_\infty(t,x_j), x_j)dt &\le \left|-\langle J'(x_j), x_j\rangle + \left\langle (A-B_\infty)\left(x_j^+ + x_j^-\right), \left(x_j^+ + x_j^-\right)\right\rangle\right| \\ &\le \|x_j\| + M_1\left\|x_j^+ + x_j^-\right\|^2 \le \left\|x_j^0\right\| + M_2\left\|x_j^0\right\|^\delta + M_3\left\|x_j^0\right\|^{2\delta} + M_4; \\ \int_0^{2\pi} (G'_\infty(t,x_j), x_j)dt &\ge b_2\int_0^{2\pi} |x_j|^\gamma dt - M_5 \ge M_6\left\|x_j^0\right\|^\gamma - M_5.\end{aligned} \tag{3.57}$$

Since $\gamma > 2\delta$, $\{x_j^0\}$ is bounded, so is $\{x_j\}$. Therefore, J satisfies (J2).

In the case that $(\mathbf{f}_9)^-$ holds, using a similar argument, we can verify (J2). This completes the proof. □

Example 3.13. Consider the following nonautonomous delay differential equation

$$x'(t) = -Mx\left(t - \frac{\pi}{2}\right)\frac{a + b(t)|x(t-(\pi/2))|^{5/2} + c|x(t-(\pi/2))|^4}{1 + |x(t-(\pi/2))|^4}, \tag{3.58}$$

where M is a 4×4 matrix, a, c are constants, $b \in C([0, \pi/2]\ , \mathbb{R}^+)$.

Case 1. Let $A = \mathrm{diag}(0.3,\ 2.7,\ 7.3,\ 9.3)$, $a = 1$, $c = 2$, and b arbitrary. Computing directly, we have $I(B_0(t), B_\infty(t)) = 10$. Applying Theorem 3.6, equation (3.58) has at least 10 pairs of 2π-periodic solutions.

Case 2. Let $A = \mathrm{diag}(0.3,\ 2.7,\ 5,\ 10.5)$, $a = 2$, $c = 1$, and b arbitrary. Then by Theorem 3.8, (3.58) has at least 8 pairs of nonconstant 2π-periodic solutions.

Acknowledgments

This paper is supported by Xinmiao Program of Guangzhou University and National Natural Science Foundation of China (no. 11026059), SRF of Guangzhou Education Bureau (no. 10A012), and Foundation for Distinguished Young Talents in Higher Education of Guangdong (no. LYM09105).

References

[1] G. S. Jones, "The existence of periodic solutions of $f'(x) = -af(x(t-1))[1+f(x)]$," *Journal of Mathematical Analysis and Applications*, vol. 5, pp. 435–450, 1962.

[2] R. B. Grafton, "A periodicity theorem for autonomous functional differential equations," *Journal of Differential Equations*, vol. 6, pp. 87–109, 1969.

[3] Z. Guo and J. Yu, "Multiplicity results for periodic solutions to delay differential equations via critical point theory," *Journal of Differential Equations*, vol. 218, no. 1, pp. 15–35, 2005.

[4] J. L. Kaplan and J. A. Yorke, "Ordinary differential equations which yield periodic solutions of differential delay equations," *Journal of Mathematical Analysis and Applications*, vol. 48, pp. 317–324, 1974.

[5] G. Fei, "Multiple periodic solutions of differential delay equations via Hamiltonian systems. I," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 1, pp. 25–39, 2006.

[6] G. Fei, "Multiple periodic solutions of differential delay equations via Hamiltonian systems. II," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 1, pp. 40–58, 2006.

[7] J. Li and X.-Z. He, "Multiple periodic solutions of differential delay equations created by asymptotically linear Hamiltonian systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 31, no. 1-2, pp. 45–54, 1998.

[8] J. Li and X. He, "Proof and generalization of Kaplan-Yorke's conjecture on periodic solution of differential delay equations,," *Science in China Series A*, vol. 42, no. 9, pp. 957–964, 1999.

[9] J. Mallet-Paret and G. R. Sell, "The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay," *Journal of Differential Equations*, vol. 125, no. 2, pp. 441–489, 1996.

[10] J. Mawhin, "Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces," *Journal of Differential Equations*, vol. 12, pp. 610–636, 1972.

[11] D. O'Regan and H. Wang, "Positive periodic solutions of systems of first order ordinary differential equations," *Results in Mathematics. Resultate der Mathematik*, vol. 48, no. 3-4, pp. 310–325, 2005.

[12] V. Benci, "On critical point theory for indefinite functionals in the presence of symmetries," *Transactions of the American Mathematical Society*, vol. 274, no. 2, pp. 533–572, 1982.

[13] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, CBMS, New York, NY, USA, 1986.

[14] A. Szulkin and W. Zou, "Infinite dimensional cohomology groups and periodic solutions of asymptotically linear Hamiltonian systems," *Journal of Differential Equations*, vol. 174, no. 2, pp. 369–391, 2001.

[15] M. Schechter, *Spectra of Partial Differential Operators*, vol. 14 of *North-Holland Series in Applied Mathematics and Mechanics*, North-Holland, Amsterdam, The Netherlands, 2nd edition, 1986.