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Research Article

Spectral Shifted Jacobi Tau and Collocation Methods for Solving Fifth-Order Boundary Value Problems

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We have presented an efficient spectral algorithm based on shifted Jacobi tau method of linear fifth-order two-point boundary value problems (BVPs). An approach that is implementing the shifted Jacobi tau method in combination with the shifted Jacobi collocation technique is introduced for the numerical solution of fifth-order differential equations with variable coefficients. The main characteristic behind this approach is that it reduces such problems to those of solving a system of algebraic equations which greatly simplify the problem. Shifted Jacobi collocation method is developed for solving nonlinear fifth-order BVPs. Numerical examples are performed to show the validity and applicability of the techniques. A comparison has been made with the existing results. The method is easy to implement and gives very accurate results.

1. Introduction

The solutions of fifth-order BVPs have been the subject of active research. These problems generally arise in mathematical modeling of viscoelastic flows, physics, engineering, and other disciplines, (see, e.g., [1–4]). Agarwal's book [5] contains some theorems that discuss the conditions for existence and uniqueness of the solutions of fifth-order BVPs in detail.

Recently, various powerful mathematical methods such as the sixth-degree B-spline [6], Adomian decomposition method [7], nonpolynomial sextic spline functions [8–12], local polynomial regression [13], and others [14, 15] have been proposed to obtain exact and approximate analytic solutions for linear and nonlinear problems.

Spectral methods (see, e.g., [16–18]) provide a computational approach that has achieved substantial popularity over the last four decades. The main advantage of these methods lies in their high accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy. In contrast, finite-difference and finite-element methods yield only algebraic convergence rates. The three most widely used spectral versions are the Galerkin, collocation, and tau methods. Recently, Doha et al. [19, 20] introduced and developed direct solution algorithms for solving high even-order differential equations using Bernstein Dual petrov-Galerkin method. Also, Bhrawy and Abd-Elhameed in [21] developed new algorithm for solving the general nonlinear third-order differential equation by means of a shifted Jacobi-Gauss collocation spectral method. The shifted Jacobi-Gauss points are used as collocation nodes. Moreover, Bhrawy and Alofi [22] used the Jacobi-Gauss collocation method to solve effectively and accurately a nonlinear singular Lane-Emden-type equation with approximation converging rapidly to exact solution.

The use of general Jacobi polynomials has the advantage of obtaining the solutions of differential equations in terms of the Jacobi parameters α and β (see, [23–25]). In the present paper, we intend to extend application of Jacobi polynomials from Galerkin method for solving second- and fourth-order linear problems (see, [23, 24, 26]) to tau and collocation methods to solve linear and nonlinear fifth-order BVPs.

In the tau method (see, e.g., [16, 27]), the approximate solution $u_N(x)$ of an equation on the interval $(0, L)$ is represented as a finite series $u_N(x) = \sum_{j=0}^{N} a_j \phi_j(x)$, where the $\phi_j(x)$ are global (base) functions on $(0, L)$. The coefficients a_j are the unknown one solves for. One characteristic of the tau method is that the expansion functions $\phi_j(x)$ do not satisfy boundary conditions in relation to the supplementary conditions imposed together with the differential equation.

Since 1960 the nonlinear problems have attracted much attention. In fact, there are many analytic asymptotic methods that have been proposed for addressing the nonlinear problems. Recently, the collocation method [21, 28] has been used to solve effectively and accurately a large class of nonlinear problems with approximations converging rapidly to exact solutions.

The fundamental goal of this paper is to develop a direct solution algorithm for approximating the linear two-point fifth-order differential equations by shifted Jacobi tau (SJT) method that can be implemented efficiently. Moreover, we introduce the pseudospectral shifted Jacobi tau (P-SJT) method in order to deal with fifth-order BVPs of variable coefficients. This method is basically formulated in the shifted Jacobi tau spectral form with general indexes $\alpha, \beta > -1$. However, the variable coefficients terms and the right-hand side of fifth-order BVPs are treated by the shifted Jacobi collocation method with the same indexes $\alpha, \beta > -1$ so that the the schemes can be implemented at shifted Jacobi-Gauss points efficiently.

For nonlinear fifth-order problems on the interval $(0, L)$, we propose the spectral shifted Jacobi collocation (SJC) method to find the solution $u_N(x)$. The nonlinear BVP is collocated only at the $(N - 4)$ points. For suitable collocation points, we use the $(N - 4)$ nodes of the shifted Jacobi-Gauss interpolation on $(0, L)$. These equations together with five boundary conditions generate $(N + 1)$ nonlinear algebraic equations which can be solved using Newton's iterative method. Finally, the accuracy of the proposed methods are demonstrated by test problems.

The remainder of this paper is organized as follows. In Section 2, we give an overview of shifted Jacobi polynomials and their relevant properties needed hereafter. Sections 3 and 4 are devoted to the theoretical derivation of the SJT and P-SJT methods for fifth-order differential equations with constant and variable coefficients. Section 5 is devoted to applying the SJC method for solving nonlinear fifth-order differential equations. In Section 6, the proposed methods are applied to several examples. Also, a conclusion is given in Section 7.

2. Preliminaries

Let $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ be a weight function in the usual sense for $\alpha, \beta > -1$. The set of Jacobi polynomials $\{P_k^{(\alpha,\beta)}(x)\}_{k=0}^{\infty}$ forms a complete $L^2_{w^{(\alpha,\beta)}}(-1,1)$-orthogonal system, and

$$\left\| P_k^{(\alpha,\beta)} \right\|^2_{w^{(\alpha,\beta)}} = h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}. \tag{2.1}$$

Here, $L^2_{w^{(\alpha,\beta)}}(-1,1)$ is a weighted space defined by

$$L^2_{w^{(\alpha,\beta)}}(-1,1) = \{v : v \text{ is measurable and } \|v\|_{w^{(\alpha,\beta)}} < \infty\}, \tag{2.2}$$

equipped with the norm

$$\|v\|_{w^{(\alpha,\beta)}} = \left(\int_{-1}^{1} |v(x)|^2 w^{(\alpha,\beta)} dx \right)^{1/2}, \tag{2.3}$$

and the inner product

$$(u,v)_{w^{(\alpha,\beta)}} = \int_{-1}^{1} u(x)\ v(x)\ w^{(\alpha,\beta)}(x)dx, \quad \forall u,v \in L^2_{w^{(\alpha,\beta)}}(-1,1). \tag{2.4}$$

It is well known that

$$\begin{gathered} P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\beta,\alpha)}(x), \qquad P_k^{(\alpha,\beta)}(-1) = \frac{(-1)^k\Gamma(k+\beta+1)}{k!\Gamma(\beta+1)}, \qquad P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)}, \\ D^m P_k^{(\alpha,\beta)}(x) = 2^{-m}\frac{\Gamma(m+k+\alpha+\beta+1)}{\Gamma(k+\alpha+\beta+1)} P_{k-m}^{(\alpha+m,\beta+m)}(x). \end{gathered} \tag{2.5}$$

If we define the shifted Jacobi polynomial of degree k by $P_{L,k}^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}(2x/L) - 1$, $L > 0$, and in virtue of (2.5), then it can easily be shown that

$$\begin{aligned} D^q P_{L,k}^{(\alpha,\beta)}(0) &= \frac{(-1)^{k-q}\Gamma(k+\beta+1)(k+\alpha+\beta+1)_q}{L^q\Gamma(k-q+1)\Gamma(q+\beta+1)}, \\ D^q P_{L,k}^{(\alpha,\beta)}(L) &= \frac{\Gamma(k+\alpha+1)(k+\alpha+\beta+1)_q}{L^q\Gamma(k-q+1)\Gamma(q+\alpha+1)}. \end{aligned} \tag{2.6}$$

Now, let $w_L^{(\alpha,\beta)}(x) = (L-x)^{\alpha}x^{\beta}$. The set of shifted Jacobi polynomials $\{P_{L,k}^{(\alpha,\beta)}(x)\}_{k=0}^{\infty}$ forms a complete $L^2_{w_L^{(\alpha,\beta)}}(0,L)$-orthogonal system. Moreover, and due to (2.1), we get

$$\left\| P_{L,k}^{(\alpha,\beta)} \right\|^2_{w_L^{(\alpha,\beta)}} = \left(\frac{L}{2}\right)^{\alpha+\beta+1} h_k^{(\alpha,\beta)} = h_{L,k}^{(\alpha,\beta)}, \tag{2.7}$$

where $L^2_{w_L^{(\alpha,\beta)}}(0,L)$ is a weighted space defined by

$$L^2_{w_L^{(\alpha,\beta)}}(0,L) = \left\{ v : v \text{ is measurable and } \|v\|_{w_L^{(\alpha,\beta)}} < \infty \right\}, \tag{2.8}$$

equipped with the norm

$$\|v\|_{w_L^{(\alpha,\beta)}} = \left(\int_0^L |v(x)|^2 w_L^{(\alpha,\beta)} dx \right)^{1/2}, \tag{2.9}$$

and the inner product

$$(u,v)_{w_L^{(\alpha,\beta)}} = \int_0^L u(x)v(x)w_L^{(\alpha,\beta)}(x)dx, \quad \forall u,v \in L^2_{w_L^{(\alpha,\beta)}}(0,L). \tag{2.10}$$

For $\alpha = \beta$, one can obtain the shifted ultraspherical polynomials (symmetric Jacobi polynomials). For $\alpha = \beta = \mp 1/2$, the shifted Chebyshev polynomials of first and second kinds. For $\alpha = \beta = 0$, one can obtain the shifted Legendre polynomials. For the two important special cases $\alpha = -\beta = \pm 1/2$, the shifted Chebyshev polynomials of third and fourth kinds are also obtained.

We denote by $x_{N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$ the nodes of the standard Jacobi-Gauss interpolation on the interval $(-1,1)$. Their corresponding Christoffel numbers are $\varpi_{N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$. The nodes of the shifted Jacobi-Gauss interpolation on the interval $(0,L)$ are the zeros of $P_{L,N+1}^{(\alpha,\beta)}(x)$, which are denoted by $x_{L,N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$. It is clear that $x_{L,N,j}^{(\alpha,\beta)} = (L/2)(x_{N,j}^{(\alpha,\beta)} + 1)$, and their corresponding Christoffel numbers are $\varpi_{L,N,j}^{(\alpha,\beta)} = (L/2)^{\alpha+\beta+1}\varpi_{N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$.

Let $S_N(0,L)$ be the set of polynomials of degree at most N. Regarding, to the property of the standard Jacobi-Gauss quadrature, then it can be easily shown that for any $\phi \in S_{2N+1}(0,L)$,

$$\begin{aligned} \int_0^L (L-x)^{\alpha}x^{\beta}\phi(x)dx &= \left(\frac{L}{2}\right)^{\alpha+\beta+1} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\phi\left(\frac{L}{2}(x+1)\right)dx \\ &= \left(\frac{L}{2}\right)^{\alpha+\beta+1} \sum_{j=0}^{N} \varpi_{N,j}^{(\alpha,\beta)} \phi\left(\frac{L}{2}\left(x_{N,j}^{(\alpha,\beta)}+1\right)\right) = \sum_{j=0}^{N} \varpi_{L,N,j}^{(\alpha,\beta)} \phi\left(x_{L,N,j}^{(\alpha,\beta)}\right). \end{aligned} \tag{2.11}$$

The qth derivative of shifted Jacobi polynomial can be written in terms of the shifted Jacobi polynomials themselves as (see, Doha [29])

$$D^q P_{L,k}^{(\alpha,\beta)}(x) = \sum_{i=0}^{k-q} C_q(k,i,\alpha,\beta) P_{L,i}^{(\alpha,\beta)}(x), \tag{2.12}$$

where

$$\begin{aligned} C_q(k,i,\alpha,\beta) =& \frac{(k+\lambda)_q (k+\lambda+q)_i (i+\alpha+q+1)_{k-i-q} \Gamma(i+\lambda)}{L^q (k-i-q)! \Gamma(2i+\lambda)} \\ & \times {}_3F_2\left(\begin{matrix} -k+i+q, \quad k+i+\lambda+q, \; i+\alpha+1 \\ \\ i+\alpha+q+1, \quad 2i+\lambda+1 \end{matrix} ;1 \right). \end{aligned} \tag{2.13}$$

For the general definition of a generalized hypergeometric series and special ${}_3F_2$, see [30], p. 41 and pp. 103-104, respectively.

3. Fifth-Order BVPs with Constant Coefficients

In this section, we are intending to use the SJT method to solve the fifth-order boundary value problems

$$D^{(5)} u(x) + \sum_{i=1}^{4} \gamma_i D^{(i)} u(x) + \gamma_5 u(x) = f(x), \quad \text{in } I = (0, L), \tag{3.1}$$

with boundary conditions

$$u(0) = \alpha_0, \qquad u^{(1)}(0) = \alpha_1, \qquad u^{(2)}(0) = \alpha_2, \qquad u(L) = \beta_1, \qquad u^{(1)}(L) = \beta_1, \tag{3.2}$$

where $(\gamma_i, i = 1, 2, \ldots, 5)$, α_0, α_1, α_2, β_0, and β_1 are real constants, and $f(x)$ is a given source function.

Let us first introduce some basic notation that will be used in this section. We set

$$S_N(0,L) = \operatorname{span}\left\{ P_{L,0}^{(\alpha,\beta)}(x), P_{L,1}^{(\alpha,\beta)}(x), \ldots, P_{L,N}^{(\alpha,\beta)}(x) \right\}, \tag{3.3}$$

then the shifted Jacobi-tau approximation to (3.1) is to find $u_N \in S_N(0,L)$ such that

$$\begin{gathered} \left(D^{(5)} u_N, P_{L,k}^{(\alpha,\beta)}(x) \right)_{w_L^{(\alpha,\beta)}} + \sum_{i=1}^{4} \gamma_i \left(D^{(i)} u_N, P_{L,k}^{(\alpha,\beta)}(x) \right)_{w_L^{(\alpha,\beta)}} + \gamma_5 \left(u_N, P_{L,k}^{(\alpha,\beta)}(x) \right)_{w_L^{(\alpha,\beta)}} \\ = \left(f, P_{L,k}^{(\alpha,\beta)}(x) \right)_{w_L^{(\alpha,\beta)}}, \quad k = 0, 1, \ldots, N-5, \\ u(0) = \alpha_0, \qquad u^{(1)}(0) = \alpha_1, \qquad u^{(2)}(0) = \alpha_2, \qquad u(L) = \beta_1, \qquad u^{(1)}(L) = \beta_1. \end{gathered} \tag{3.4}$$

If we assume that

$$u_n(x) = \sum_{j=0}^{N} a_j P_{L,j}^{(\alpha,\beta)}(x), \quad \mathbf{a} = (a_0, a_1, \dots, a_N)^T,$$
$$f_k = \left(f, P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)}}, \quad k = 0, 1, \dots, N-5, \tag{3.5}$$
$$\mathbf{f} = \left(f_0, f_1, \dots, f_{N-5}, \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1\right),$$

then (3.4) with its boundary conditions give

$$\begin{aligned}\sum_{j=0}^{N} a_j \Bigg[& \left(D^{(5)} P_{L,j}^{(\alpha,\beta)}(x), P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)}} + \sum_{i=1}^{4} \gamma_i \left(D^{(i)} P_{L,j}^{(\alpha,\beta)}(x), P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)}} \\ & + \gamma_5 \left(P_{L,j}^{(\alpha,\beta)}(x), P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)}} \Bigg] \\ = & \left(f, P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)}}, \quad k = 0, 1, \dots, N-5,\end{aligned} \tag{3.6}$$

$$\begin{gathered}\sum_{j=0}^{N} a_j P_{L,j}^{(\alpha,\beta)}(0) = \alpha_0, \qquad \sum_{j=0}^{N} a_j D^{(1)} P_{L,j}^{(\alpha,\beta)}(0) = \alpha_1, \qquad \sum_{j=0}^{N} a_j D^{(2)} P_{L,j}^{(\alpha,\beta)}(0) = \alpha_2, \\ \sum_{j=0}^{N} a_j P_{L,j}^{(\alpha,\beta)}(L) = \beta_0, \qquad \sum_{j=0}^{N} a_j D^{(1)} P_{L,j}^{(\alpha,\beta)}(L) = \beta_1.\end{gathered} \tag{3.7}$$

Now, let us denote

$$\begin{gathered}A = \left(a_{kj}\right)_{0 \le k,j \le N}, \qquad B^i = \left(b_{kj}^i\right)_{0 \le k,j \le N,\ i=1,2,3,4}, \qquad C = \left(c_{kj}\right)_{0 \le k,j \le N}, \\
a_{kj} = \begin{cases} \left(D^{(5)} P_{L,j}^{(\alpha,\beta)}(x), P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)}}, & k = 0, 1, \dots, N-5,\ j = 0, 1, \dots, N, \\ P_{L,j}^{(\alpha,\beta)}(0), & k = N-4,\ j = 0, 1, \dots, N, \\ D^{(1)} P_{L,j}^{(\alpha,\beta)}(0), & k = N-3,\ j = 0, 1, \dots, N, \\ D^{(2)} P_{L,j}^{(\alpha,\beta)}(0), & k = N-2,\ j = 0, 1, \dots, N, \\ P_{L,j}^{(\alpha,\beta)}(L), & k = N-1,\ j = 0, 1, \dots, N, \\ D^{(1)} P_{L,j}^{(\alpha,\beta)}(L), & k = N,\ j = 0, 1, \dots, N, \end{cases} \\
b_{kj}^i = \begin{cases} \left(D^{(i)} P_{L,j}^{(\alpha,\beta)}(x),\ P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)}}, & k = 0, 1, \dots, N-5,\ j = 0, 1, \dots, N,\ i = 1, 2, 3, 4, \\ 0, & \text{otherwise}, \end{cases} \\
c_{kj} = \begin{cases} \left(P_{L,j}^{(\alpha,\beta)}(x), P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)}}, & k = 0, 1, \dots, N-5,\ j = 0, 1, \dots, N, \\ 0, & \text{otherwise}. \end{cases}\end{gathered} \tag{3.8}$$

Then, recalling (2.6) and (2.12) and making use of the orthogonality relation of shifted Jacobi polynomials (2.7), and after performing some rather calculations, the nonzero elements of a_{kj}, b^i_{kj} and c_{kj} for $0 \le k, j \le N$ are given by

$$a_{kj} = \begin{cases} C_5(j,k,\alpha,\beta)h^{(\alpha,\beta)}_{L,k}, & k = 0,1,\dots,N-5,\ j = 5,6,\dots,N, \\ \dfrac{(-1)^j\Gamma(j+\beta+1)}{j!\ \Gamma(\beta+1)}, & k = N-4,\ j = 0,1,\dots,N, \\ \dfrac{(-1)^{j-1}(j+\alpha+\beta+1)\Gamma(j+\beta+1)}{L\ (j-1)!\ \Gamma(\beta+2)}, & k = N-3,\ j = 1,2,\dots,N, \\ \dfrac{(-1)^{j-2}(j+\alpha+\beta+1)_2\Gamma(j+\beta+1)}{L^2\ (j-2)!\ \Gamma(\beta+3)}, & k = N-2,\ j = 2,3,\dots,N, \\ \dfrac{\Gamma(j+\alpha+1)}{j!\Gamma(\alpha+1)}, & k = N-1,\ j = 0,1,\dots,N, \\ \dfrac{(j+\alpha+\beta+1)\Gamma(j+\alpha+1)}{L(j-1)!\ \Gamma(\alpha+2)}, & k = N,\ j = 1,2,\dots,N, \end{cases} \tag{3.9}$$

$$b^i_{kj} = C_i(j,k,\alpha,\beta)\ h^{(\alpha,\beta)}_{L,k}, \quad k = 0,1,\dots,N-5,\ j = i, i+1,\dots,N,\ i = 1,2,3,4,$$

$$c_{kj} = h^{(\alpha,\beta)}_{L,k}, \quad k = j = 0,1,\dots,N-5,$$

and consequently, (3.6) may be put in the form

$$\left(A + \sum_{i=1}^{4}\gamma_i B^i + \gamma_5 C\right)\mathbf{a} = \mathbf{f}. \tag{3.10}$$

4. Fifth-Order BVPs with Variable Coefficients

In this section, we use the pseudospectral shifted Jacobi tau (P-SJT) method to numerically solve the following fifth-order boundary value problem with variable coefficients

$$\begin{gathered} D^{(5)}u(x) + \sum_{i=1}^{4}\gamma_i(x)D^{(i)}u(x) + \gamma_5(x)u(x) = f(x), \quad x \in I, \\ u(0) = \alpha_0, \quad u^{(1)}(0) = \alpha_1, \quad u^{(2)}(0) = \alpha_2, \quad u(L) = \beta_1, \quad u^{(1)}(L) = \beta_1. \end{gathered} \tag{4.1}$$

We define the discrete inner product and norm as follows:

$$(u,v)_{w^{(\alpha,\beta)}_L,N} = \sum_{j=0}^{N} u\left(x^{(\alpha,\beta)}_{L,N,j}\right)v\left(x^{(\alpha,\beta)}_{L,N,j}\right)\varpi^{(\alpha,\beta)}_{L,N,j}, \qquad \|u\|_{w^{(\alpha,\beta)}_L,N} = \sqrt{(u,u)_{w^{(\alpha,\beta)}_L,N}}, \tag{4.2}$$

where $x_{L,N,j}^{(\alpha,\beta)}$ and $\varpi_{L,N,j}^{(\alpha,\beta)}$ are the nodes and the corresponding weights of shifted Jacobi-Gauss-quadrature formula on the interval $(0,L)$, respectively.

Obviously,

$$(u,v)_{w_L^{(\alpha,\beta)},N} = (u,\mathrm{v})_{w_L^{(\alpha,\beta)}}, \quad \forall u,v \in S_{2N-1}. \tag{4.3}$$

Thus, for any $u \in S_N(0,L)$, the norms $\|u\|_{w_L^{(\alpha,\beta)},N}$ and $\|u\|_{w_L^{(\alpha,\beta)}}$ coincide.

The pseudospectral tau method for (4.1) is to find $u_N \in S_N(0,L)$ such that

$$\begin{aligned}&\left(D^{(5)}u_N, P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)}} + \sum_{i=1}^{4}\left(\gamma_i(x)D^{(i)}u_N, P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)},N} + \left(\gamma_5(x)u_N, P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)},N}\\ &\quad = \left(f, P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)},N}, \quad k = 0,1,\ldots,N-5,\end{aligned} \tag{4.4}$$

where $(u,v)_{w_L^{(\alpha,\beta)},N}$ is the discrete inner product of u and v associated with the shifted Jacobi-Gauss quadrature, and the boundary conditions can easily be treated as in (3.7).

Hence, by setting

$$\begin{aligned}
&u_N(x) = \sum_{j=0}^{N}\tilde{a}_j P_{L,j}^{(\alpha,\beta)}(x), \quad \bar{\mathbf{a}} = (\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_N)^T,\\
&\tilde{f}_k = \left(f, P_{L,k}^{(\alpha,\beta)}\right)_{w_L^{(\alpha,\beta)},N}, \quad k = 0,1,\ldots,N-5, \qquad \bar{\mathbf{f}} = \left(\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{N-5}, \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1\right)^T,\\
&\tilde{b}_{kj}^i = \begin{cases}\left(\gamma_i(x)D^{(i)}P_{L,j}^{(\alpha,\beta)}(x), P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)},N}, & k = 0,1,\ldots,N-5,\ j = 0,1,\ldots,N,\\ & i = 1,2,3,4,\\ 0, & \text{otherwise},\end{cases}\\
&\tilde{c}_{kj} = \begin{cases}\left(\gamma_5(x)P_{L,j}^{(\alpha,\beta)}(x), P_{L,k}^{(\alpha,\beta)}(x)\right)_{w_L^{(\alpha,\beta)},N}, & k = 0,1,\ldots,N-5,\ j = 0,1,\ldots,N,\\ 0, & \text{otherwise},\end{cases}\\
&\tilde{B}^i = \left(\tilde{b}_{kj}^i\right), \quad \tilde{C} = (\tilde{c}_{kj}), \quad 0 \le k,j \le N,\ i = 1,2,3,4,
\end{aligned} \tag{4.5}$$

then (4.4) and (3.7) may be put in the following matrix form

$$\left(A + \sum_{i=1}^{4}\gamma_i\tilde{B}^i + \gamma_5\tilde{C}\right)\bar{\mathbf{a}} = \bar{\mathbf{f}}, \tag{4.6}$$

where the matrix A is defined as in (3.10).

5. Nonlinear Fifth-Order BVPs

In this section, we are interested in solving numerically the nonlinear fifth-order boundary value problem

$$D^{(5)}u(x) = F\Big(x, u(x), D^{(1)}u(x), D^{(2)}u(x), D^{(3)}u(x), D^{(4)}u(x)\Big), \tag{5.1}$$

with boundary conditions

$$u(0) = \alpha_0, \qquad u^{(1)}(0) = \alpha_1, \qquad u^{(2)}(0) = \alpha_2, \qquad u(L) = \beta_1, \qquad u^{(1)}(L) = \beta_1. \tag{5.2}$$

It is well known that one can convert (5.1) with its boundary conditions (5.2) into a fifth-order system of first-order boundary value problems. Methods to solve such system are simply generalizations of the methods for a single first-order equation, for example, the classical Runge-Kutta of order four. Another alternative spectral method is to use the shifted Jacobi collocation method. Let

$$u_N(x) = \sum_{j=0}^{N} b_j P_{L,j}^{(\alpha,\beta)}(x), \tag{5.3}$$

then, making use of formula (2.12), one can express explicitly the derivatives $D^{(i)}u(x)$, $(i = 0, 1, 2, 3, 4)$ in terms of the expansion coefficients b_j. The criterion of spectral shifted Jacobi collocation method for solving approximately (5.1)–(5.2) is to find $u_N(x) \in S_N(0, L)$ such that

$$D^{(5)}u_N(x) = F\Big(x, u_N(x), D^{(1)}u_N(x), D^{(2)}u_N(x), D^{(3)}u_N(x), D^{(4)}u_N(x)\Big), \tag{5.4}$$

is satisfied exactly at the collocation points $x_{L,N,k}^{(\alpha,\beta)}$, $k = 0, 1, \ldots, N-5$. In other words, we have to collocate (5.4) at the $(N-4)$ shifted Jacobi roots $x_{L,N,k}^{(\alpha,\beta)}$, which immediately yields

$$\begin{aligned}\sum_{j=0}^{N} b_j D^{(5)} P_{L,j}^{(\alpha,\beta)}(x) = F\Bigg(& x, \sum_{j=0}^{N} b_j P_{L,j}^{(\alpha,\beta)}(x), \sum_{j=0}^{N} b_j D^{(1)} P_{L,j}^{(\alpha,\beta)}(x), \sum_{j=0}^{N} b_j D^{(2)} P_{L,j}^{(\alpha,\beta)}(x), \\ & \sum_{j=0}^{N} b_j D^{(3)} P_{L,j}^{(\alpha,\beta)}(x), \sum_{j=0}^{N} b_j D^{(4)} P_{L,j}^{(\alpha,\beta)}(x) \Bigg),\end{aligned} \tag{5.5}$$

with the boundary conditions (5.2) written in the form

$$\begin{aligned}\sum_{j=0}^{N} b_j P_{L,j}^{(\alpha,\beta)}(0) &= \alpha_0,\\ \sum_{j=1}^{N} b_j D^{(1)} P_{L,j}^{(\alpha,\beta)}(0) &= \alpha_1,\\ \sum_{j=2}^{N} b_j D^{(2)} P_{L,j}^{(\alpha,\beta)}(0) &= \alpha_2,\\ \sum_{j=0}^{N} b_j P_{L,j}^{(\alpha,\beta)}(L) &= \beta_0,\\ \sum_{j=1}^{N} b_j D^{(1)} P_{L,j}^{(\alpha,\beta)}(L) &= \beta_1.\end{aligned} \tag{5.6}$$

This forms a system of $(N+1)$ nonlinear algebraic equations in the unknown expansion coefficients b_j $(j = 0, 1, \ldots, N)$, which can be solved by using any standard iteration technique, like Newton's iteration method. For more detail, see [21] for the numerical solution of nonlinear third-order differential equation using Jacobi-Gauss collocation method.

6. Numerical Results

In this section, we apply shifted Jacobi tau (SJT) method for solving the fifth-order boundary value problems. Numerical results are very encouraging. For the purpose of comparison, we took the same examples as used in [6, 7, 31–34].

Example 6.1. Consider the following linear boundary value problem of fifth-order (see [6, 7, 31–34])

$$u^{(5)}(x) - u(x) = f(x), \tag{6.1}$$

with boundary conditions

$$u(0) = 0, \qquad u^{(1)}(0) = 1, \qquad u^{(2)}(0) = 0, \qquad u(1) = 0, \qquad u^{(1)}(1) = -e, \tag{6.2}$$

where f is selected such that exact solution is

$$u(x) = x(1-x)e^{x}. \tag{6.3}$$

Table 1 shows the absolute errors obtained by using the shifted Jacobi tau (SJT) method, the variational iteration method using He's polynomials (VIMHP) [34], homotopy perturbation method (HPM) [31], variational iteration method (VIM) [32], decomposition

Table 1: Absolute errors using SJT method for $N = 14$.

x	α	β	SJT method	VIMHP	B-spline	VIM	ADM	ITM	HPM
	$-\frac{1}{2}$	$-\frac{1}{2}$	$3.1 \cdot 10^{-15}$						
0.0	0	0	$2.0 \cdot 10^{-16}$	0.000	0.000	0.000	0.000	0.000	0.000
	$\frac{1}{2}$	$\frac{1}{2}$	$9.1 \cdot 10^{-15}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	$1.9 \cdot 10^{-15}$						
0.2	0	0	$2.5 \cdot 10^{-16}$	$2.0 \cdot 10^{-10}$	$1.2 \cdot 10^{-3}$	$2.0 \cdot 10^{-10}$	$2.0 \cdot 10^{-10}$	$2.0 \cdot 10^{-10}$	$2.0 \cdot 10^{-10}$
	$\frac{1}{2}$	$\frac{1}{2}$	$8.2 \cdot 10^{-15}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	$1.7 \cdot 10^{-16}$						
0.4	0	0	$1.6 \cdot 10^{-15}$	$8.0 \cdot 10^{-10}$	$3.0 \cdot 10^{-3}$	$8.0 \cdot 10^{-10}$	$8.0 \cdot 10^{-10}$	$8.0 \cdot 10^{-10}$	$8.0 \cdot 10^{-10}$
	$\frac{1}{2}$	$\frac{1}{2}$	$5.4 \cdot 10^{-15}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	$3.5 \cdot 10^{-16}$						
0.6	0	0	$2.9 \cdot 10^{-15}$	$2.0 \cdot 10^{-9}$	$6.0 \cdot 10^{-3}$	$2.0 \cdot 10^{-9}$	$2.0 \cdot 10^{-9}$	$2.0 \cdot 10^{-9}$	$2.0 \cdot 10^{-9}$
	$\frac{1}{2}$	$\frac{1}{2}$	$1.3 \cdot 10^{-15}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	$1.3 \cdot 10^{-15}$						
0.8	0	0	$4.3 \cdot 10^{-15}$	$1.9 \cdot 10^{-9}$	$9.0 \cdot 10^{-3}$	$1.9 \cdot 10^{-9}$	$1.9 \cdot 10^{-9}$	$1.9 \cdot 10^{-9}$	$1.9 \cdot 10^{-9}$
	$\frac{1}{2}$	$\frac{1}{2}$	$2.7 \cdot 10^{-15}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	$3.5 \cdot 10^{-15}$						
1.0	0	0	$4.8 \cdot 10^{-15}$	0.000	0.000	0.000	0.000	0.000	0.000
	$\frac{1}{2}$	$\frac{1}{2}$	$5.0 \cdot 10^{-15}$						

method (ADM) [7], the sixth-degree B-spline method [6], and iterative method (ITM) [33] for $x = 0.0\ (0.2)\ 1.0$. The numerical results show that SJT method is more accurate than the existing methods for all $\alpha,\ \beta > -1$.

Example 6.2. Consider the following nonlinear boundary value problem of fifth-order (see [6, 7, 31–34])

$$u^{(5)}(x) = e^{-x} y^2(x), \tag{6.4}$$

with boundary conditions

$$u(0) = 0 = u^{(1)}(0) = u^{(2)}(0) = 1; \qquad u(1) = u^{(1)}(1) = e. \tag{6.5}$$

Table 2: Absolute errors using SJC method for $N = 14$.

x	α	β	SJC method	VIMHP	B-spline	VIM	ADM	ITM	HPM
	$-\frac{1}{2}$	$-\frac{1}{2}$	$7.0 \cdot 10^{-17}$						
0.0	0	0	$6.2 \cdot 10^{-17}$	0.000	0.000	0.000	0.000	0.000	0.000
	$\frac{1}{2}$	$\frac{1}{2}$	$5.5 \cdot 10^{-17}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	$2.2 \cdot 10^{-16}$						
0.2	0	0	$2.2 \cdot 10^{-16}$	$2.0 \cdot 10^{-9}$	$7.2 \cdot 10^{-4}$	$2.0 \cdot 10^{-9}$	$2.0 \cdot 10^{-9}$	$2.0 \cdot 10^{-9}$	$2.0 \cdot 10^{-9}$
	$\frac{1}{2}$	$\frac{1}{2}$	$2.2 \cdot 10^{-16}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	0						
0.4	0	0	$2.2 \cdot 10^{-16}$	$2.0 \cdot 10^{-8}$	$4.6 \cdot 10^{-4}$	$2.0 \cdot 10^{-8}$	$2.0 \cdot 10^{-8}$	$2.0 \cdot 10^{-8}$	$2.0 \cdot 10^{-8}$
	$\frac{1}{2}$	$\frac{1}{2}$	$4.4 \cdot 10^{-16}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	$2.2 \cdot 10^{-16}$						
0.6	0	0	$4.4 \cdot 10^{-16}$	$3.7 \cdot 10^{-8}$	$4.8 \cdot 10^{-4}$	$3.7 \cdot 10^{-8}$	$3.7 \cdot 10^{-8}$	$3.7 \cdot 10^{-8}$	$3.7 \cdot 10^{-8}$
	$\frac{1}{2}$	$\frac{1}{2}$	$2.2 \cdot 10^{-16}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	$4.4 \cdot 10^{-16}$						
0.8	0	0	$4.4 \cdot 10^{-15}$	$3.1 \cdot 10^{-8}$	$3.1 \cdot 10^{-4}$	$3.1 \cdot 10^{-8}$	$3.1 \cdot 10^{-8}$	$3.1 \cdot 10^{-8}$	$3.1 \cdot 10^{-8}$
	$\frac{1}{2}$	$\frac{1}{2}$	$4.4 \cdot 10^{-16}$						
	$-\frac{1}{2}$	$-\frac{1}{2}$	$8.8 \cdot 10^{-16}$						
1.0	0	0	0	0.000	0.000	0.000	0.000	0.000	0.000
	$\frac{1}{2}$	$\frac{1}{2}$	0						

The analytic solution for this problem is

$$u(x) = e^{x}. \tag{6.6}$$

In Table 2, we list the results obtained by using the shifted Jacobi collocation (SJC) method with three choices of α, β and one choice of N ($N = 14$), and we compare our results with variational iteration method using He's polynomials (VIMHP) [34], homotopy perturbation method (HPM) [31], variational iteration method (VIM) [32], decomposition method (ADM) [7], the sixth degree B-spline method [6] and iterative method (ITM) [33], respectively at $x = 0.0$ (0.2) 1.0. As we see from this Table, it is clear that the results obtained by the present method are superior to those obtained by the numerical methods given in [6, 7, 31–34].

7. Conclusion

In this paper, we have presented some efficient direct solvers for the general fifth-order BVPs by using Jacobi-tau approximation. Moreover, we developed a new approach implementing shifted Jacobi tau method in combination with the shifted Jacobi collocation technique for the numerical solution of fifth-order BVPs with variable coefficients. Furthermore, we proposed a numerical algorithm to solve the general nonlinear fifth-order differential equations by using Gauss-collocation points and approximating directly the solution using the shifted Jacobi polynomials. The numerical results in this paper demonstrate the high accuracy of these algorithms.

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