Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 759175, 13 pages
doi:10.1155/2011/759175

Research Article

A Study on Becker's Univalence Criteria

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Received 26 January 2011; Accepted 11 May 2011

Academic Editor: Allan C. Peterson

We study univalence properties for certain subclasses of univalent functions $\mathfrak{K}$, $\mathfrak{K}_2$, $\mathfrak{K}_{2,\mu}$, and $S(p)$, respectively. These subclasses are associated with a generalized integral operator. The extended Becker-typed univalence criteria will be studied for these subclasses.

1. Introduction and Preliminaries

Let A denote the class of analytic functions f in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Thus, each $f \in A$ has a Taylor series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Let A_2 be the subclass of A consisting of functions of the form

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k. \tag{1.2}$$

Let $\mathfrak{K}$ be the univalent subclass of A which satisfies

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1, \quad z \in \mathbb{U}. \tag{1.3}$$

Let $\mathfrak{K}_2$ be the subclass of $\mathfrak{K}$ for which $f''(0) = 0$. Let $\mathfrak{K}_{2,\mu}$ be the subclass of $\mathfrak{K}_2$ consisting of functions of the form (1.2) which satisfy

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < \mu, \quad 0 < \mu \leq 1, \ z \in \mathbb{U}. \tag{1.4}$$

Next, we define a subclass $S(p)$ of A consisting of all functions $f(z)$ that satisfy

$$\left| \left(\frac{z}{(f(z))} \right)'' \right| \leq p, \quad 0 < p \leq 2, \ p \in \Re, \ z \in \mathbb{U}. \tag{1.5}$$

For functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) $f * g$ is defined as usual by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \cdot \tag{1.6}$$

Define the function $\varphi(a, c; z)$ by

$$\varphi(a, c; z) = z_2F(1, a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k-1}, \quad c \neq 0, -1-, 2, \ldots, \tag{1.7}$$

where $(a)_k k$ is the famous Pochhammer symbol defined in terms of Gamma function. It is easily seen that $\varphi(2-\alpha, 2; z)$ is a convex function, since $z\varphi'(z) = \varphi(2-\alpha, 1; z) \in \mathrm{SV}^\star(\alpha)\cdot$

Using the fractional derivative of order α, D_z^α [1], Owa and Srivastava [2] introduced the operator $\Omega^\alpha : A \to A$ which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned} \Omega^\alpha f(z) &= \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z), \quad \alpha \neq 2, 3, 4, \ldots \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k \\ &= \varphi(2, 2-\alpha; z) * f(z) \cdot \end{aligned} \tag{1.8}$$

Note that $\Omega^0 f(z) = f(z) \cdot$

For a function f in A, we define $D_\lambda^{n,\nu}(\alpha, \beta, \mu) f(z) : A \to A$, the linear fractional differential operator, as follows:

$$I_\lambda^{0,\nu}(\alpha, \beta, \mu) f(z) = f(z),$$
$$I_\lambda^{1,\nu}(\alpha, \beta, \mu) f(z) = \left(\frac{\nu - \mu + \beta - \lambda}{\nu + \beta} \right) (\Omega^\alpha f(z)) + \left(\frac{\mu + \lambda}{\nu + \beta} \right) z (\Omega^\alpha f(z)),$$

$$
\begin{aligned}
I_{\lambda}^{2,\nu}(\alpha,\beta,\mu)f(z) &= I_{\lambda}^{\alpha}\left(I_{\lambda}^{1,\nu}(\alpha,\beta,\mu)f(z)\right)\\
&\vdots\\
I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z) &= I_{\lambda}^{\alpha}\left(I_{\lambda}^{n-1,\nu}(\alpha,\beta,\mu)f(z)\right).
\end{aligned}
\tag{1.9}
$$

If f is given by (1.1), then by (1.8) and (1), we see that

$$
I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z) = z + \sum_{k=2}^{\infty}\left(\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right)\right)^{n} a_k z^k. \tag{1.10}
$$

From (1.8) and (1), $D_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z)$ can be written in terms of convolution as

$$
I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z) = \underbrace{\left[\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)\cdots\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)\right]} * f(z), \tag{1.11}
$$

where

$$
\begin{aligned}
g_{\beta,\lambda}^{\mu,\nu}(z) &= \frac{z-((\nu-\mu+\beta-\lambda)/(\nu+\beta))z^2}{(1-z)^2}\\
&= z-\left(\frac{\nu-\mu+\beta-\lambda}{\nu+\beta}\right)z^2\left(1+2z+3z^2+\cdots\right)\\
&= z+\left(1+\frac{\mu+\lambda}{\nu+\beta}\right)z^2+\left(1+2\frac{\mu+\lambda}{\nu+\beta}\right)z^3\cdots
\end{aligned}
\tag{1.12}
$$

$$
\begin{aligned}
&\vdots\\
g_{\beta,\lambda}^{\mu,\nu}(z) &= z+\sum_{k=2}^{\infty}\left(\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right)\right)z^k,\\
\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)\cdots\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z) &= n\text{-times product},
\end{aligned}
\tag{1.13}
$$

which generalizes many operators. Indeed, if we choose suitably values of α, β, μ, and ν in (1.12), we have the following.

(i) $\beta=1$, $\mu=0$, and $\alpha=0$, we obtain $D_{\alpha,\lambda}^{m}f(z)$ given by Aouf et al. [3].

(ii) $\nu=1$, $\beta=0$, $\mu=0$, and $\alpha=0$, we obtain $D_{\lambda}^{m}f(z)$ given by Al-Oboudi [4].

(iii) $\nu=1$, $\beta=0$, $\mu=0$, $\lambda=1$, and $\alpha=0$, we obtain $D^{m}f(z)$ given by Sălăgean [5].

(iv) $\nu=1$, $\beta=1$, $\lambda=1$, $\mu=0$, and $\alpha=0$, we obtain $I^{m}f(z)$ given by Uralegaddi and Somanatha [6].

(v) $\beta=1$, $\lambda=1$, $\mu=0$, and $\alpha=0$, we obtain $I^{m}(\ell)f(z)$ given by Cho and Srivastava [7] and Cho and Kim [8].

(vi) $\nu = 1, \beta = 0, \mu = 0, \lambda = 0$, and $n = 1$, we obtain Owa and Srivastava differential operator [2].

(vii) $\nu = 1, \beta = 0$, and $\mu = 0$, we obtain $D_\lambda^{n,\alpha} f(z)$ given by Al-Oboudi and Al-Amoudi [9, 10].

(viii) $\beta = l$, $\mu = 0$, and $\alpha = p$, we obtain $I_p^n(\lambda, l) f(z)$ given by Catas [11].

(ix) $\beta = l$, $\mu = 0$, $\alpha = p$, and $\lambda = 1$, we obtain $I_p^n(\lambda, l) f(z)$ given by Kumar et al. and Srivastava et al., respectively [12, 13].

Next, we introduce a new family of integral operator by using generalized differential operator already defined above.

For $m \in \mathbb{N} \cup \{0\}$ and $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n, \rho \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, we define a family of integral operators $\Upsilon_{\gamma_i,\eta,\lambda}(n, \rho, \nu, \alpha, \beta) : A^m \to A^m$ by

$$\Upsilon_{\gamma_i,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z) = \left\{ \rho \int_0^z t^{\rho-1} \prod_{i=1}^{m} \left(\frac{I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(t)}{t} \right)^{1/\gamma_i} dt \right\}^{1/\rho}, \quad f_i \in A, \tag{1.14}$$

which generalize many integral operators. In fact, if we choose suitable values of parameters in this type of operator, we get the following interesting operators.

(i) $\nu = 1, \beta = 0, \mu = 0, \alpha = 0, \gamma_i = 1/\alpha_i$, and $\rho = 1$, we obtain $I(f_1, \ldots, f_m)$ given by Bulut [14].

(ii) $n = 0, \nu = 1,\ \beta = 0,\ \mu = 0, \alpha = 0, \gamma_i = 1/(\alpha - 1)$, and $\rho = n(\alpha - 1) + 1$, we obtain $F_{n,\alpha}(z)$ given by Breaz et al. [15].

(iii) $n = 0, \nu = 1, \beta = 0, \mu = 0, \alpha = 0, \gamma_i = 1/\alpha_i$, and $\rho = 1$, we obtain $F_\alpha(z)$ given by D. Breaz and N. Breaz [16].

For our main result, we need the following lemmas.

Lemma 1.1 (see [17, 18]). *Let c be a complex number, $|c| \le 1$, $c \ne -1$. If $f(z) = z + a_2 z^2 + \cdots$ is a regular function in $\mathbb{U}$ and*

$$\left| c|z|^2 + \left(1 - |z|^2\right) \frac{z f''(z)}{f'(z)} \right| \le 1, \quad \forall z \in \mathbb{U}, \tag{1.15}$$

then the function f is regular and univalent in $\mathbb{U}$.

Lemma 1.2 (Schwarz Lemma). *Let the function $f(z)$ be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$. If $f(z)$ has one zero with multiply $\ge m$ for $z = 0$, then*

$$\left| f(z) \right| \le \frac{M}{R^m} |z|^m, \quad \forall z \in \mathbb{U}_R, \tag{1.16}$$

and equality holds only if $f(z) = e^{i\theta}(M/R^m)|z|^m$, where θ is constant.

Lemma 1.3 (see [19]). *Let δ be a complex number with* $\operatorname{Re}\delta > 0$ *such that* $c \in \mathbb{C}$, $|c| \le 1$, $c \ne -1$. *If* $f \in A$ *satisfies the condition*

$$\left| c|z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{zf''(z)}{\delta f'(z)} \right| \le 1, \quad \forall z \in \mathbb{U}, \tag{1.17}$$

then the function

$$F_\delta(z) = \left\{ \delta \int_0^z t^{\delta-1} f'(t)\, dt \right\}^{1/\delta} \tag{1.18}$$

is analytic and univalent in $\mathbb{U}$.

Lemma 1.4 (see [20]). *If a function* $f \in S(p)$, *then*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \le p|z|^2, \quad \forall z \in \mathbb{U}. \tag{1.19}$$

2. Univalence Properties

In this section, we will discuss the univalence properties of the new family of integral operators mentioned above.

Theorem 2.1. *Let c be a complex number,* $|I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z)| \le M_i$, $M_i \ge 1$ *for all* $i = \{1,2,3,\ldots\}$ *and* $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(t) \in S(p_i)$ *for* $i = \{1,2,3,\ldots\}$ *such that*

$$\Re(\rho) \ge \sum_{i=1}^{\infty} \frac{((M_i - 1)p_i + 2)M_i - 1}{|\gamma_i|(M_i 1)}, \tag{2.1}$$

where ρ, γ_i *are complex numbers. If*

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{\infty} \frac{((M_i - 1)p_i + 2)M_i - 1}{|\gamma_i|(M_i - 1)}, \quad M_i \ge 1, \tag{2.2}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ *is univalent.*

Proof. Since $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(t) \in S(p_i)$, so by Lemma 1.4, we have

$$\left| \frac{z^2 (I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(t))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(t))^2} - 1 \right| \le p_i |z|^2, \quad \forall z \in \mathbb{U}. \tag{2.3}$$

Now, by using hypothesis, we have

$$\left| I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z) \right| \le M_i, \tag{2.4}$$

so by Lemma 1.3, we get

$$\left|I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right| \le M_i z, \quad \because R=1. \tag{2.5}$$

Let

$$\begin{aligned}\frac{I_\lambda^{n,\nu}(\alpha,\beta,\mu)f(z)}{z} &= 1+\sum_{k=2}^{\infty}\left(\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right)\right)^n a_k z^{k-1} \neq 0\\ &= 1 \quad \text{if } z=0,\end{aligned} \tag{2.6}$$

so

$$\begin{aligned}\prod_{i=1}^{m}\left(\frac{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)}{z}\right)^{1/\gamma_i} &= \left(\frac{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(z)}{z}\right)^{1/\gamma_1}\cdots\left(\frac{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(z)}{z}\right)^{1/\gamma_m}\\ &= 1.\end{aligned} \tag{2.7}$$

Let

$$F(z)=\int_0^z\left(\left(\frac{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(t)}{t}\right)^{1/\gamma_1}\cdots\left(\frac{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(t)}{t}\right)^{1/\gamma_m}\right)dt, \tag{2.8}$$

which implies that

$$\begin{aligned}F'(z) &= \left(\left(\frac{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(t)}{t}\right)^{1/\gamma_1}\cdots\left(\frac{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(t)}{t}\right)^{1/\gamma_m}\right)',\\ \frac{zF''(z)}{F'(z)} &= \frac{1}{\gamma_1}\left(\frac{z\left(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(z)\right)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(z)}-1\right)+\cdots+\frac{1}{\gamma_m}\left(\frac{z\left(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(z)\right)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(z)}-1\right),\end{aligned} \tag{2.9}$$

$$\frac{zF''(z)}{F'(z)}=\sum_{i=1}^{m}\frac{1}{\gamma_i}\left(\frac{z\left(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)}-1\right). \tag{2.10}$$

This implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \le \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)'}{I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)}\right|+1\right), \tag{2.11}$$

or

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)'}{\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)^2}\right|\left|\frac{\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)}{(z)}\right|+1\right). \tag{2.12}$$

Using (2.5), we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)'}{\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)^2}\right|M_i+1\right). \tag{2.13}$$

This implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)'}{\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)^2}-1\right|M_i+M_i+1\right). \tag{2.14}$$

By using (2.3), we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_i|z|^2M_i+M_i+1\right), \tag{2.15}$$

which implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_iM_i+\left(M_i+M_i^2+M_i^3+\cdots\right)+1\right), \tag{2.16}$$

because $M_i, M_i^2, M_i^3, \ldots, \geq 1$ implies that

$$\begin{gathered}\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_iM_i+\left(\frac{M_i}{M_i-1}\right)+1\right)=\sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_iM_i+\left(\frac{2M_i-1}{M_i-1}\right)\right),\\ \left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{p_iM_i^2-p_iM_i+2M_i-1}{M_i-1}\right)\right) \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{(p_iM_i-p_i+2)M_i-1}{M_i-1}\right)\right),\\ \left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{((M_i-1)p_i+2)M_i-1}{M_i-1}\right)\right).\end{gathered} \tag{2.17}$$

Now, we calculate

$$\left|c|z|^{2\rho}+\left(1-|z|^{2\rho}\right)\frac{zF''(z)}{\rho F'(z)}\right| \leq |c|+\frac{1}{|\rho|}\left|\frac{zF''(z)}{F'(z)}\right| \leq |c|+\frac{1}{\Re(\rho)}\left|\frac{zF''(z)}{F'(z)}\right|. \tag{2.18}$$

This implies that

$$\left| c|z|^{2\rho} + \left(1 - |z|^{2\rho}\right) \frac{zF''(z)}{\rho F'(z)} \right| < |c| + \frac{1}{\Re(\rho)} \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} \left(\left(\frac{((M_i - 1)p_i + 2)M_i - 1}{M_i - 1} \right) \right) \cdot \tag{2.19}$$

By using (2.46), we conclude that

$$\left| c|z|^{2\rho} + \left(1 - |z|^{2\rho}\right) \frac{zF'(z)}{\rho F(z)} \right| \le |c| + \frac{1}{|\rho|} \left| \frac{zF''(z)}{F'(z)} \right| \le 1 \cdot \tag{2.20}$$

Hence, by Lemma 1.3, the family of integral operators $\Upsilon_{\gamma_i,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z)$ is univalent. □

Corollary 2.2. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z)| \le M$, $M \ge 1$ for all $i = \{1, 2, 3, \ldots\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(t) \in S(p)$, $M_i = M \ge 1$, for all $i = \{1, 2, 3, \ldots\}$ such that*

$$\Re(\rho) \ge \sum_{i=1}^{\infty} \frac{((M - 1)p + 2)M - 1}{|\gamma_i|(M - 1)}, \tag{2.21}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\mathrm{Re}(\rho)} \sum_{i=1}^{\infty} \frac{((M - 1)p_i + 2)M - 1}{|\gamma_i|(M - 1)}, \quad M \ge 1, \tag{2.22}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z)$ is univalent.

Corollary 2.3. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z)| \le M$, $M \ge 1$, for all $i = \{1, 2, 3, \ldots\}$ and the family $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(t) \in S(p)$, $M_i = M \ge 1$, $|\gamma_i| = |\gamma|$, for all $i = \{1, 2, 3, \ldots\}$ such that*

$$\Re(\rho) \ge \sum_{i=1}^{\infty} \frac{((M - 1)p + 2)M - 1}{|\gamma|(M - 1)}, \tag{2.23}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\mathrm{Re}(\rho)} \sum_{i=1}^{\infty} \frac{((M - 1)p_i + 2)M - 1}{|\gamma|(M - 1)}, \quad M \ge 1, \tag{2.24}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z)$ is univalent.

Using the method given in the proof of Theorem 2.1, one can prove the following results.

Theorem 2.4. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots\}$ and the family $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in S(p_i)$ for $i = \{1,2,3,\ldots\}$ and c such that*

$$\Re(\rho) \geq \sum_{i=1}^{\infty} \frac{(p_i M_i - 1)M_i + 1}{|\gamma_i| p_i M_i}, \tag{2.25}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{\infty} \frac{(p_i M_i - 1)M_i + 1}{|\gamma_i|(p_i M_i)}, \quad M_i \geq 1, \tag{2.26}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Theorem 2.5. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in S(p_i)$ for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(p_i(M_i - 1) + M_i^n - 2)M_i + 1}{|\gamma_i|(M_i - 1)}, \tag{2.27}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(p_i(M_i - 1) + M_i^n - 2)M_i + 1}{|\gamma_i|(M_i - 1)}, \quad M_i \geq 1, \tag{2.28}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Theorem 2.6. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in S(p_i)$ for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(p_i + (n(n+1)/2))M_i - 1}{|\gamma_i|}, \tag{2.29}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(p_i + (n(n+1)/2))M_i - 1}{|\gamma_i|}, \quad M_i \geq 1, \tag{2.30}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Theorem 2.7. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M_i}{|\gamma_i|}, \tag{2.31}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M_i}{|\gamma_i|}, \quad M_i \geq 1, \tag{2.32}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Proof. Using the proof of Theorem 2.1, we have

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|}\left(\left|\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))^2}\right|M_i + 1\right). \tag{2.33}$$

Since $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in \mathfrak{K}_{2,\mu_i}$, so by using (1.4), we get

$$\left|\frac{z^2(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))^2} - 1\right| < \mu_i, \quad 0 < \mu \leq 1, \ z \in \mathbb{U}. \tag{2.34}$$

So from (2.33), we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|}\left(\left|\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))^2} - 1\right|M_i + M_i + 1\right), \tag{2.35}$$

or

$$\begin{gathered}\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|}(\mu_i M_i + 2M_i), \quad M_i > 1, \\ \left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|}(\mu_i M_i + 2M_i + 4M_i + \cdots + n\text{-times}), \quad M_i > 1, \\ \left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|}(\mu_i M_i + n(n+1)M_i), \quad M_i > 1.\end{gathered} \tag{2.36}$$

Now, we evaluate the expression

$$\begin{gathered}\left|c|z|^{2\rho} + \left(1 - |z|^{2\rho}\right)\frac{zF''(z)}{\rho F'(z)}\right| \leq |c| + \frac{1}{|\rho|}\left|\frac{zF''(z)}{F'(z)}\right| \leq |c| + \frac{1}{\Re(\rho)}\left|\frac{zF''(z)}{F'(z)}\right|, \\ \left|c|z|^{2\rho} + \left(1 - |z|^{2\rho}\right)\frac{zF''(z)}{\rho F'(z)}\right| \leq |c| + \frac{1}{\Re(\rho)}\sum_{i=1}^{\infty} \frac{1}{|\gamma_i|}(\mu_i M_i + n(n+1)M_i).\end{gathered} \tag{2.37}$$

Using (2.45) and (2.46), we conclude that

$$\left| c|z|^{2\rho} + \left(1 - |z|^{2\rho}\right) \frac{zF'(z)}{\rho F(z)} \right| \leq 1. \tag{2.38}$$

Hence by using Lemma 1.3, the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent. □

Corollary 2.8. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M$, $M \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M}{|\gamma_i|}, \tag{2.39}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M}{|\gamma_i|}, \quad M \geq 1, \tag{2.40}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Corollary 2.9. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M$, $M \geq 1$, $|\gamma_i| = |\gamma|$ for all $i = \{1,2,3,\ldots,n\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M}{|\gamma|}, \tag{2.41}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M}{|\gamma|}, \quad M \geq 1, \tag{2.42}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Using a similar method as in the proof of Theorem 2.7, one can prove the following results.

Theorem 2.10. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i M_i - 1)M_i + M_i^n M_i}{|\gamma_i|(M_i - 1)}, \tag{2.43}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i M_i - 1) M_i + M_i^n M_i}{|\gamma_i| (M_i - 1)}, \quad M_i \geq 1, \tag{2.44}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z)$ is univalent.

Theorem 2.11. *Let c be a complex number, $|I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1, 2, 3, \ldots\}$ and $I_\lambda^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1, 2, 3, \ldots\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i M_i - \mu_i + 2) M_i - 1}{|\gamma_i| (M_i - 1)}, \tag{2.45}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i M_i - \mu_i + 2) M_i - 1}{|\gamma_i| (M_i - 1)}, \quad M_i \geq 1, \tag{2.46}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z)$ is univalent.

Note that some other related work involving integral operators regarding univalence criteria can also be found in [21–23].

Acknowledgment

The work presented here was partially supported by UKM-ST-06-FRGS0244-2010.

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