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Research Article

Dynamic Analysis of a Nonlinear Timoshenko Equation

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We characterize the global and nonglobal solutions of the Timoshenko equation in a bounded domain. We consider nonlinear dissipation and a nonlinear source term. We prove blowup of solutions as well as convergence to the zero and nonzero equilibria, and we give rates of decay to the zero equilibrium. In particular, we prove instability of the ground state. We show existence of global solutions without a uniform bound in time for the equation with nonlinear damping. We define and use a potential well and positive invariant sets.

1. Introduction

We consider

$$u_{tt} + k\Delta^2 u - M\left(\|\nabla u\|_2^2\right)\Delta u + g(u_t) = f(u) \quad \text{in } \Omega, \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0, \quad u_t(x,0) = v_0, \quad x \in \Omega, \tag{1.2}$$

and with one set of the following boundary conditions:

$$u = 0, \quad \Delta u = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

or

$$u = 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, $\|\cdot\|_2$ is the norm in $L^2(\Omega)$,

$$M\left(s^2\right) = \alpha + \beta s^{2\gamma}, \quad \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta > 0,\ \gamma \geq 1,\ k = 1, \tag{1.5}$$

$$g(u_t) = \delta u_t |u_t|^{\lambda-2}, \quad \delta > 0,\ \lambda \geq 2, \tag{1.6}$$

$$f(u) = \mu u |u|^{r-2}, \quad \mu > 0,\ r > 0. \tag{1.7}$$

When the source term $f \equiv 0$, there is a considerable set of works studying several properties of equation (1.1), see for instance, the early papers by Ball [1, 2], Haraux and Zuazua [3], and the books by Hale [4], Haraux [5], and references therein. For a destabilizing source term, $sf(s) > 0,\ s \in \mathbb{R} \setminus \{0\}$, in the works of Payne and Sattinger [6], Georgiev and Todorova [7], and Ikehata [8], qualitative properties of (1.1) are studied, when $k = 0 = \beta$. To understand the dynamics of second-order equations in time, similar to (1.1), active research is reported in Alves and Cavalcanti [9], Barbu et al. [10], Cavalcanti et al. [11–16], Rammaha [17] Rammaha and Sakuntasathien [18], and Todorova and Vitillaro [19], Vitillaro [20]. For the Timoshenko equation, with $g \equiv 0$, Bainov and Minchev [21] gave sufficient conditions for the nonexistence of smooth solutions of (1.1), with negative initial energy, and gave an upper bound of the maximal time of existence. For positive and sufficiently small initial energy, blowup and globality properties are characterized in Esquivel-Avila [22]. For the Kirchhoff equation, that is, (1.1) with $k = 0$, the nonexistence of global solutions is studied in [23]. In [24, 25], we characterized properties such as blowup and asymptotic behavior of solutions, for (1.1) with $k = 0$ and $\beta = 0$. To the knowledge of the author, such problems are still open for the Timoshenko equation (1.1). Here, we want to give some results about the dynamics of problem (1.1). To do that we will generalize the concept of the depth of the potential well in such manner that our results of the dynamics be as sharp as the ones in [24, 25]. Furthermore, for particular cases, our definition of depth of the potential well will coincide with the one introduced in [6].

2. Preliminaries and Framework

We begin this section with an existence, uniqueness, and continuation theorem for (1.1). The proof is similar to the ones in [7, 8], where semilinear wave equations are studied.

Theorem 2.1. *Assume that $r > 2$ and $r \leq 2(n-2)/(n-4)$ if $n \geq 5$. For every initial data $(u_0, v_0) \in H \equiv B \times L^2(\Omega)$, where B is defined either by $B \equiv H^2(\Omega) \cap H_0^1(\Omega)$, or $B \equiv H_0^2(\Omega)$, there exists a unique (local) weak solution $(u(t), v(t)) \equiv S(t)(u_0, v_0)$ of problem* (1.1), *that is,*

$$\frac{d}{dt}(v(t), w)_2 + (\Delta u(t), \Delta w)_2 + M\left(\|\nabla u(t)\|_2^2\right)(\nabla u(t), \nabla w)_2 + \left(g(v(t)), w\right)_2 = \left(f(u(t)), w\right)_2, \tag{2.1}$$

a.e. in $(0, T)$ and for every $w \in B \cap L^\lambda(\Omega)$, such that

$$u \in C([0,T]; B) \cap C^1\left([0,T); L^2(\Omega)\right), \qquad v \equiv u_t \in L^\lambda((0,T) \times \Omega). \tag{2.2}$$

Here, $S(t)$ denotes the corresponding semigroup on H, generated by problem (1.1), *and* $(\cdot,\cdot)_2$ *is the inner product in* $L^2(\Omega)$.

The following energy equation holds:

$$E_0 = E(t) + \int_0^t \delta \|v(\tau)\|_\lambda^\lambda d\tau, \tag{2.3}$$

where

$$E(t) \equiv E(u(t), v(t)) \equiv \frac{1}{2}\|v(t)\|_2^2 + J(u(t)), \tag{2.4}$$

$$J(u) \equiv \frac{1}{2}a(u) + \frac{1}{2(\gamma+1)}c(u) - \frac{1}{r}b(u), \tag{2.5}$$

with

$$a(u) \equiv \|u\|_B^2, \qquad b(u) \equiv \mu\|u\|_r^r, \qquad c(u) \equiv \beta\|\nabla u\|_2^{2(\gamma+1)}. \tag{2.6}$$

Here, $E_0 \equiv E(u_0, v_0)$ *is the initial energy, and* $\|\cdot\|_q$ *denotes the norm in the* $L^q(\Omega)$ *space.*

If the maximal time of existence $T_M < \infty$, *then* $S(t)(u_0, v_0) \to \infty$ *as* $t \nearrow T_M$, *in the norm of* H:

$$\|(u,v)\|_H^2 \equiv \|u\|_B^2 + \|v\|_2^2 \equiv \|\Delta u\|_2^2 + \alpha\|\nabla u\|_2^2 + \|v\|_2^2. \tag{2.7}$$

In that case, from (2.3)–(2.6), $\|u(t)\|_r \to \infty$ *as* $t \nearrow T_M$.

Now, we define, respectively, the stable (potential well) and unstable sets:

$$\begin{gathered} W \equiv ([I(u) > 0] \cup \{0\}) \cap [J(u) < d], \\ V \equiv [I(u) < 0] \cap [J(u) < d], \end{gathered} \tag{2.8}$$

where

$$I(u) \equiv a(u) + c(u) - b(u). \tag{2.9}$$

Here, $[I(u) < 0]$ denotes the set of $u \in B$ with that property, and the depth of the potential well is defined as follows:

$$d \equiv \frac{r-2}{2r} S^{r/(r-2)}, \tag{2.10}$$

where

$$\sqrt{S} \equiv \inf_{\substack{0 \neq u \in B \\ \widehat{b}(u) > 0}} \left(\frac{\widehat{a}(u)^{1/2}}{\widehat{b}(u)^{1/r}} \right), \tag{2.11}$$

$$\widehat{a}(u) \equiv a(u) + \kappa_1 c(u), \qquad \widehat{b}(u) \equiv b(u) + \kappa_2 c(u). \tag{2.12}$$

with

$$\kappa_1 \equiv \frac{r - 2(\gamma + 1)}{(r-2)(\gamma+1)}, \qquad \kappa_2 \equiv \kappa_1 - 1 = \frac{-r\gamma}{(r-2)(\gamma+1)}. \tag{2.13}$$

We assume that $r \geq 2(\gamma + 1)$, and since $\gamma \geq 1$, then $\kappa_1 \in [0, 1/2)$, and $\kappa_2 \in [-1, -1/2)$. Also note that if $r = 2(\gamma+1)$, then $\kappa_1 = 0$, $\kappa_2 = -1$, and we have the following characterization of the depth of the potential well (2.10)-(2.11):

$$d = \inf_{0 \neq u \in B} \sup_{\lambda \geq 0} J(\lambda u), \tag{2.14}$$

which is the definition given in [6], where a nondissipative nonlinear wave equation is studied.

Consider any $u \in B$, $r > 2$, and $r \leq 2n/(n-4)$ if $n \geq 5$, then

$$\widehat{a}(u) \geq C(\Omega) b(u)^{2/r} + \kappa_1 c(u) \geq C(\Omega) b(u)^{2/r}, \tag{2.15}$$

where $C(\Omega) > 0$, is any constant in the Sobolev-Poincaré's inequality

$$\left(\|\Delta u\|_2^2 + \alpha \|\nabla u\|_2^2 \right)^{1/2} \geq \sqrt{C(\Omega)} \mu^{1/r} \|u\|_r. \tag{2.16}$$

Moreover, if $\widehat{b}(u) > 0$, from (2.15) and since $b(u) \geq \widehat{b}(u)$,

$$\widehat{a}(u) \geq C(\Omega) \widehat{b}(u)^{2/r}. \tag{2.17}$$

Hence, $S \geq C(\Omega)$, and $d \geq D \equiv ((r-2)/2r) C(\Omega)^{r/(r-2)} > 0$.

If u_e denotes any nonzero equilibria of equation (1.1),

$$\mathcal{E} \equiv \left[0 \neq u_e \in B : \Delta^2 u_e - M\left(\|u\|_2^2 \right) \Delta u_e = f(u_e) \right], \tag{2.18}$$

then, by (2.1) in Theorem 2.1 with $u(t) = u_e = w$, we get that u_e belongs to the Nehari manifold, $\mathcal{N}$, that is,

$$\mathcal{E} \subset \mathcal{N} \equiv \left[0 \neq u \in B : \widehat{I}(u) = I(u) = 0 \right], \tag{2.19}$$

where $\widehat{I}(u) \equiv \widehat{a}(u) - \widehat{b}(u)$.

Consequently, $\widehat{b}(u_e) = \widehat{a}(u_e) > 0$. Furthermore, from (2.17) which is an equality when $C(\Omega) = S$, we conclude that the Nehari manifold can be represented by the line: $y = x$, in the plane with axes $x = \widehat{b}(u)$ and $y = \widehat{a}(u)$, beginning at the point: $y = x = S^{r/(r-2)} = (2r/(r-2))d$. We also note that

$$J(u) = \frac{1}{2}\widehat{a}(u) - \frac{1}{r}\widehat{b}(u). \tag{2.20}$$

From these facts it follows that the depth of the potential well (2.10) is characterized by

$$d = \inf_{u \in \mathcal{N}} J(u) = \frac{r-2}{2r}\varrho, \tag{2.21}$$

where

$$0 < \varrho \equiv \inf_{u \in \mathcal{N}} \widehat{a}(u) = \inf_{u \in \mathcal{N}} \widehat{b}(u). \tag{2.22}$$

Hence, any equilibrium is such that $u_e \in [J(u) \geq d]$. Moreover, like in [6], the set of extremals of (2.21) is characterized by set of equilibria with least energy, that is the ground state

$$\mathcal{N}^* \equiv [u_e \in \mathcal{E} : J(u_e) = d] = \left[u_e \in \mathcal{E} : \widehat{a}(u_e) = \widehat{b}(u_e) = \varrho\right]. \tag{2.23}$$

Observe that $J(u) = d$ is a tangent line to the curve defined by the equality in (2.17) with $C(\Omega) = S$, at the point $\mathcal{N}^*$, which holds if $\widehat{b}(u) > 0$. On the other hand, we notice that

$$\kappa_1 \widehat{b}(u) - \kappa_2 \widehat{a}(u) = \kappa_1 b(u) - \kappa_2 a(u) > 0, \tag{2.24}$$

and is equal to zero if and only if $a(u) = 0 = b(u)$. Hence, if $\widehat{b}(u) < 0$, then

$$\widehat{a}(u) > \frac{r - 2(\gamma + 1)}{-r\gamma}\widehat{b}(u). \tag{2.25}$$

Therefore, next results about the stable and unstable sets follow.

Lemma 2.2. *The following properties of V and W hold:*

(i) *W is a neighborhood of $0 \in B$.*

(ii) $0 \notin \overline{[I(u) < 0]}$ *(closure in B), in particular* $0 \notin \overline{V}$.

(iii) $W = W_+ \cup W_- \cup \{0\}$, *where*

$$W_+ \equiv W \cap \left[\widehat{b}(u) > 0\right] = \left[\varrho^{(r-2)/r}\widehat{b}(u)^{2/r} \le \widehat{a}(u) < \frac{2}{r}\widehat{b}(u) + \frac{r-2}{r}\varrho,\ 0 < \widehat{b}(u) < \varrho\right],$$
$$W_- \equiv W \cap \left[\widehat{b}(u) < 0\right] \subset \left[\frac{r-2(\gamma+1)}{-r\gamma}\widehat{b}(u) < \widehat{a}(u) < \frac{2}{r}\widehat{b}(u) + \frac{r-2}{r}\varrho,\ -\gamma\varrho < \widehat{b}(u) < 0\right]. \tag{2.26}$$

(iv) $V = \left[\varrho^{(r-2)/r}\widehat{b}(u)^{2/r} \le \widehat{a}(u) < \frac{2}{r}\widehat{b}(u) + \frac{r-2}{r}\varrho,\ \widehat{b}(u) > \varrho\right]$.

(v) $\mathcal{N}^* = \overline{W} \cap \overline{V} = \overline{W_+} \cap \overline{V} = [u_e \in \mathcal{N},\ \widehat{a}(u_e) = \widehat{b}(u_e) = \varrho]$.

(vi) $W = [I(u) < 0]^c \cap [J(u) < d]$, $V = ([I(u) > 0] \cup \{0\})^c \cap [J(u) < d]$.

The following result follows easily like in [23].

Lemma 2.3. *One has that*

$$J(u) > \frac{r-2}{2r}\widehat{a}(u) > \frac{r-2}{2r}\widehat{b}(u), \tag{2.27}$$

$$\begin{aligned} J(u) &> \frac{r-2}{2r}a(u) + \frac{r-2(\gamma+1)}{2r(\gamma+1)}c(u) \\ &> \frac{\gamma}{2(\gamma+1)}a(u) + \frac{r-2(\gamma+1)}{2r(\gamma+1)}b(u), \end{aligned} \tag{2.28}$$

for any $u \in B$ such that $I(u) > 0$, in particular if $0 \ne u \in W$, and

$$d < \frac{r-2}{2r}\widehat{a}(u) < \frac{r-2}{2r}\widehat{b}(u), \tag{2.29}$$

$$\begin{aligned} d &< \frac{r-2}{2r}a(u) + \frac{r-2(\gamma+1)}{2r(\gamma+1)}c(u) \\ &< \frac{\gamma}{2(\gamma+1)}a(u) + \frac{r-2(\gamma+1)}{2r(\gamma+1)}b(u), \end{aligned} \tag{2.30}$$

for any $u \in B$, such that $I(u) < 0$, in particular if $u \in V$.

A set $\mho \subset H$ is positive invariant, with respect to problem (1.1), if the corresponding generated semigroup $S(t)$ on H is such that

$$S(t)\mho \subset \mho. \tag{2.31}$$

Lemma 2.4. *Let (u, v) denote any solution of (1.1), given by Theorem 2.1. Then, the sets*

$$\mathcal{S} \equiv [E(u,v) < d] \cap [(u,v) \in H : u \in W], \tag{2.32}$$

$$\mathcal{U} \equiv [E(u,v) < d] \cap [(u,v) \in H : u \in V], \tag{2.33}$$

are positive invariant.

Proof. First, we show that $\mathcal{S}$ is positive invariant. In order to do that, we take $(u_0, v_0) \in \mathcal{S}$. Then, by (2.4), $J(u(t)) \leq E(u(t), v(t)) \leq E_0 < d$, for any $t \geq 0$. Now, if $\mathcal{S}$ is not positive invariant, there exists some $\hat{t} > 0$, such that $I(u(\hat{t})) = 0$, with $u(\hat{t}) \neq 0$. Then, by (2.21), $d \leq J(u(\hat{t}))$. But this is impossible because $J(u(\hat{t})) < d$. The proof of the positive invariance of $\mathcal{U}$ is quite similar. Indeed, if this is not true there exists some $\hat{t} > 0$, such that $I(u(\hat{t})) = 0$. From (ii) of Lemma 2.2 $u(\hat{t}) \neq 0$, and this implies the same contradiction as before. □

Next result gives an interpretation of sets $\mathcal{S}$ and $\mathcal{U}$ and follows from Lemma 2.2.

Lemma 2.5. *The sets $\mathcal{S}$ and $\mathcal{U}$ have the properties*

$$\mathcal{S} \subset [E(u,v) < d] \cap \left[(u,v) \in H : \varrho^{(r-2)/r} \hat{b}(u)^{2/r} \leq \hat{a}(u) < \frac{2}{r}\hat{b}(u) + \frac{r-2}{r}\varrho,\ 0 < \hat{b}(u) < \varrho\right],$$
$$\cup [E(u,v) < d] \cap \left[(u,v) \in H : \frac{r-2(\gamma+1)}{-r\gamma}\hat{b}(u) < \hat{a}(u) < \frac{2}{r}\hat{b}(u) + \frac{r-2}{r}\varrho,\ -\gamma\varrho < \hat{b}(u) < 0\right],$$
$$\mathcal{U} = [E(u,v) < d] \cap \left[(u,v) \in H : \varrho^{(r-2)/r} \hat{b}(u)^{2/r} \leq \hat{a}(u) < \frac{2}{r}\hat{b}(u) + \frac{r-2}{r}\varrho,\ \hat{b}(u) > \varrho\right], \tag{2.34}$$

$$\begin{aligned} \overline{\mathcal{S}} \cap \overline{\mathcal{U}} &= [(u_e, 0) \in H : u_e \in \mathcal{N}^*] \\ &= \left[(u_e, 0) \in H : u_e \in \mathcal{N},\ \hat{a}(u_e) = \hat{b}(u_e) = \varrho\right]. \end{aligned} \tag{2.35}$$

The following result is a direct consequence of (vi) in Lemma 2.2 and Lemma 2.4.

Lemma 2.6. *For every solution of* (1.1), *only one of the following holds:*

(i) *there exists some $t_0 \geq 0$ such that $(u(t_0), v(t_0)) \in \mathcal{S}$, and remains there for every $t > t_0$,*

(ii) *there exists some $t_0 \geq 0$ such that $(u(t_0), v(t_0)) \in \mathcal{U}$, and remains there for every $t > t_0$,*

(iii) *$(u(t), v(t)) \in [E(u,v) \geq d]$ for every $t \geq 0$.*

Hence, we notice that the sets $\mathcal{S}$ and $\mathcal{U}$ play an important role in the dynamics of (1.1). Moreover, we will prove that any solution eventually contained in $\mathcal{S}$ converges to the zero equilibrium. If enters in $\mathcal{U}$, either blowups in a finite time or it is global but without a uniform bound in H for every $t \geq 0$, in the case that $\lambda > 2$, in (1.6). Also, we will prove that any solution with $(u(t), v(t)) \in [E(u,v) \geq d]$, for every $t \geq 0$, is bounded and converges to the set of nonzero equilibria $\mathcal{E}$.

We will need the following inequalities to show blowup and convergence to the zero equilibrium, respectively, in the dissipative case.

Lemma 2.7. *Let $\mathcal{F} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+)$ be a nonnegative function such that*

$$\dot{\mathcal{F}}(t) \geq C\mathcal{F}^a(t) \quad \text{a.e. for } t \geq 0, \tag{2.36}$$

with $a > 1$ and $C > 0$.

Then, there exists some $T^* > 0$ such that $\lim_{t \nearrow T^*} \mathcal{F}(t) = \infty$.

Proof. Define $\mathcal{G}(t) \equiv \mathcal{F}^{1-a}(t)$, then

$$\dot{\mathcal{G}}(t) \leq (1-a)C < 0 \quad \text{a.e. for } t \geq 0. \tag{2.37}$$

Hence, $0 < G(0) + (1-a)Ct$, which is only possible if $t < T^* \equiv (1/C(a-1))\mathcal{F}^{1-a}(0)$. □

Lemma 2.8. *Let $\mathcal{F} \in W^{1,1}_{\text{loc}}(\mathbb{R}^+)$ be a nonnegative function such that*

$$\dot{\mathcal{F}}(t) \leq -C\mathcal{F}^a(t) \quad \textit{a.e. for } t \geq 0, \tag{2.38}$$

with $a \geq 1$ and $C > 0$.

Then, for $t \geq 0$, if $a > 1$

$$\mathcal{F}(t) \leq \mathcal{F}_0\left\{1 + tC(a-1)\mathcal{F}_0^{a-1}\right\}^{-1/(a-1)}, \tag{2.39}$$

and, if $a = 1$

$$\mathcal{F}(t) \leq \mathcal{F}_0 e^{-Ct}. \tag{2.40}$$

Proof. Consider $a > 1$, and notice that $(\mathcal{F}^{(1-a)})\dot{}(t) \geq (a-1)C$. Then, we integrate and obtain the first inequality. Now, let $a \to 1$, and the second one follows. □

3. Timoshenko Equation

Due to our assumptions on r and γ, we restrict our analysis to dimensions $n \leq 5$. Indeed, since $\gamma \geq 1$, $2(\gamma+1) < r$ and $r \leq 2(n-2)/(n-4)$, if $n \geq 5$, then our analysis considers, $n = 5$ whenever $\gamma < 2$. We also notice that in any case we do not consider the interval $2 < r \leq 4$. Moreover, $r \leq 6$ whenever $n = 5$. We begin with a characterization of blowup when $\delta > 0$ and $\lambda \geq 2$.

Theorem 3.1. *Let $(u(t), v(t)) = S(t)(u_0, v_0)$ be a solution of problem* (1.1)*, and suppose that $r > 2(\gamma+1)$. A necessary and sufficient condition for nonglobality, blowup by Theorem 2.1, is that $\lambda < r$ and there exists $t_0 \geq 0$ such that $(u(t_0), v(t_0)) \in \mathcal{U}$.*

Proof. Sufficiency

By Lemma 2.4, $(u(t), v(t)) \in \mathcal{U}$ for all $t > t_0$.

Now, we consider the function defined, along the solution, by

$$\mathcal{V}(t) \equiv d - E(t), \tag{3.1}$$

and notice that because of energy equation (2.3),

$$\mathcal{V}(t) \geq d - E_0 \equiv \mathcal{V}_0 > 0, \tag{3.2}$$

where, now $E_0 \equiv E(u(t_0), v(t_0))$.

Notice that from (2.29) in Lemma 2.3,

$$\begin{aligned} \mathcal{V}(t) &\le d - J(u(t)) \\ &\le d - \frac{r}{r-2}d + \frac{1}{r}\widehat{b}(u(t)) \\ &= -\frac{2}{r-2}d + \frac{1}{r}b(u(t)) - \frac{\gamma}{(r-2)(\gamma+1)}c(u(t)). \end{aligned} \tag{3.3}$$

We will need some estimates. First, we notice that from energy equation in terms of $\mathcal{V}(t)$ and (3.3),

$$\begin{aligned} \left|\delta\left(u(t), v(t)|v(t)|^{\lambda-2}\right)_2\right| &\le \delta\|u(t)\|_\lambda \|v(t)\|_\lambda^{\lambda-1} \\ &\le C(\Omega)\delta\|u(t)\|_r \|v(t)\|_\lambda^{\lambda-1} \\ &\le C(\Omega)\delta\|u(t)\|_r^{1-k}\|u(t)\|_r^k \|v(t)\|_\lambda^{\lambda-1} \\ &\le C(\Omega)\delta\|u(t)\|_r^{1-k}\left[\nu\|u(t)\|_r^{k\lambda} + \frac{1}{C(\nu)}\|v(t)\|_\lambda^\lambda\right] \\ &< C\mathcal{V}^{(1-k)/r}(t)\left[\nu\delta\|u(t)\|_r^{k\lambda} + \frac{1}{C(\nu)}\dot{\mathcal{V}}(t)\right], \end{aligned} \tag{3.4}$$

where $k \in (1, r/\lambda)$, $C \equiv C(\Omega)(r/\mu)^{(1-k)/r}$, $C(\Omega) > 0$ is the constant in the continuous embedding $L^r(\Omega) \subset L^\lambda(\Omega)$, $C(\nu) > 0$, and $\nu > 0$ will be chosen later.

Consider a positive number q to be chosen later, from (3.2)-(3.3), we obtain

$$\begin{aligned} -I(u(t)) &= -qJ(u(t)) + \frac{q-2}{2}a(u(t)) + \frac{q-2(\gamma+1)}{2(\gamma+1)}c(u(t)) + \frac{r-q}{r}b(u(t)) \\ &\ge q(\mathcal{V}_0 - d) + \frac{q-2}{2}a(u(t)) + \frac{q-2(\gamma+1)}{2(\gamma+1)}c(u(t)) + \frac{r-q}{r}b(u(t)). \end{aligned} \tag{3.5}$$

If $\mathcal{V}_0 \ge d$, we choose $q \equiv 2(\gamma+1)$, and from (3.5) we get

$$-I(u(t)) \ge \frac{r-2(\gamma+1)}{r}b(u(t)). \tag{3.6}$$

If $\mathcal{V}_0 < d$, then we notice that from (3.2)-(3.3),

$$\mathcal{V}_0 - d \ge \frac{(r-2)(\mathcal{V}_0 - d)}{(r-2)\mathcal{V}_0 + 2d}\left(\frac{1}{r}b(u(t)) - \frac{\gamma}{(r-2)(\gamma+1)}c(u(t))\right). \tag{3.7}$$

Hence and from (3.5), we have the estimate

$$-I(u(t)) \geq \frac{q-2}{2} a(u(t)) + \frac{q}{2(\gamma+1)} \left(\frac{q-2(\gamma+1)}{q} + \frac{2\gamma(d-\mathcal{V}_0)}{(r-2)\mathcal{V}_0+2d} \right) c(u(t)) + \frac{q}{r} \left(\frac{r-q}{q} - \frac{(r-2)(d-\mathcal{V}_0)}{(r-2)\mathcal{V}_0+2d} \right) b(u(t)). \tag{3.8}$$

In this case, we choose the number q so that the coefficient of $c(u(t))$ in (3.8) be equal to zero, then

$$q \equiv \frac{2(\gamma+1)((r-2)\mathcal{V}_0+2d)}{(r-2(\gamma+1))\mathcal{V}_0+2(\gamma+1)d}. \tag{3.9}$$

We note that $2 < q < 2(\gamma+1)$, and we get

$$-I(u(t)) \geq \frac{\gamma r a(u(t)) + (r-2(\gamma+1)) b(u(t))}{(r-2(\gamma+1))\mathcal{V}_0+2(\gamma+1)d} \mathcal{V}_0. \tag{3.10}$$

Therefore, from (3.6) and (3.10),

$$-I(u(t)) \geq \widehat{C} b(u(t)), \tag{3.11}$$

where

$$\widehat{C} \equiv \frac{r-2(\gamma+1)}{r} \min\left(1, \frac{r\mathcal{V}_0}{(r-2(\gamma+1))\mathcal{V}_0+2(\gamma+1)d}\right) > 0. \tag{3.12}$$

Now, we define the function, along the solution, by

$$\mathcal{F}(t) \equiv \mathcal{V}^{1/a}(t) + \epsilon(u(t), v(t))_2, \tag{3.13}$$

where $a \equiv (1+(1-k)/r)^{-1} \in (1,2)$ and $\epsilon > 0$ will be choosen later.

We intend to apply Lemma 2.7 to functional (3.13). First, we calculate the derivative, along solutions, with respect to t. Let us start with the second term of (3.13). From (3.2)–(3.4) and (3.11), one has

$$\begin{aligned}\frac{d}{dt}(u(t),v(t))_2 &= \|v(t)\|_2^2 - I(u(t)) - \delta\left(u(t),v(t)|v(t)|^{\lambda-2}\right)_2 \\ &\geq \|v(t)\|_2^2 + \widehat{C}\mu\|u(t)\|_r^r - C\mathcal{V}^{(1-k)/r}(t)\left[\nu\delta\|u(t)\|_r^{k\lambda} + \frac{1}{C(\nu)}\dot{\mathcal{V}}(t)\right] \\ &\geq \|v(t)\|_2^2 + \left[\widehat{C}\mu - \nu C\delta\left(\frac{r}{\mu}\right)^{(k\lambda-r)/r}\mathcal{V}_0^b\right]\|u(t)\|_r^r - \frac{Ca}{C(\nu)}\dot{\mathcal{V}}^{1/a}(t) \\ &\geq \|v(t)\|_2^2 + \frac{\widehat{C}\mu}{2}\|u(t)\|_r^r - \frac{Ca}{C(\nu)}\dot{\mathcal{V}}^{1/a}(t),\end{aligned} \tag{3.14}$$

where $b \equiv (k(\lambda-1)-(r-1))/r < 0$, and $\nu > 0$ is sufficiently small.

Consequently, if $\epsilon > 0$ is sufficiently small,

$$\dot{\mathcal{F}}(t) \geq \widetilde{C}\left[\|v(t)\|_2^2 + \|u(t)\|_r^r\right] > 0, \tag{3.15}$$

where $\widetilde{C} \equiv \epsilon\min(1, \widehat{C}\mu/2) > 0$.

From (3.15) and choosing $\epsilon > 0$ small enough, we get

$$\mathcal{F}(t) \geq \mathcal{F}_0 \equiv \mathcal{V}_0 + \epsilon(u(t_0), v(t_0))_2 > 0. \tag{3.16}$$

Utilizing two times (3.3), we get

$$\begin{aligned}\mathcal{F}^a(t) &\leq 2^{a-1}\left[\mathcal{V}(t) + \epsilon^a|(u(t),v(t))_2|^a\right] \\ &\leq 2^{a-1}\left[\frac{\mu}{r}\|u(t)\|_r^r + \epsilon^a C(\Omega)^a\|u(t)\|_r^a\|v(t)\|_2^a\right] \\ &\leq 2^{a-1}\left[\frac{\mu}{r}\|u(t)\|_r^r + \epsilon^a C(\Omega)^a\left(\|u(t)\|_r^{2a/(2-a)} + \|v(t)\|_2^2\right)\right] \\ &\leq 2^{a-1}\left[\frac{\mu}{r}\|u(t)\|_r^r + \epsilon^a C(\Omega)^a\left(\left(\frac{\mu(r-2)}{2rd}\right)^c\|u(t)\|_r^r + \|v(t)\|_2^2\right)\right] \\ &\leq \overline{C}\left[\|u(t)\|_r^r + \|v(t)\|_2^2\right],\end{aligned} \tag{3.17}$$

where $C(\Omega) > 0$ is the imbedding constant of $L^r(\Omega) \subset L^2(\Omega)$, $c \equiv (1-2a)/r(2-a) > 0$, and $\overline{C} > 0$.

Hence and from (3.15), we obtain the inequality in order to apply Lemma 2.7. Therefore, the maximal time of existence is finite: $T < \infty$.

Necessity

Suppose that $\lambda \geq r$. Define the function, along the solution, by

$$\mathcal{W}(t) \equiv E(t) + \frac{2\mu}{r}\|u(t)\|_r^r. \tag{3.18}$$

Then,

$$\begin{aligned}\dot{\mathcal{W}}(t) &= \dot{E}(t) + 2\mu\left(u(t)|u(t)|^{r-2}, v(t)\right)_2 \\ &\leq -\delta\|v(t)\|_\lambda^\lambda + \frac{\delta}{2}\|v(t)\|_\lambda^r + C\|u(t)\|_r^r \\ &\leq \widehat{C}(\mathcal{W}(t) + 1),\end{aligned} \tag{3.19}$$

where $\widehat{C} \equiv \max(\delta/2, Cr/\mu)$, $C \equiv 2\mu C(\Omega)(\delta/2)^r$, and $C(\Omega) > 0$ is the imbedding constant of $L^\lambda(\Omega) \subset L^r(\Omega)$.

Hence, by Gronwall inequality, it follows that (u, v) is bounded in H for any finite time. A contradiction.

Proceeding again by contradiction suppose that, for all $t \geq 0$, $(u(t), v(t)) \notin \mathcal{U}$. Then, by Lemma 2.6, we have either $u(t) \in \mathcal{S}$ for all $t \geq 0$, or $E(t) \geq d$ for all $t \geq 0$. In the first case, from (2.28) in Lemma 2.3

$$E_0 \geq \frac{1}{2}\|v(t)\|_2^2 + \frac{r-2}{2r}a(u(t)), \tag{3.20}$$

that is, $(u(t), v(t))$ is bounded in H. This is not possible. In the second case,

$$\delta\int_0^t \|v(\tau)\|_\lambda^\lambda d\tau \leq E_0 - d, \tag{3.21}$$

where $E_0 \equiv E(u_0, v_0)$. Hence, by the Hölder inequality,

$$\delta t^{-(\lambda-1)}\left\|\int_0^t v(\tau)d\tau\right\|_\lambda^\lambda \leq E_0 - d, \tag{3.22}$$

and consequently

$$\|u(t)\|_\lambda \leq C(T), \tag{3.23}$$

for $t \in [0, T]$, where $C(T) \equiv \|u_0\|_\lambda + ((E_0 - d)/\delta)^{1/\lambda}T^{(\lambda-1)/\lambda}$.

From Theorem 2.1,

$$\lim_{t\nearrow T_{\text{MAX}}} \|u(t)\|_r = \infty; \tag{3.24}$$

hence, by Sobolev-Poincaré's inequality (2.16), for every $M > E_0$, there exists some $\widehat{t} > 0$, such that

$$M < \frac{r-2}{2r} a(u(t)), \tag{3.25}$$

for every $t \geq \widehat{t}$. This implies the first inequality of (2.29) in Lemma 2.3, replacing d by M. Now, we consider the function (3.13)

$$\mathcal{F}(t) \equiv \mathcal{V}(t)^{1/a} + \epsilon (u(t), v(t))_2, \tag{3.26}$$

defined for $t \geq \widehat{t}$, where $a \in (1,2)$, $\epsilon > 0$ is sufficiently small, and here

$$\mathcal{V}(t) \equiv M - E(t) > 0, \tag{3.27}$$

and repeat the sufficiency part of the proof. Then, by Lemma 2.7, $\mathcal{F}(t)$ blowups as $t \nearrow T^*$, $T^* > \widehat{t}$. Moreover, for $\widehat{t} \leq t < T^*$,

$$\mathcal{F}(t) \geq \frac{\mathcal{F}\left(\widehat{t}\right)}{\left(1 - \left(t - \widehat{t}\right) / \left(T^* - \widehat{t}\right)\right)^{1/(a-1)}}, \tag{3.28}$$

hence and from (3.26), (3.27), and since $E(t) \geq d$,

$$\begin{aligned} \|u(t)\|_2^2 &\geq \left\|u\left(\widehat{t}\right)\right\|_2^2 + \frac{2}{\epsilon}\left\{\int_{\widehat{t}}^{t}\left(\frac{\mathcal{F}\left(\widehat{t}\right)}{\left(1 - \left(\tau - \widehat{t}\right) / \left(T^* - \widehat{t}\right)\right)^{1/(a-1)}} - \mathcal{V}^{1/a}(\tau)\right) d\tau\right\} \\ &\geq \left\|u\left(\widehat{t}\right)\right\|_2^2 - \frac{2\left(t - \widehat{t}\right)}{\epsilon}(M-d)^{1/a} \\ &\quad + \frac{2(a-1)}{\epsilon(2-a)} \mathcal{F}\left(\widehat{t}\right)\left(T^* - \widehat{t}\right)\left\{\left(1 - \frac{t - \widehat{t}}{T^* - \widehat{t}}\right)^{-((2-a)/(a-1))} - 1\right\}. \end{aligned} \tag{3.29}$$

Consequently,

$$\lim_{t \nearrow T^*} \|u(t)\|_2^2 = \infty. \tag{3.30}$$

But this contradicts (3.23), since $L^{\lambda}(\Omega) \subset L^2(\Omega)$. The proof is complete. □

Remark 3.2. From the last result, if $\lambda = 2$ and $r > 2(\gamma + 1)$, any solution of problem (1.1), $(u(t), v(t))$, is global if and only if either (i) there exists $t_0 \geq 0$ such that $(u(t_0), v(t_0)) \in \mathcal{S}$ or (ii) $(u(t), v(t)) \in [E(u,v) \geq d]$, for every $t \geq 0$. On the other hand, if $\lambda > 2$ and $r > 2(\gamma + 1)$, then any solution is global if and only if one of the following holds: (i), (ii), or (iii) $\lambda \geq r$ and there exists $t_0 \geq 0$ such that $(u(t_0), v(t_0)) \in \mathcal{U}$.

We next prove a characterization of convergence to the zero equilibrium, and we give rates of decay.

Theorem 3.3. *Let* $(u(t), v(t)) = S(t)(u_0, v_0)$ *be a solution of problem* (1.1) *with* $\lambda \geq 2$*. Suppose that* $r > 2(\gamma + 1)$ *and that* $\lambda \leq 10$*, if* $n = 5$*. A necessary and sufficient condition for* $(u(t), v(t)) \to (0,0)$*, strongly in* H *as* $t \to \infty$*, is that there exists* $t_0 \geq 0$ *such that* $(u(t_0), v(t_0)) \in \mathcal{S}$*.*

In this case, if $\mathcal{F}(t)$ *denotes either the energy*

$$E(t) \equiv E(u(t), v(t)), \tag{3.31}$$

or the norm of the solution in H

$$\omega(t) \equiv \|(u(t), v(t))\|_H^2 \equiv a(u(t)) + \|v(t)\|_2^2, \tag{3.32}$$

One has the rates of decay, for $t \geq T$*,*

$$\mathcal{F}(t) \leq K_0 \left\{ 1 + t\left(\frac{\lambda - 2}{2}\right) K_1 K_0^{(\lambda-2)/2} \right\}^{-2/(\lambda-2)}, \tag{3.33}$$

and, for linear dissipation, $\lambda = 2$*,*

$$\mathcal{F}(t) \leq K_0 e^{-K_1 t}, \tag{3.34}$$

where $T > 0$ *is sufficiently large, and* $K_0 > 0$*,* $K_1 > 0$ *are constants depending only on initial conditions.*

Proof. Necessity

By (ii) in Lemma 2.2, $(0,0) \notin \overline{\mathcal{U}}$, and since the equilibrium $(0,0) \notin \overline{[E(u(t), v(t)) \geq d]}$, strong closures in H, then, by Lemma 2.6, the solution must eventually enter in $\mathcal{S}$.

Sufficiency

By energy equation and (2.27) in Lemma 2.3, the solution must be global and uniformly bounded in the norm of H, that is $\omega(t) < 2rd/(r-2)$, for any $t \geq 0$. Hence, there exists a sequence of times, $\{t_n\}$, such that if $n \to \infty$ then $t_n \to \infty$, $(u(t_n), v(t_n)) \to (\hat{u}, \hat{v})$ weakly in H and, since the embedding $B \subset L^r(\Omega)$ is compact, $b(u(t_n)) \to b(\hat{u})$. Also, notice that the energy is such that

$$0 \leq E_\infty \equiv \lim_{t \to \infty} E(t) = \inf_{t \geq 0} E(t) < \infty. \tag{3.35}$$

Consequently, from the energy equation and the continuous embedding $L^\lambda(\Omega) \subset L^2(\Omega)$,

$$\lim_{t \to \infty} \int_t^{t+1} \|v(\tau)\|_2^\lambda d\tau = 0, \tag{3.36}$$

in particular, for any sequence of times $\{s_n\}$ such that $s_n \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty}\int_0^1 h_n(\tau)d\tau = 0, \tag{3.37}$$

where $h_n(\tau) \equiv \|v(s_n+\tau)\|_2^\lambda$, for $\tau \in [0,1]$. By Fatou Lemma,

$$\liminf_{n\to\infty}\|v(s_n+\tau)\|_2^\lambda = \liminf_{n\to\infty} h_n(\tau) = 0, \tag{3.38}$$

for a.e. $\tau \in [0,1]$, and by the weak convergence to $\widehat{v}$,

$$\|\widehat{v}\|_2 \le \liminf_{n\to\infty}\|v(t_n)\|_2 = 0, \tag{3.39}$$

where we choose $\{s_n\}$ such that $t_n = s_n + \tau_0$, for some $\tau_0 \in [0,1]$.

It can be shown that the semigroup generated by problem (1.1) is continuous in H with the weak topology, and then that the weak limit set is positive invariant, see Ball [26]. Consequently $(\widehat{u},\widehat{v}) = (u_e,0)$ must be an equilibrium of (1.1). Furthermore, by the lower-semicontinuity of the norm in H, one has

$$\begin{aligned}\widehat{b}(u_e) = \widehat{a}(u_e) &\le \liminf_{n\to\infty}\left\{\|v(t_n)\|_2^2 + \widehat{a}(u(t_n))\right\}\\ &= \lim_{n\to\infty}\left\{2E(t_n) + \frac{2}{r}\widehat{b}(u(t_n))\right\}\\ &= 2E_\infty + \frac{2}{r}\widehat{b}(u_e).\end{aligned} \tag{3.40}$$

Hence,

$$\frac{r-2}{2r}\widehat{b}(u_e) \le E_\infty < d. \tag{3.41}$$

Then, by (2.19) and (2.21), $u_e = 0$, and

$$\lim_{t\to\infty}(u(t),v(t)) = (0,0) \quad \text{weakly in } H. \tag{3.42}$$

Strong convergence follows if we get the rates of decay in our statement. Here, we will adapt the technique used in Haraux and Zuazua [3], to (1.1). That technique is based on the construction of suitable Liapunov functions defined along solutions and the application of Lemma 2.8. One of them is the energy, and we will need one more, defined by

$$W(t) \equiv E(t) + \kappa E(t)^{\lambda/2-1}(u(t),v(t))_2, \tag{3.43}$$

where $\kappa > 0$ is a constant to be chosen later. We next prove that $W(t)$ is equivalent to both, the energy $E(t)$ and the norm $\omega(t)$ of the solution, in the sense of (2.30) and (2.28) below. First we note that from (2.27) in Lemma 2.3,

$$\frac{r-2}{2r}\omega(t) \le E(t) \le \frac{K}{2}\omega(t), \tag{3.44}$$

where $K - 1 \equiv (\beta/\alpha(\gamma+1))(2rd/\alpha(r-2))^{\gamma}$.

Also, notice that from (3.43),

$$|W(t) - E(t)| \le \kappa E_0^{\lambda/2-1}\|u(t)\|_2\|v(t)\|_2 \le \kappa E_0^{\lambda/2-1}C_1(\Omega)\omega(t), \tag{3.45}$$

where $C_1(\Omega) > 0$ is a constant that depends on the continuous embedding $B \subset L^2(\Omega)$. Hence and from (3.44), if κ is sufficiently small, then

$$\frac{1}{2}E(t) \le W(t) \le \frac{3}{2}E(t). \tag{3.46}$$

We will need the following estimate:

$$\begin{aligned}
\delta\left(u(t), v(t)|v(t)|^{\lambda-1}\right)_2 &\le \delta\|u(t)\|_\lambda\|v(t)\|_\lambda^{\lambda-1} \\
&\le \delta C_2(\Omega)a(u(t))^{(\lambda-2)/2\lambda}a(u(t))^{1/\lambda}\|v(t)\|_\lambda^{\lambda-1} \\
&\le \delta C_2(\Omega)\left(\frac{2r}{r-2}E_0\right)^{(\lambda-2)/2\lambda}a(u(t))^{1/\lambda}\left(-\dot{E}(t)\right)^{(\lambda-1)/\lambda} \\
&\le \frac{1}{\lambda}a(u(t)) - \widehat{C}\dot{E}(t),
\end{aligned} \tag{3.47}$$

where we applied (3.44) in the third step and Young inequality in last step, and the constants $C_2(\Omega) > 0$, $\widehat{C} > 0$ depend on the continuous embedding $B \subset L^\lambda(\Omega)$, and $\widehat{C}$ also depends on E_0.

It follows that, by (3.42) and since $B \subset L^r(\Omega)$ is compact, for any $\epsilon > 0$, there exists some $T > 0$ such that for any $t > T$

$$b(u(t)) \le C_3(\Omega)b(u(t))^{(r-2)/r}a(u(t)) \le \epsilon E(t), \tag{3.48}$$

where $C_3(\Omega) > 0$ is the corresponding embedding constant and we used (3.44) in the last step.

Since we will apply Lemma 2.8, we need to calculate the time derivative of (3.43) and we begin with

$$\begin{aligned}\frac{d}{dt}(u(t),v(t))_2 &= \|v(t)\|_2^2 - a(u(t)) - c(u(t)) + b(u(t)) - \delta\Big(u(t), v(t)|v(t)|^{\lambda-1}\Big)_2 \\ &\le \|v(t)\|_2^2 - \frac{1}{2}a(u(t)) - c(u(t)) - \widehat{C}\dot{E}(t) + \epsilon E(t) \\ &\le \frac{3}{2}\|v(t)\|_2^2 - (1-\epsilon)E(t) - \widehat{C}\dot{E}(t),\end{aligned} \tag{3.49}$$

which holds for any $t > T$, and where we used (3.47), (3.48) and definition of $E(t)$.

We notice that for any small $\eta > 0$, and by Young inequality and energy equation

$$E^{\lambda/2-1}(t)\frac{3}{2}\|v(t)\|_2^2 \le E^{\lambda/2-1}(t)C_4(\Omega)\|v(t)\|_\lambda^2 \le \eta E^{\lambda/2}(t) - C(\eta)\dot{E}(t), \tag{3.50}$$

where $C_4(\Omega) > 0$, $C(\eta) > 0$ depend on the continuous embedding $L^\lambda(\Omega) \subset L^2(\Omega)$, and $C(\eta)$ depends on η.

Then, for ϵ and η sufficiently small, (3.49) and (3.50), imply

$$E^{\lambda/2-1}(t)\frac{d}{dt}(u(t),v(t))_2 \le -\frac{1}{2}E^{\lambda/2}(t) - \widetilde{C}\dot{E}(t), \tag{3.51}$$

for any $t > T$, where $\widetilde{C} \equiv C(\eta) + \widehat{C}E_0^{\lambda/2-1}$.

Consequently, for κ sufficiently small and any $t > T$

$$\dot{W}(t) \le \dot{E}(t) - \kappa\frac{\lambda-2}{r-2}rC_1(\Omega)E_0^{\lambda/2-1}\dot{E}(t) - \kappa\left(\frac{1}{2}E^{\lambda/2}(t) + \widetilde{C}\dot{E}(t)\right) \le -\frac{\kappa}{2}E^{\lambda/2}(t) \le -\kappa_0 W^{\lambda/2}(t), \tag{3.52}$$

where $\kappa_0 \equiv (\kappa/2)(2/3)^{\lambda/2}$ and $C_1(\Omega) > 0$ is the constant in (3.45); also we used (3.44), the fact that the energy is decreasing and (3.46). Then, from (3.52) and Lemma 2.8, we obtain the desired rates of decay for $W(t)$. The result now follows by (3.46) and (3.44), and the proof is complete. □

Remark 3.4. By (2.35), the ground state is: $[(u,0) \in H : u \in \mathcal{N}^*] = \overline{\mathcal{S}} \cap \overline{\mathcal{U}}$. Then, in any H-neighborhood of that subset of nonzero equilibria, one can choose initial conditions either in $\mathcal{U}$ or in $\mathcal{S}$. Hence, by Theorem 3.1 and (3.3), the ground state is unstable in the sense of Liapunov when the dissipation term $g(u_t)$ is either linear or nonlinear.

Next we will study the behavior of solutions such that $(u(t), v(t)) \in [E(u,v) \ge d]$ for all $t \ge 0$. First, we prove that those solutions are uniformly bounded in time. To that end we will study the cases: $\lambda = 2$ and $\lambda > 2$ separately, First, we consider the case $\lambda = 2$.

Theorem 3.5. *Let $(u(t), v(t)) = S(t)(u_0, v_0)$ be a solution of problem* (1.1). *Assume that $r > 2(\gamma+1)$, and $\lambda = 2$. Also, assume that $r \le 2(6/n+1)$ if $n \ge 2$. If $(u(t), v(t)) \in [E(u,v) \ge d]$ for all $t \ge 0$, then the solution is global and uniformly bounded in H, for all $t \ge 0$.*

Proof. Suppose that $(u(t), v(t))$ is not global, then by Theorem 2.1 blowups and by Theorem 3.1, $(u(t_0), v(t_0)) \in \mathcal{U}$ for some $t_0 \geq 0$. Hence, $(u(t), v(t)) \in [E(u,v) < d]$ for all $t \geq t_0$. A contradiction.

Next, we will prove that $\|u(t)\|_2$ is uniformly bounded for all $t \geq 0$.

Let $\mathcal{F}(t) \equiv (1/2)\|u(t)\|_2^2 - C$, where $C > 0$ is the constant given below. Then, we obtain

$$\begin{aligned} \ddot{\mathcal{F}}(t) + \delta\dot{\mathcal{F}}(t) &= \|v(t)\|_2^2 - \widehat{a}(u(t)) + \widehat{b}(u(t)) \\ &= \frac{r+2}{2}\|v(t)\|_2^2 + \frac{r-2}{2}\widehat{a}(u(t)) - rE(t) \\ &\geq C(\Omega)(r-2)\mathcal{F}(t), \end{aligned} \tag{3.53}$$

where $C(\Omega) > 0$ is the imbedding constant of $B \subset L^2(\Omega)$, and $C \equiv rE_0/(r-2)C(\Omega)$.

We define $\mathcal{W}(t) \equiv \mathcal{F}^+(t) \equiv \sup\{\mathcal{F}(t), 0\}$, the positive part of $\mathcal{F}(t)$. We claim that, along solutions of (1.1), the time derivative satisfies $\dot{\mathcal{W}}(t) \leq 0$. Indeed, if this is no the case, there exists some $t_0 > 0$ such that

$$\mathcal{F}(t_0) > 0, \qquad \dot{\mathcal{F}}(t_0) > 0. \tag{3.54}$$

By a standard comparison result for ordinary differential equations, (3.53) and (3.54) imply that $\mathcal{F}(t) \to \infty$ as $t \to \infty$. Consequently, for any constant $C > E_0$, there exists some $t_0 > 0$, such that for $t \geq t_0$

$$C < \frac{r-2}{2r}\widehat{a}(u(t)). \tag{3.55}$$

This is (2.29) in Lemma 2.3, replacing d by C. If we now define, for $t \geq t_0$, the function

$$\mathcal{V}(t) \equiv C - E(t), \tag{3.56}$$

we can repeat the sufficiency part of the proof of Theorem 3.1 and show that the solution blowups in a finite time, consequently is nonglobal. A contradiction. Then, $\|u(t)\|_2 \leq C < \infty$, for all $t \in \mathbb{R}^+$, and some constant $C > 0$.

Next, we will prove that uniform boundedness of $\|u(t)\|_2$ implies uniform boundedness of $(u(t), v(t))$ in H, for all $t \in \mathbb{R}^+$. To that end, we consider the functions $\mathcal{H}(t) \equiv \mathcal{G}(t) - kE_0 \equiv \dot{\mathcal{F}}(t) + \delta\mathcal{F}(t) - kE_0$, where now $\mathcal{F}(t) \equiv (1/2)\|u(t)\|_2^2$ and $k > 0$ is defined below. From the second line in (3.53),

$$\dot{\mathcal{H}}(t) \geq K\mathcal{H}(t), \tag{3.57}$$

where $K \equiv \min\{r+2, \alpha C(\Omega)(r-2)/(1+\delta)\} > 0$ and $k \equiv r/K$.

Hence, for $0 \leq s \leq \tau$,

$$\mathcal{H}(\tau) \geq \mathcal{H}(s)e^{(K(\tau-s))}, \tag{3.58}$$

and consequently, from definition of $\mathcal{H}(t)$, for $0 \le s \le \tau \le t$,

$$\begin{aligned}
\mathcal{F}(t) &= \mathcal{F}(s)e^{-\delta(t-s)} + \int_s^t (\mathcal{H}(\tau) + kE_0)e^{\delta(\tau-t)}d\tau \\
&\ge \mathcal{F}(s)e^{-\delta(t-s)} + \int_s^t \left(\mathcal{H}(s)e^{K(\tau-s)} + kE_0\right)e^{\delta(\tau-t)}d\tau \\
&\ge \frac{\mathcal{H}(s)}{\delta+K}\left(e^{K(t-s)} - e^{\delta(s-t)}\right) + \frac{kE_0}{\delta}\left(1-e^{\delta(s-t)}\right).
\end{aligned} \tag{3.59}$$

Notice that if $\mathcal{H}(s) > 0$, for some $s \ge 0$, we obtain from (3.59) that $\lim_{t\to\infty}\mathcal{F}(t) = \infty$. A contradiction. Then, for all $t \ge 0$,

$$\mathcal{G}(t) \le kE_0. \tag{3.60}$$

Now, we define $\mathcal{L}(t) \equiv \mathcal{G}(t) + \tilde{k}E_0$, and like in (3.57)

$$\dot{\mathcal{L}}(t) \ge -\tilde{K}\mathcal{L}(t), \tag{3.61}$$

where $\tilde{K} \equiv \min\{r+2, C(\Omega)(r-2)\} > 0$ and $\tilde{k} \equiv r/\tilde{K}$. Hence,

$$\mathcal{L}(t) \ge \mathcal{L}(0)e^{-\tilde{K}t} \ge \min\{\mathcal{L}(0), 0\}, \tag{3.62}$$

and consequently,

$$\mathcal{G}(t) \ge \min\left\{\mathcal{G}(0), -\tilde{k}E_0\right\}. \tag{3.63}$$

Hence and from (3.60), $\mathcal{G}(t)$ is uniformly bounded in time.

We integrate the second line of (3.53) in terms of $\mathcal{G}(t)$ and, by the energy equation, we obtain

$$\mathcal{G}(t+1) - \mathcal{G}(t) \ge \frac{r-2(\gamma+1)}{2}\int_t^{t+1}\left(\|v(\tau)\|_2^2 + a(u(\tau)) + \frac{1}{\gamma+1}c(u(\tau))\right)d\tau - rE_0. \tag{3.64}$$

Hence and since $\mathcal{G}(t)$ is uniformly bounded in time,

$$\int_t^{t+1}\omega(\tau)d\tau \le C, \tag{3.65}$$

where $C > 0$ is a constant, and

$$\omega(t) \equiv \frac{1}{2}\|v(t)\|_2^2 + \frac{1}{2}a(u(t)) + \frac{1}{2(\gamma+1)}c(u(t)) = E(t) + \frac{1}{r}b(u(t)). \tag{3.66}$$

Next, we will show that there exists a constant $\kappa > 0$, such that

$$\omega(t) \leq \kappa(\omega(s) + 1), \tag{3.67}$$

for any $0 \leq s \leq t \leq s+1$.

To this end, we calculate

$$\dot{\omega}(t) = \left(v(t), f(u(t))\right)_2 - \delta\|v(t)\|_2^2 \leq \mu\|v(t)\|_2\|u(t)\|_{2(r-1)}^{r-1}. \tag{3.68}$$

If $n = 1$, we integrate and obtain, for $0 \leq s \leq t \leq s+1$, that

$$\begin{aligned}
\omega(t) &\leq \omega(s) + \mu\int_s^t \|v(\tau)\|_2\|u(\tau)\|_{2(r-1)}^{r-1}d\tau \\
&\leq \omega(s) + \frac{\mu}{2}\int_s^t\left(\|v(\tau)\|_2^2 + \|u(\tau)\|_{2(r-1)}^{2(r-1)}\right)d\tau \\
&\leq \omega(s) + \frac{\mu}{2}\int_s^{s+1}\|u(\tau)\|_{2(r-1)}^{2(r-1)}d\tau + \mu\int_s^t\omega(\tau)d\tau.
\end{aligned} \tag{3.69}$$

By Gronwall inequality and (3.65),

$$\begin{aligned}
\omega(t) &\leq \left(\omega(s) + \frac{\mu}{2}\|u\|_{L^{2(r-1)}(\Omega\times(s,s+1))}^{2(r-1)}\right)e^{\mu} \\
&\leq \widetilde{C}\left\{\omega(s) + \left[\int_s^{s+1}\left(\|u(\tau)\|_2^2 + \alpha\|\nabla u(\tau)\|_2^2 + \|v(\tau)\|_2^2\right)d\tau\right]^{r-1}\right\} \\
&\leq \widehat{C}\left\{\omega(s) + \left[\int_s^{s+1}\omega(\tau)d\tau\right]^{r-1}\right\} \\
&\leq \widehat{C}\left\{\omega(s) + C^{r-1}\right\},
\end{aligned} \tag{3.70}$$

where $\widetilde{C} > 0$, $\widehat{C} > 0$ depend on the continuous embeddings $B \subset L^2(\Omega)$, and $H^1(\Omega\times(s,s+1)) \subset L^{2(r-1)}(\Omega\times(s,s+1))$.

If $2 \leq n \leq 5$, we use Galiardo-Niremberg's inequality,

$$\|u\|_{2(r-1)}^{r-1} \leq C(\Omega)\|u\|_B^{(r-1)a}\|u\|_2^{(r-1)(1-a)}, \tag{3.71}$$

where $C(\Omega) > 0$, and $a = n(r-2)/4(r-1)$. Notice that $a < 1$ if $2 \leq n \leq 4$, and $a \leq 1$ if $n = 5$, because $r \leq 2(n-2)/(n-4) = 6$. Then, from

$$\dot{\omega}(t) \leq \mu\|v(t)\|_2\|u(t)\|_{2(r-1)}^{r-1}, \tag{3.72}$$

we integrate and apply Gronwall inequality, for $0 \le s \le t \le s+1$,

$$\begin{aligned}
\omega(t) &\le \omega(s) + \mu \int_s^t \|v(\tau)\|_2 \|u(\tau)\|_{2(r-1)}^{r-1} d\tau \\
&\le \omega(s) + \mu C(\Omega) \int_s^t \|v(\tau)\|_2 \|u(\tau)\|_B^{(r-1)a} \|u(\tau)\|_2^{(r-1)(1-a)} d\tau \\
&\le \omega(s) + \mu C(\Omega) \int_s^t \omega(\tau) \|u(\tau)\|_B^{(r-1)a-1} \|u(\tau)\|_2^{(r-1)(1-a)} d\tau \\
&\le \omega(s) \exp\left\{ \mu C(\Omega) \int_s^t \|u(\tau)\|_B^{(r-1)a-1} \|u(\tau)\|_2^{(r-1)(1-a)} d\tau \right\} \\
&\le \omega(s) \exp\left\{ \widehat{C} \int_s^t \|u(\tau)\|_B^{(r-1)a-1} d\tau \right\},
\end{aligned} \tag{3.73}$$

where $\widehat{C} \equiv \mu C(\Omega) \sup_{t\ge 0} \|u(t)\|_2^{(r-1)(1-a)}$.

Notice that $(r-1)a - 1 \le 2$ because by hypothesis $r \le 2(6/n+1)$, then we use the Hölder inequality, and from (3.65) we get

$$\begin{aligned}
\omega(t) &\le \omega(s) \exp\left\{ \widehat{C} \left(\int_s^t \|u(\tau)\|_B^2 d\tau \right)^{\{(r-1)a-1\}/2} \right\} \\
&\le \omega(s) \exp\left\{ \widetilde{C} \left(\int_s^{s+1} \omega(\tau) d\tau \right)^{\{(r-1)a-1\}/2} \right\} \\
&\le \omega(s) \exp\left\{ \widetilde{C} C^{\{(r-1)a-1\}/2} \right\}.
\end{aligned} \tag{3.74}$$

Then (3.67) holds for any $n \ge 1$, under our assumptions on r.

Consequently, (3.65) and (3.67) imply that

$$\begin{aligned}
\|v(t)\|_2^2 + \|u(t)\|_B^2 &\le \int_{t-1}^t 2\omega(t) ds \\
&\le 2\kappa \int_{t-1}^t (\omega(s)+1) ds \\
&\le 2\kappa (C+1),
\end{aligned} \tag{3.75}$$

and the proof is complete. □

Next, we consider the case $\lambda > 2$. Due to our assumptions on r, we restrict our analysis to $n\gamma < 4$. Since $\gamma \ge 1$, our analysis considers, at most, dimensions $n \le 3$, whenever $\gamma < 4/3$.

Theorem 3.6. *Let $(u(t), v(t)) = S(t)(u_0, v_0)$ be a solution of problem* (1.1). *Assume that $r > 2(\gamma + 1)$, and $\lambda > 2$ with $\lambda \le 2(n+1)/(n-1)$ if $n \ge 2$. Also assume that $r < 2(4/n+1)$ for $n \ge 1$. If*

$(u(t), v(t)) \in [E(u, v) \geq d]$ *for all* $t \geq 0$*, then the solution is global and uniformly bounded in* H*, for all* $t \geq 0$.

Proof. Globality follows like in last Theorem. Suppose that

$$\|u(t)\|_2^2 \leq K, \tag{3.76}$$

for some constant $K > 0$, and every $t \geq 0$.

We recall Galiardo-Nirenberg's inequality

$$\|u\|_r^r \leq C(\Omega) a(u)^{ar/2} \|u\|_2^{r(1-a)}, \tag{3.77}$$

where $C(\Omega) > 0$ is a constant, $a = (n/2)((r-2)/2r)$, and $a \in (0, 1]$.

Hence and from (3.76) in the energy equation, we obtain, for any time $t \geq 0$,

$$E_0 \geq \frac{1}{2}\|v(t)\|_2^2 + \frac{1}{2}a(u(t)) - \widehat{C} a(u(t))^{ar/2}, \tag{3.78}$$

where $\widehat{C} \equiv K^{r(1-a)/2} C(\Omega)\mu/r$, and therefore $(u(t), v(t))$, $t \geq 0$, is uniformly bounded in H, since $ar < 2$ if and only if $r < 2(4/n + 1)$. This implies, since $r > 2(\gamma + 1)$, that $n\gamma < 4$.

Now, we prove (3.76). First, we notice that from energy equation,

$$\delta \int_0^t \|v(\tau)\|_\lambda^\lambda d\tau \leq E_0 - d. \tag{3.79}$$

Hence, by Hölder inequality,

$$\delta t^{-(\lambda-1)} \left\| \int_0^t v(\tau) d\tau \right\|_\lambda^\lambda \leq E_0 - d, \tag{3.80}$$

and then

$$\|u(t)\|_\lambda \leq C(t), \tag{3.81}$$

for every $t \geq 0$, where $C(t) \equiv \|u_0\|_\lambda + ((E_0 - d)/\delta)^{1/\lambda} t^{(\lambda-1)/\lambda}$.

Next, we define

$$2\omega(t) \equiv \|(u(t), v(t))\|_H^2, \tag{3.82}$$

and obtain the following estimate for $t \in (0,T)$, and $T > 0$ finite and arbitrary:

$$\begin{aligned}
\int_0^t \delta\Big(u(\tau), |v(\tau)|^{\lambda-2}v(\tau)\Big)_2 d\tau &\le \int_0^t \delta\|u(\tau)\|_\lambda^\lambda \|v(\tau)\|_\lambda^{\lambda-1} d\tau \\
&\le \left(\int_0^t \|u(\tau)\|_\lambda^\lambda d\tau\right)^{1/\lambda} \delta\left(\int_0^t \|v(\tau)\|_\lambda^\lambda d\tau\right)^{(\lambda-1)/\lambda} \\
&\le C_0\|u\|_{L^\lambda(\Omega\times(0,T))} \\
&\le \widehat{C}\left(\int_0^t \Big(\|u(\tau)\|_2^2 + \alpha\|\nabla u(\tau)\|_2^2 + \|v(\tau)\|_2^2\Big)d\tau\right)^{1/2} \\
&\le \widetilde{C}\left(\int_0^t \omega(\tau)d\tau\right)^{1/2}.
\end{aligned} \tag{3.83}$$

Here, we used the Hölder inequality, the energy equation and the fact that $E(t) \ge d$, also $C_0 \equiv (E_0 - d)^{(\lambda-1)/\lambda}\delta^{1/\lambda} > 0$, and $\widehat{C} > 0$, $\widetilde{C} > 0$ depend on the continuous embedding $H^1(\Omega\times(0,T)) \subset L^\lambda(\Omega\times(0,T))$, valid for $\lambda > 2$ and $\lambda \le 2(n+1)/(n-1)$ if $n \ge 2$. $\widetilde{C}$ also depends on the embedding $B \subset L^2(\Omega)$.

Now, we define the function

$$\mathcal{F}(t) \equiv \frac{1}{2}\|u(t)\|_2^2 - \kappa, \tag{3.84}$$

where $\kappa > 0$ is the constant given below. Then, the second derivative is

$$\begin{aligned}
\ddot{\mathcal{F}}(t) &= \|v(t)\|_2^2 - a(u(t)) - c(u(t)) - \delta\Big(u(t), |v(t)|^{\lambda-2}v(t)\Big)_2 + b(u(t)) \\
&= \frac{r+2}{2}\|v(t)\|_2^2 + \frac{r-2}{2}a(u(t)) + \frac{r-2(\gamma+1)}{2(\gamma+1)}c(u(t)) - \delta\Big(u(t), |v(t)|^{\lambda-2}v(t)\Big)_2 - rE(t) \\
&\ge (r-2)\omega(t) - \delta\Big(u(t), |v(t)|^{\lambda-2}v(t)\Big)_2 - rE_0.
\end{aligned} \tag{3.85}$$

If we integrate (3.85), and we use (3.83) and that $\omega(t) \ge E(t) \ge d$, we obtain

$$\begin{aligned}
\dot{\mathcal{F}}(t) &\ge \dot{\mathcal{F}}(0) + (r-2)\int_0^t \omega(\tau)d\tau - \widetilde{C}\left(\int_0^t \omega(\tau)d\tau\right)^{1/2} - rE_0 t \\
&\ge \dot{\mathcal{F}}(0) + (r-2)\int_0^t \omega(\tau)d\tau\left\{1 - \frac{\widetilde{C}}{(r-2)\sqrt{T_m d}}\right\} - rE_0 t
\end{aligned}$$

$$\begin{aligned}
&= \dot{\mathcal{F}}(0) + \frac{(r-2)}{2}\int_0^t \omega(\tau)d\tau - rE_0 t \\
&\geq \dot{\mathcal{F}}(0) + \frac{(r-2)C(\Omega)}{2}\int_0^t \mathcal{F}(\tau)d\tau,
\end{aligned} \tag{3.86}$$

for $t \in (T_m, T)$, where $\sqrt{T_m d} \equiv 2\widetilde{C}/(r-2) > 0$, $C(\Omega) > 0$ is the embedding constant of $B \subset L^2(\Omega)$, and $\kappa \equiv 2rE_0/(r-2)C(\Omega) > 0$.

Now, we define for every $t \geq 0$,

$$\mathcal{G}(t) \equiv \int_0^t \mathcal{F}(\tau)d\tau + \frac{\dot{\mathcal{F}}(0)}{C}, \tag{3.87}$$

where $C \equiv (r-2)C(\Omega)/2$. Hence, (3.86) has the form

$$\ddot{\mathcal{G}}(t) - C\mathcal{G}(t) \geq 0, \tag{3.88}$$

for every $t \in (T_m, T)$.

We define $\mathscr{H}(t) \equiv \mathcal{F}^+(t) \equiv \sup\{\mathcal{F}(t), 0\}$, the positive part of $\mathcal{F}(t)$. We claim that, for every $t > T_m$, the time derivative satisfies $\dot{\mathscr{H}}(t) \leq 0$. Otherwise, there exists some $t_0 \in (T_m, T)$ such that

$$\mathcal{F}(t_0) > 0, \qquad \dot{\mathcal{F}}(t_0) > 0, \tag{3.89}$$

that is,

$$\dot{\mathcal{G}}(t_0) > 0, \qquad \ddot{\mathcal{G}}(t_0) > 0. \tag{3.90}$$

By a standard comparison result for ordinary differential equations, (3.88) and (3.90) imply that

$$\mathcal{F}(t) = \dot{\mathcal{G}}(t) \geq M_0 e^{\sqrt{C}t}, \tag{3.91}$$

for $t \in (t_1, T)$ and some $t_1 \in (t_0, T)$, where $M_0 > 0$ depends on $\mathcal{F}(t_0)$ and $\dot{\mathcal{F}}(t_0)$. Furthermore, for any constant $M > 0$, there exists some $t_2 \in (t_1, T)$, such that $M_0 e^{\sqrt{C}t} > M(1 + t^{2(\lambda-1)/\lambda})$, for $t \in (t_2, T)$. Then, (3.91) contradicts (3.81), since $L^\lambda(\Omega) \subset L^2(\Omega)$. Consequently, (3.76) holds with $K \equiv \max\{\max_{0 \leq t \leq T_m} \|u(t)\|_2^2, 2\kappa\}$, and the proof is complete. □

From Remark 3.2 and Theorems 3.1, 3.3, and 3.5, any global solution of (1.1), with $\lambda = 2$, is necessarily uniformly bounded in H, for all $t \geq 0$. However, this is false when $\lambda > 2$, as we show next.

Theorem 3.7. *Let $(u(t), v(t)) = S(t)(u_0, v_0)$ be a solution of problem* (1.1). *Assume the conditions, on r and λ, made in Theorem 3.6. Then, $(u(t), v(t))$ is global and not uniformly bounded in H, for $t \geq 0$, if and only if $\lambda \geq r$ and there exists $t_0 \geq 0$ such that $(u(t_0), v(t_0)) \in \mathcal{U}$.*

Proof. *Sufficiency*

By Lemma 2.4, $(u(t), v(t)) \in \mathcal{U}$, for every $t \geq t_0$. This solution must be global. Otherwise, by Theorem 2.1, blowups in a finite time and, by Theorem 3.1, necessarily $\lambda < r$. A contradiction.

Now, suppose that, for $t \geq t_0$, the solution is uniformly bounded in H. Hence, there exists a sequence of times $\{t_n\}$, such that if $n \to \infty$, then $t_n \to \infty$, $(u(t_n), v(t_n)) \to (\widehat{u}, \widehat{v})$ weakly in H and, since the imbedding $B \subset H^1(\Omega) \cap L^r(\Omega)$ is compact, $\widehat{b}(u(t_n)) \to \widehat{b}(\widehat{u})$. Moreover, since the energy is nonincreasing and bounded, $E_\infty \equiv \lim_{t\to\infty} E(t) \in \mathbb{R}$. Consequently, from the energy equation and since $L^\lambda(\Omega) \subset L^2(\Omega)$,

$$\lim_{t\to\infty} \int_t^{t+1} \|v(\tau)\|_2^\lambda d\tau = 0, \tag{3.92}$$

in particular, for any sequence of times $\{s_n\}$ such that $s_n \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \int_0^1 h_n(\tau) d\tau = 0, \tag{3.93}$$

where $h_n(\tau) \equiv \|v(s_n + \tau)\|_2^\lambda$, for $\tau \in [0, 1]$. By Fatou Lemma,

$$\liminf_{n\to\infty} \|v(s_n + \tau)\|_2^\lambda = \liminf_{n\to\infty} h_n(\tau) = 0, \tag{3.94}$$

for a.e. $\tau \in [0, 1]$, and by the weak convergence $v(t_n) \to \widehat{v}$, in $L^2(\Omega)$,

$$\|\widehat{v}\|_2 \leq \liminf_{n\to\infty} \|v(t_n)\|_2 = 0, \tag{3.95}$$

where we choose $\{s_n\}$ such that $t_n = s_n + \tau_0$, for some $\tau_0 \in [0, 1]$.

It can be shown that the semigroup generated by problem (1.1) is continuous in H with the weak topology, and then that the weak limit set is positive invariant; see Ball [26]. Consequently $(\widehat{u}, \widehat{v}) = (u_e, 0)$ must be an equilibrium of (1.1). Since $(u(t_n), v(t_n)) \in \mathcal{U}$ then, by definition (2.33) and (2.29),

$$\widehat{b}(u_e) = \lim_{n\to\infty} \widehat{b}(u(t_n)) \geq \frac{2rd}{r-2}. \tag{3.96}$$

Since the norm in H is weak lower-semicontinuous, from (2.4), (2.19), and (2.20), we get that

$$\begin{aligned}\widehat{b}(u_e) &= \widehat{a}(u_e)\\ &\leq \liminf_{n\to\infty}\left\{\|v(t_n)\|_2^2 + \widehat{a}(u(t_n))\right\}\\ &= \lim_{n\to\infty}\left\{2E(t_n) + \frac{2}{r}\widehat{b}(u(t_n))\right\}\\ &= 2E_\infty + \frac{2}{r}\widehat{b}(u_e).\end{aligned} \tag{3.97}$$

Hence,

$$\frac{r-2}{2r}\widehat{b}(u_e) \leq E_\infty < d. \tag{3.98}$$

This contradicts (3.96). Then, the solution cannot be uniformly bounded in H, for all $t \geq 0$.

Necessity

This follows from Lemma 2.6 and Theorems 3.1, 3.3, and 3.6. The proof is complete. □

Next, we characterize the convergence to the set of nonzero equilibria of equation (1.1). Due to our assumptions on r and γ, our result considers, at most, dimensions $n \leq 3$ for $\lambda > 2$, and $n \leq 5$ for $\lambda = 2$.

Theorem 3.8. *Let $(u(t), v(t)) = S(t)(u_0, v_0)$ be a solution of problem* (1.1). *If $\lambda > 2$, assume the conditions, on r and λ, made in Theorem 3.6. If $\lambda = 2$, and assume the conditions, on r made in Theorem 3.5. Then, $(u(t), v(t)) \to \mathcal{E}_\infty$, strongly in H as $t \to \infty$, if and only if $(u(t), v(t)) \in [E(u,v) \geq d]$ for all $t \geq 0$, where $\mathcal{E}_\infty \equiv [(u_e, 0) \in \mathcal{E} : J(u_e) = E_\infty \geq d]$ and $E_\infty \equiv \lim_{t\to\infty} E(u(t), v(t))$.*

Proof. Sufficiency

By Theorems 3.5 and 3.6, the solution is global and uniformly bounded in H, that is, $\omega(t) \equiv \|(u(t), v(t))\|_H^2 \leq K$, for all $t \geq 0$, and some constant $K > 0$. Then, like in the sufficiency part of the proof of Theorem 3.7, there exists a sequence of times such that $t_n \to \infty$ and $(u(t_n), v(t_n)) \to (u_e, 0)$, weakly in H, as $n \to \infty$, where $(u_e, 0)$ is an equilibrium. If

$$\{(u(t), v(t))\}_{t\geq 0} \quad \text{is precompact in } H, \tag{3.99}$$

then strong convergence to $(u_e, 0)$ follows. In this situation, $(u(t_n), v(t_n))$ converges to the set $\mathcal{E}_\infty$, because $(0,0) \notin \overline{[E(u,v) \geq d]}$, strong closure in H.

Haraux [5], developed a technique to prove precompactness of bounded orbits of some kind of semilinear wave equations. We will follow that method to prove (3.99). To this end we define, for every $\epsilon > 0$, and $t \geq 0$,

$$u_\epsilon(t) \equiv u(t+\epsilon) - u(t), \qquad v_\epsilon(t) \equiv v(t+\epsilon) - v(t), \qquad 2\omega_\epsilon(t) \equiv \|(u_\epsilon(t), v_\epsilon(t))\|_H^2, \tag{3.100}$$

and note that from (2.1), we get the energy equation for $(u_\epsilon(t), v_\epsilon(t))$

$$\omega_\epsilon(0) = \omega_\epsilon(t) + \int_0^t \left(g_\epsilon(\tau) - f_\epsilon(\tau) - \widehat{m}_\epsilon(\tau), v_\epsilon(\tau)\right)_2 d\tau, \tag{3.101}$$

where,

$$\begin{gathered} g_\epsilon(t) \equiv g(v(t+\epsilon)) - g(v(t)), \qquad f_\epsilon(t) \equiv f(u(t+\epsilon)) - f(u(t)), \\ \widehat{m}_\epsilon(t) \equiv m\left(\|u(t+\epsilon)\|_2^2\right)\Delta u(t+\epsilon) - m\left(\|u(t)\|_2^2\right)\Delta u(t), \qquad m\left(\|\nabla u(t)\|_2^2\right) \equiv \beta\|\nabla u(t)\|_2^{2\gamma}. \end{gathered} \tag{3.102}$$

However, in order to handle the nonlinearity $\widehat{m}_\epsilon(t)$, we need to introduce the function

$$W_\epsilon(t) \equiv \omega_\epsilon(t) + \frac{1}{2} m\left(\|\nabla u(t+\epsilon)\|_2^2\right)\|\nabla u_\epsilon(t)\|_2^2. \tag{3.103}$$

Hence, the corresponding energy equation for $W_\epsilon(t)$ is

$$W_\epsilon(0) = W_\epsilon(t) + \int_0^t \left(g_\epsilon(\tau) - f_\epsilon(\tau) - m_\epsilon(\tau), v_\epsilon(\tau)\right)_2 d\tau - \int_0^t (n_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau, \tag{3.104}$$

where,

$$\begin{gathered} m_\epsilon(t) \equiv \left(m\left(\|\nabla u(t+\epsilon)\|_2^2\right) - m\left(\|\nabla u(t)\|_2^2\right)\right)\Delta u(t), \\ n_\epsilon(t) \equiv m'\left(\|\nabla u(t+\epsilon)\|_2^2\right)(\Delta u(t+\epsilon), v(t+\epsilon))_2 \Delta u_\epsilon(t). \end{gathered} \tag{3.105}$$

Notice that since the solution is uniformly bounded by K, there exists a constant $\widehat{K} > 0$, depending on K, such that

$$\omega_\epsilon(t) \leq W_\epsilon(t) \leq \widehat{K}\omega_\epsilon(t). \tag{3.106}$$

We will prove that for any $\eta > 0$, there exists $\epsilon(\eta) > 0$, such that

$$\omega_\epsilon(t) \leq \eta, \tag{3.107}$$

for every $t \geq 0$, and $\epsilon \in (0, \epsilon(\eta))$, that is, $t \mapsto (u(t), v(t)) \in H$, is uniformly continuous.

For every $t \geq 0$, we have one of the following two cases:

$$W_\epsilon(t+1) \leq W_\epsilon(t), \tag{3.108}$$

$$W_\epsilon(t+1) > W_\epsilon(t). \tag{3.109}$$

From (3.109) and (3.104),

$$0 > W_\epsilon(t) - W_\epsilon(t+1) = \int_t^{t+1} \left(g_\epsilon(\tau) - f_\epsilon(\tau) - m_\epsilon(\tau), v_\epsilon(\tau)\right)_2 d\tau - \int_t^{t+1} (n_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau. \tag{3.110}$$

Notice that, by a well-known inequality,

$$\left(g_\epsilon(t), v_\epsilon(t)\right)_2 \geq 2^{2-\lambda}\delta \|v_\epsilon(t)\|_\lambda^\lambda. \tag{3.111}$$

Also, we recall the inequality

$$\left|f_\epsilon(t)\right| \leq \sigma(r)\mu\left(|u(t+\epsilon)|^{r-2} + |u(t)|^{r-2}\right)|u_\epsilon(t)|, \tag{3.112}$$

where $\sigma(r) = 1$, if $r \in [2,3]$ and $\sigma(r) = (r-1)/2$, if $r > 3$.

From the Hölder inequality, (3.112) yields

$$\begin{aligned}\left(\int_t^{t+1} \|f_\epsilon(\tau)\|_2^2 d\tau\right)^{1/2} &\leq 2\sigma(r)\mu C(\Omega) \sup_{t\geq 0} a(u(t))^{(r-2)/2} \\ &\quad \times \left(\int_t^{t+1} \|u_\epsilon(\tau)\|_{2(r-1)}^{2(r-1)} d\tau\right)^{1/2(r-1)},\end{aligned} \tag{3.113}$$

where $C(\Omega) > 0$, is an embedding constant in $B \subset L^{2(r-1)}(\Omega)$.

We claim that $t \mapsto u(t) \in L^{2(r-1)}(\Omega)$ must be uniformly continuous. Otherwise, there exists some $\eta_0 > 0$, and sequences $\{\epsilon_n\}_{n\geq 1}$, $\{t_n\}_{n\geq 1}$, such that $\epsilon_n \rightarrow 0$, and $t_n \rightarrow \infty$, as $n \rightarrow \infty$, and

$$\|u_{\epsilon_n}(t_n)\|_{2(r-1)} > \eta_0, \tag{3.114}$$

for every $n \geq 1$. By assumption, $B \subset L^{2(r-1)}(\Omega)$ is compact, then $\{u(t_n + \epsilon_n)\}_{n\geq 1}$, $\{u(t_n)\}_{n\geq 1}$ are precompact in $L^{2(r-1)}(\Omega)$, and we can extract subsequences $\{u(t'_n + \epsilon'_n)\}_{n\geq 1}$, $\{u(t'_n)\}_{n\geq 1}$, such that for some fixed n_0, sufficiently big, and every $n \geq n_0$,

$$\begin{aligned}\left\|u(t'_n + \epsilon'_n) - u(t'_n)\right\|_{2(r-1)} &\leq \left\|u(t'_n + \epsilon'_n) - u(t_{n_0} + \epsilon_{n_0})\right\|_{2(r-1)} \\ &\quad + \|u(t_{n_0} + \epsilon_{n_0}) - u(t_{n_0})\|_{2(r-1)} \\ &\quad + \left\|u(t_{n_0}) - u(t'_n)\right\|_{2(r-1)} \\ &\leq \frac{\eta_0}{3} + \frac{\eta_0}{3} + \frac{\eta_0}{3} = \eta_0.\end{aligned} \tag{3.115}$$

This contradicts (3.114). Hence, for any $\eta > 0$ there exists some $\widehat{\epsilon}(\eta) > 0$, such that for every $t \geq 0$, and every $\epsilon \in (0, \widehat{\epsilon}(\eta))$,

$$\left(\int_t^{t+1} \| u_\epsilon(\tau)\|_{2(r-1)}^{2(r-1)} d\tau\right)^{1/2(r-1)} \leq \eta^{4(\lambda-1)}. \tag{3.116}$$

Consequently, from (3.113), (3.116) and the Hölder inequality

$$\left|\int_t^{t+1} \left(f_\epsilon(\tau), v_\epsilon(\tau)\right)_2 d\tau\right| \leq C\eta^{4(\lambda-1)}\left(\int_t^{t+1} \|v_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{1/\lambda}, \tag{3.117}$$

where $C > 0$ depends on K, μ, r, $C(\Omega)$ and the inclusion $L^\lambda(\Omega) \subset L^2(\Omega)$.

Now notice that,

$$\begin{aligned}\|m_\epsilon(t)\|_2 &\leq \sup_{t\geq 0}\left\{m'\left(\|\nabla u(t)\|_2^2\right)\right\}\left|\|\nabla u(t+\epsilon)\|_2^2 - \|\nabla u(t)\|_2^2\right|\|\Delta u(t)\|_2 \\ &= \sup_{t\geq 0}\left\{m'\left(\|\nabla u(t)\|_2^2\right)\right\}|(\nabla u(t+\epsilon) + \nabla u(t), \nabla u(t+\epsilon) - \nabla u(t))_2|\|\Delta u(t)\|_2 \\ &\leq \sup_{t\geq 0}\left\{m'\left(\|\nabla u(t)\|_2^2\right)\|\Delta u(t+\epsilon) + \Delta u(t)\|_2\|\Delta u(t)\|_2\right\}\|u_\epsilon(t)\|_2 \\ &\leq C(K)\|u_\epsilon(t)\|_2,\end{aligned} \tag{3.118}$$

and that

$$\begin{aligned}|(n_\epsilon(t), u_\epsilon(t))_2| &\leq m'\left(\|\nabla u(t+\epsilon)\|_2^2\right)\|\Delta u(t+\epsilon)\|_2\|v(t+\epsilon)\|_2\|\Delta u_\epsilon(t)\|_2\|u_\epsilon(t)\|_2 \\ &\leq C(K)\|u_\epsilon(t)\|_2,\end{aligned} \tag{3.119}$$

where $C(K) > 0$ is a constant.

Since $B \subset L^2(\Omega)$ is compact, we show like in (3.116), that

$$\left(\int_t^{t+1} \|u_\epsilon(\tau)\|_2^2 d\tau\right)^{1/2} \leq \eta^{4(\lambda-1)}. \tag{3.120}$$

Consequently, from (3.118)–(3.120) and the Hölder inequality, we obtain

$$\left|\int_t^{t+1} \{(m_\epsilon(\tau), v_\epsilon(\tau))_2 + (n_\epsilon(\tau), u_\epsilon(\tau))_2\} d\tau\right| \leq \widehat{C}\eta^{4(\lambda-1)} \left\{\left(\int_t^{t+1} \|v_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{1/\lambda} + 1\right\}, \tag{3.121}$$

where $\widehat{C} > 0$ depends on K and the inclusion $L^\lambda(\Omega) \subset L^2(\Omega)$.

From (3.117), (3.121), and (3.111) in (3.110), we have

$$\int_t^{t+1} \|v_\epsilon(\tau)\|_\lambda^\lambda d\tau \leq \widetilde{C}\eta^{4(\lambda-1)} \left\{\left(\int_t^{t+1} \|v_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{1/\lambda} + 1\right\}, \tag{3.122}$$

where $\widetilde{C} > 0$ is a constant. Consequently, for η sufficiently small, we obtain

$$\int_t^{t+1} \|v_\epsilon(\tau)\|_\lambda^\lambda d\tau \leq \eta^{3\lambda}, \tag{3.123}$$

$$\int_t^{t+1} \|v_\epsilon(\tau)\|_2^2 d\tau \leq \frac{\eta^2}{5}. \tag{3.124}$$

We apply inequality (3.112) to g_ϵ, and by the Hölder inequality we get

$$\begin{aligned}\int_t^{t+1} (g_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau \leq \sigma(\lambda)\delta\left(\int_t^{t+1} \|v_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{1/\lambda} \left(\int_t^{t+1} \|u_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{1/\lambda} \\ \times \left[\left(\int_t^{t+1} \|v(\tau+\epsilon)\|_\lambda^\lambda d\tau\right)^{(\lambda-2)/\lambda} + \left(\int_t^{t+1} \|v(\tau)\|_\lambda^\lambda d\tau\right)^{(\lambda-2)/\lambda}\right].\end{aligned} \tag{3.125}$$

Notice that by (2.3) and since $E(t) \geq d$,

$$\int_0^\infty \|v(t)\|_\lambda^\lambda dt \leq \frac{E_0 - d}{\delta}. \tag{3.126}$$

By assumption $B \subset L^{\lambda}(\Omega)$, then

$$\left(\int_t^{t+1} \|u_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{1/\lambda} \leq 2C(\Omega) \sup_{t\geq 0} \|u(t)\|_B \leq C, \tag{3.127}$$

where $C > 0$ depends on the embedding constant $C(\Omega)$ and K.

Therefore, from (3.123), (3.126), and (3.127) in (3.125), we get, for η sufficiently small

$$\left|\int_t^{t+1} \left(g_\epsilon(\tau), u_\epsilon(\tau)\right)_2 d\tau\right| \leq 2\sigma(\lambda)\delta C\left(\frac{E_0 - d}{\delta}\right)^{(\lambda-2)/\lambda} \eta^3 \leq \frac{\eta^2}{5}. \tag{3.128}$$

By (3.113), (3.116), (3.120) and the Hölder inequality, for small η, we obtain

$$\int_t^{t+1} \left(f_\epsilon(\tau), u_\epsilon(\tau)\right)_2 d\tau \leq C\eta^{8(\lambda-1)} \leq \frac{\eta^2}{5}. \tag{3.129}$$

one has that

$$\begin{aligned}(\widehat{m}_\epsilon(t), u_\epsilon(t))_2 &= -m\left(\|\nabla u(t+\epsilon)\|_2^2\right)\|\nabla u_\epsilon(t)\|_2^2 \\ &\quad + \left\{m\left(\|\nabla u(t+\epsilon)\|_2^2\right) - m\left(\|\nabla u(t)\|_2^2\right)\right\}(\Delta u(t), u_\epsilon(t))_2 \\ &\leq C(K)\|u_\epsilon(t)\|_2,\end{aligned} \tag{3.130}$$

where $C(K)$ depends on K. Hence, by (3.120) and the Hölder inequality, and again for small η,

$$\int_t^{t+1} (\widehat{m}_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau \leq \frac{\eta^2}{5}. \tag{3.131}$$

From (2.1), one has the identity

$$\begin{aligned}&\frac{d}{dt}(v_\epsilon(t), u(t)_\epsilon)_2 - \|v_\epsilon(t)\|_2^2 + \|\Delta u_\epsilon(t)\|_2^2 + \alpha\|\nabla u_\epsilon(t)\|_2^2 + \left(g_\epsilon(t), u_\epsilon(t)\right)_2 \\ &\quad = (\widehat{m}_\epsilon(t), u_\epsilon(t))_2 + \left(f_\epsilon(t), u_\epsilon(t)\right)_2.\end{aligned} \tag{3.132}$$

Hence, (3.124), (3.128), (3.129), and (3.131) in (3.132) yield

$$\int_t^{t+1} \|u_\epsilon(\tau)\|_B^2 d\tau \leq 2\sup_{t\geq 0}\{\|v_\epsilon(t)\|_2 \|u_\epsilon(t)\|_2\} + \frac{4\eta^2}{5}. \tag{3.133}$$

For every $t \geq 0$, one has

$$\|u_\epsilon(t)\|_2 \leq \epsilon \sup_{s\in[t,t+1]} \|v(s)\|_2 \leq \epsilon\sqrt{2K},$$
$$\|v_\epsilon(t)\|_2 \leq 2\sup_{t\geq 0} \|v(t)\|_2 \leq 2\sqrt{2K} \tag{3.134}$$

then, for $\epsilon \in (0, \widehat{\epsilon}(\eta))$, with $\widehat{\epsilon}(\eta)$ sufficiently small, (3.133) is

$$\int_t^{t+1} \|u_\epsilon(\tau)\|_B^2 d\tau \leq \eta^2. \tag{3.135}$$

Hence, in case (3.109), from (3.124) and (3.135), we conclude that

$$\int_t^{t+1} \omega_\epsilon(\tau) d\tau \leq \frac{3\eta^2}{5}. \tag{3.136}$$

And, from (3.106) with η small,

$$\int_t^{t+1} W_\epsilon(\tau) d\tau \leq \frac{3\eta}{5}. \tag{3.137}$$

From (3.104), (3.111), (3.117), (3.121) and (3.124), we have, for any $s \in [t, t+1]$, $t \geq 0$, and η sufficiently small, that

$$\begin{aligned} W_\epsilon(t+1) &= W_\epsilon(s) + \int_s^{t+1} \left(m_\epsilon(\tau) + f_\epsilon(\tau) - g_\epsilon(\tau), v_\epsilon(\tau)\right)_2 d\tau + \int_0^t (n_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau \\ &\leq W_\epsilon(s) + \left|\int_t^{t+1} \left(f_\epsilon(\tau), v_\epsilon(\tau)\right)_2 d\tau\right| + \left|\int_t^{t+1} (m_\epsilon(\tau), v_\epsilon(\tau))_2 d\tau\right| \\ &\quad + \left|\int_t^{t+1} (n_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau\right| \\ &\leq W_\epsilon(s) + \frac{2\eta}{5}. \end{aligned} \tag{3.138}$$

Therefore, by (3.137), we obtain

$$W_\epsilon(t+1) \leq \int_t^{t+1} W_\epsilon(s) ds + \frac{2\eta}{5} \leq \eta. \tag{3.139}$$

Consequently, in both cases, (3.108) and (3.109),

$$W_\epsilon(t+1) \leq \max\{\eta, W_\epsilon(t)\}, \tag{3.140}$$

and then

$$W_\epsilon(t) \leq \max\left\{\eta, \max_{s\in[0,1]} W_\epsilon(s)\right\}, \tag{3.141}$$

for any $t \geq 0$.

Since the solution $(u, v) : [0,1] \to H$, is uniformly continuous, for any $\eta > 0$ there exists some $\tilde{\epsilon}(\eta) > 0$, such that

$$\max_{s\in[0,1]} W_\epsilon(s) \leq \eta, \tag{3.142}$$

for any $\epsilon \in (0, \tilde{\epsilon}(\eta))$. Then, (3.141) and (3.106) imply (3.107) for any $t \geq 0$, $\eta > 0$, and $0 < \epsilon < \epsilon(\eta) \equiv \min\{\hat{\epsilon}(\eta), \tilde{\epsilon}(\eta)\}$.

Next, we will prove that the orbit $\{(u(t), v(t))\}_{t\geq 0}$, is a precompact subset of H. We start with $\{v(t)\}_{t\geq 0} \subset L^2(\Omega)$.

Notice that because of (3.107)

$$\begin{aligned}
\left\| v(t) - \frac{1}{\epsilon}\int_t^{t+\epsilon} v(\tau)d\tau \right\|_2 &\leq \frac{1}{\epsilon}\int_t^{t+\epsilon} \|v(t) - v(\tau)\|_2 d\tau \\
&\leq \sup_{\tau\in[t,t+\epsilon]} \|v(t) - v(\tau)\|_2 \\
&\leq \sqrt{2\eta}.
\end{aligned} \tag{3.143}$$

Since $\{u(t)\}_{t\geq 0}$ is bounded in B, then

$$\begin{aligned}
\left\| \frac{1}{\epsilon}\left(\int_t^{t+\epsilon} v(\tau)d\tau\right) \right\|_B &\leq \frac{1}{\epsilon}\|u(t+\epsilon) - u(t)\|_B \\
&\leq \frac{2}{\epsilon}\sup_{t\geq 0} \|u(t)\|_B \\
&\leq \frac{2}{\epsilon}\sqrt{2K}.
\end{aligned} \tag{3.144}$$

Consequently, $\{(1/\epsilon)\int_t^{t+\epsilon} v(\tau)d\tau\}_{t\geq 0}$ is precompact or, equivalently, totally bounded in $L^2(\Omega)$, because $B \subset L^2(\Omega)$ is compact. Hence, by (3.143), $\{v(t)\}_{t\geq 0}$ is precompact in $L^2(\Omega)$.

Like in (3.143), from (3.107), we obtain that

$$\left\| u(t) - \frac{1}{\epsilon}\int_t^{t+\epsilon} u(\tau)d\tau \right\|_B \leq \sup_{\tau\in[t,t+\epsilon]} \|u(t) - u(\tau)\|_B \leq \sqrt{2\eta}. \tag{3.145}$$

If $\{\Delta^2 \int_t^{t+\epsilon} u(\tau)d\tau\}_{t\geq 0}$ is precompact in $B' \equiv$ dual space of B, then precompactness of $\{u(t)\}_{t\geq 0}$ in B follows from (3.145), since

$$\mathcal{L}^{-1} \equiv \left(\Delta^2\right)^{-1} : B' \longrightarrow B, \tag{3.146}$$

is a linear and continuous operator.

According to the dense and continuous inclusions

$$B \subset L^{\lambda}(\Omega) \subset L^2(\Omega) \subset L^{\lambda^*}(\Omega) \subset B', \tag{3.147}$$

where $L^{\lambda^*}(\Omega) = (L^{\lambda}(\Omega))'$, $\lambda^* = \lambda/(\lambda-1)$, we extend the inner product in $L^2(\Omega)$ to the duality product in $B' \times B$. Now, we integrate the wave equation, and since $\mathcal{L} : B \to B'$ is closed, we get, in the sense of B',

$$\begin{aligned}\Delta^2 \int_t^{t+\epsilon} u(\tau)d\tau = v(t) - v(t+\epsilon) + \int_t^{t+\epsilon} M\left(\|\nabla u(\tau)\|_2^2\right)\Delta u(\tau)d\tau \\ - \int_t^{t+\epsilon} g(v(\tau))d\tau + \int_t^{t+\epsilon} f(u(\tau))d\tau.\end{aligned} \tag{3.148}$$

By the Hölder inequality and (3.126),

$$\left\|\int_t^{t+\epsilon} g(v(\tau))d\tau\right\|_{\lambda^*} \leq \delta \int_t^{t+\epsilon} \|(v(\tau))\|_{\lambda}^{\lambda-1} d\tau \leq \delta^{1/\lambda}(E_0 - d)^{(\lambda-1)/\lambda}. \tag{3.149}$$

Boundedness of $u(t)$ in B, since $B \subset L^{2(r-1)}(\Omega)$, yields the estimate

$$\begin{aligned}\left\|\int_t^{t+\epsilon} f(u(\tau))d\tau\right\|_2 &\leq \mu \int_t^{t+\epsilon} \|(u(\tau))\|_{2(r-1)}^{r-1} d\tau \\ &\leq \mu C(\Omega) \sup_{t\geq 0} \|u(t)\|_B \\ &\leq \mu C(\Omega) C(K).\end{aligned} \tag{3.150}$$

Also,

$$\left\|\int_t^{t+\epsilon} M\left(\|\nabla u(\tau)\|_2^2\right)\Delta u(\tau)d\tau\right\|_2 \leq \sup_{t\geq 0}\left\{M\left(\|\nabla u(t)\|_2^2\right)\|u(t)\|_B\right\} \leq C(K), \tag{3.151}$$

Therefore, (3.149)–(3.151) in (3.148) imply that

$$\left\|\Delta^2 \int_t^{t+\epsilon} u(\tau)d\tau\right\|_{\lambda^*} \leq C, \tag{3.152}$$

for some constant $C > 0$ and every $t \geq 0$.

$B \subset L^{\lambda}(\Omega)$ is compact by assumption. By Schauder Theorem, see for instance Brézis [27], $B \subset L^{\lambda}(\Omega)$ is compact if and only if $L^{\lambda^*}(\Omega) \subset B'$ is compact. Then, (3.152) implies that $\{\Delta^2 \int_t^{t+\epsilon} u(\tau)d\tau\}_{t\geq 0}$ is precompact in B'.

Necessity

Suppose that the solution converges, strongly in H, to the set $\mathcal{E}_\infty$. Since the energy is nonincreasing, $E(t) \geq E_\infty \equiv \lim_{t\to\infty} E(t) \geq d$, for all $t \geq 0$, and the proof is complete. □

Remark 3.9. By Theorem 3.3 and (2.35), every H-neighborhood of the ground state, $\overline{\mathcal{S}} \cap \overline{\mathcal{U}} = [(u_e, 0) \in H : u_e \in \mathcal{N}^*]$, is connected, through an orbit, with $(0,0)$. Furthermore, if $E_\infty = d$ in Theorem 3.8, the ground state attracts every solution such that $(u(t), v(t)) \in [E(u,v) \geq d]$, for every $t \geq 0$.

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