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*Research Article*

# Asymptotic Convergence of the Solutions of a Discrete Equation with Two Delays in the Critical Case

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A discrete equation $\Delta y(n) = \beta(n)[y(n-j) - y(n-k)]$ with two integer delays $k$ and $j$, $k > j \geq 0$ is considered for $n \to \infty$. We assume $\beta : \mathbb{Z}_{n_0-k}^{\infty} \to (0,\infty)$, where $\mathbb{Z}_{n_0}^{\infty} = \{n_0, n_0+1, \ldots\}$, $n_0 \in \mathbb{N}$ and $n \in \mathbb{Z}_{n_0}^{\infty}$. Criteria for the existence of strictly monotone and asymptotically convergent solutions for $n \to \infty$ are presented in terms of inequalities for the function $\beta$. Results are sharp in the sense that the criteria are valid even for some functions $\beta$ with a behavior near the so-called critical value, defined by the constant $(k-j)^{-1}$. Among others, it is proved that, for the asymptotic convergence of all solutions, the existence of a strictly monotone and asymptotically convergent solution is sufficient.

## 1. Introduction

We use the following notation: for integers $s, q, s \leq q$, we define $\mathbb{Z}_s^q := \{s, s+1, \ldots, q\}$, where the cases $s = -\infty$ and $q = \infty$ are admitted too. Throughout this paper, using the notation $\mathbb{Z}_s^q$ or another one with a pair of integers $s$, $q$, we assume $s \leq q$.

In this paper we study a discrete equation with two delays

$$\Delta y(n) = \beta(n)\left[y(n-j) - y(n-k)\right] \tag{1.1}$$

as $n \to \infty$. Integers $k$ and $j$ in (1.1) satisfy the inequality $k > j \geq 0$ and $\beta : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^+ := (0,\infty)$, where $n_0 \in \mathbb{N}$ and $n \in \mathbb{Z}_{n_0}^{\infty}$. Without loss of generality, we assume $n_0 - k > 0$ throughout the paper (this is a technical detail, necessary for some expressions to be well defined).

The results concern the asymptotic convergence of all solutions of (1.1). We focus on what is called the *critical case* (with respect to the function $\beta$) which separates the case when all solutions are convergent from the case when there exist divergent solutions.

Such a critical case is characterized by the constant value

$$\beta(n) \equiv \beta_{\text{cr}} := (k-j)^{-1}, \quad n \in \mathbb{Z}^{\infty}_{n_0-k}, \tag{1.2}$$

and below we explain its meaning and importance by an analysis of the asymptotic behavior of solutions of (1.1).

Consider (1.1) with $\beta(n) = \beta_0$, where $\beta_0$ is a positive constant; that is, we consider the following equation:

$$\Delta y(n) = \beta_0 \cdot [y(n-j) - y(n-k)]. \tag{1.3}$$

Looking for a solution of (1.3) in the form $y(n) = \lambda^n$, $\lambda \in \mathbb{C} \setminus \{0\}$ using the usual procedure, we get the characteristic equation

$$\lambda^{k+1} - \lambda^k = \beta_0 \cdot \left[\lambda^{k-j} - 1\right]. \tag{1.4}$$

Denote its roots by $\lambda_i$, $i = 1, \ldots, k+1$. Then characteristic equation (1.4) has a root $\lambda_{k+1} = 1$. Related solution of (1.3) is $y_{k+1}(n) = 1$. Then there exists a one-parametric family of constant solutions of (1.3) $y(n) = c_{k+1} y_{k+1}(n) = c_{k+1}$, where $c_{k+1}$ is an arbitrary constant. Equation (1.4) can be rewritten as

$$\lambda^k(\lambda - 1) = \beta_0 \cdot (\lambda - 1)\left(\lambda^{k-j-1} + \lambda^{k-j-2} + \cdots + 1\right), \tag{1.5}$$

and, instead of (1.4), we can consider the following equation:

$$f(\lambda) := \lambda^k - \beta_0 \cdot \left(\lambda^{k-j-1} + \lambda^{k-j-2} + \cdots + 1\right) = 0. \tag{1.6}$$

Let $\beta_0 = \beta_{\text{cr}}$. Then (1.6) has a root $\lambda_k = 1$ which is a double root of (1.4). By the theory of linear difference equations, (1.3) has a solution $y_k(n) = n$, linearly independent with $y_{k+1}(n)$. There exists a two-parametric family of solutions of (1.3)

$$y(n) = c_k y_k(n) + c_{k+1} y_{k+1}(n) = c_k n + c_{k+1}, \tag{1.7}$$

where $c_k$, $c_{k+1}$ are arbitrary constants. Then $\lim_{n\to\infty} y(n) = \infty$ if $c_k \neq 0$. This means that solutions with $c_k \neq 0$ are divergent.

Let $\beta_0 < \beta_{\text{cr}}$ and $k - j > 1$. We define two functions of a complex variable $\lambda$

$$F(\lambda) := \lambda^k, \qquad \Psi(\lambda) := \beta_0 \cdot \left(\lambda^{k-j-1} + \lambda^{k-j-2} + \cdots + 1\right), \tag{1.8}$$

and (1.6) can be written as

$$F(\lambda) - \Psi(\lambda) = 0. \tag{1.9}$$

By Rouche's theorem, all roots $\lambda_i$, $i = 1, 2, \ldots, k$ of (1.6) satisfy $|\lambda_i| < 1$ because, on the boundary $C$ of a unit circle $|\lambda| < 1$, we have

$$|\Psi(\lambda)|_C = \beta_0 \cdot \left|\lambda^{k-j-1} + \lambda^{k-j-2} + \cdots + 1\right| < \frac{1}{k-j}(k-j) = 1 = |F(\lambda)|_C, \tag{1.10}$$

and the functions $F(\lambda)$, $F(\lambda) - \Psi(\lambda)$ have the same number of zeros in the domain $|\lambda| < 1$.

The case $\beta_0 < \beta_{\text{cr}}$ and $k - j = 1$ is trivial because (1.6) turns into

$$\lambda^k - \beta_0 = 0 \tag{1.11}$$

and, due to inequality $|\lambda|^k = \beta_0 < \beta_{\text{cr}} = 1$, has all its roots in the domain $|\lambda| < 1$.

Then the relevant solutions $y_i(n)$, $i = 1, 2, \ldots, k$ satisfy $\lim_{n\to\infty} y_i(n) = 0$, and the limit of the general solution of (1.3), $y(n) = \lim_{n\to\infty} \sum_{i=1}^{k+1} c_i y_i(n)$ where $c_i$ are arbitrary constants, is finite because

$$\lim_{n\to\infty} y(n) = \lim_{n\to\infty} \sum_{i=1}^{k+1} c_i y_i(n) = c_{k+1}. \tag{1.12}$$

Let $\beta_0 > \beta_{\text{cr}}$. Since $f(1) = 1 - \beta_0 \cdot (k-j) < 0$ and $f(+\infty) = +\infty$, there exists a root $\lambda = \lambda_* > 1$ of (1.6) and a solution $y_*(n) = (\lambda_*)^n$ of (1.3) satisfying $\lim_{n\to\infty} y_*(n) = \infty$. This means that solution $y_*(n)$ is divergent.

Gathering all the cases considered, we have the following:

(i) if $0 < \beta_0 < \beta_{\text{cr}}$, then all solutions of (1.3) have a finite limit as $n \to \infty$,

(ii) if $\beta_0 \geq \beta_{\text{cr}}$, then there exists a divergent solution of (1.3) when $n \to \infty$.

The above analysis is not applicable in the case of a nonconstant function $\beta(n)$ in (1.1). To overcome some difficulties, the method of auxiliary inequalities is applied to investigate (1.1). From our results it follows that, for example, all solutions of (1.1) have a finite limit for $n \to \infty$ (or, in accordance with the below definition, are asymptotically convergent) if there exists a $p > 1$ such that the inequality

$$\beta(n) \leq \frac{1}{k-j} - \frac{p(k+j+1)}{2n(k-j)} \tag{1.13}$$

holds for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$, where $n_0$ is a sufficiently large natural number. The limit of the right-hand side of (1.13) as $n \to \infty$ equals the critical value $\beta_{\text{cr}}$:

$$\lim_{n\to\infty} \left( \frac{1}{k-j} - \frac{p(k+j+1)}{2n(k-j)} \right) = \frac{1}{k-j} = \beta_{\text{cr}}. \tag{1.14}$$

It means that the function $\beta(n)$ in (1.1) can be sufficiently close to the critical value $\beta_{\mathrm{cr}}$ but such that all solutions of (1.1) are convergent as $n \to \infty$.

The proofs of the results are based on comparing the solutions of (1.1) with those of an auxiliary inequality that formally copies (1.1). First, we prove that, under certain conditions, (1.1) has an increasing and convergent solution $y = y(n)$ (i.e., there exists a finite limit $\lim_{n\to\infty} y(n)$). Then we extend this statement to all the solutions of (1.1). It is an interesting fact that, in the general case, the asymptotic convergence of all solutions is characterized by the existence of a strictly increasing and bounded solution.

The problem concerning the asymptotic convergence of solutions in the continuous case, that is, in the case of delayed differential equations or other classes of equations, is a classical one and has attracted much attention recently. The problem of the asymptotic convergence of solutions of discrete and difference equations with delay has not yet received much attention. We mention some papers from both of these fields (in most of them, equations and systems with a structure similar to the discrete equation (1.1) are considered).

Arino and Pituk [1], for example, investigate linear and nonlinear perturbations of a linear autonomous functional-differential equation which has infinitely many equilibria. Bereketoğlu and Karakoç [2] derive sufficient conditions for the asymptotic constancy and estimates of the limits of solutions for an impulsive system, and Györi et al. give sufficient conditions for the convergence of solutions of a nonhomogeneous linear system of impulsive delay differential equations and a limit formula in [3]. Bereketoğlu and Pituk [4] give sufficient conditions for the asymptotic constancy of solutions of nonhomogeneous linear delay differential equations with unbounded delay. The limits of the solutions can be computed in terms of the initial conditions and a special matrix solution of the corresponding adjoint equation. In [5] Diblík studies the scalar equation under the assumption that every constant is its solution. Criteria and sufficient conditions for the convergence of solutions are found. The paper by Diblík and Růžičková [6] deals with the asymptotic behavior of a first-order linear homogeneous differential equation with double delay. The convergence of solutions of the delay Volterra equation in the critical case is studied by Messina et al. in [7]. Berezansky and Braverman study a behavior of solutions of a food-limited population model with time delay in [8].

Bereketoğlu and Huseynov [9] give sufficient conditions for the asymptotic constancy of the solutions of a system of linear difference equations with delays. The limits of the solutions, as $t \to \infty$, can be computed in terms of the initial function and a special matrix solution of the corresponding adjoint equation. Dehghan and Douraki [10] study the global behavior of a certain difference equation and show, for example, that zero is always an equilibrium point which satisfies a necessary and suffient condition for its local asymptotic stability. Györi and Horváth [11] study a system of linear delay difference equations such that every solution has a finite limit at infinity. The stability of difference equations is studied intensively in papers by Stević [12, 13]. In [12], for example, he proves the global asymptotic stability of a class of difference equations. Baštinec and Diblík [14] study a class of positive and vanishing at infinity solutions of a linear difference equation with delay. Nonoscillatory solutions of second-order difference equations of the Poincaré type are investigated by Medina and Pituk in [15].

Comparing the known investigations with the results presented, we can see that our results can be applied to the critical case giving strong sufficient conditions of asymptotic convergence of solutions for this case. Nevertheless, we are not concerned with computing the limits of the solutions as $n \to \infty$.

The paper is organized as follows. In Section 2 auxiliary results are collected, an auxiliary inequality is studied, and the relationship of its solutions with the solutions of (1.1) is derived. The existence of a strictly increasing and convergent solution of (1.1) is established in Section 3. Section 4 contains results concerning the convergence of all solutions of (1.1). An example illustrating the sharpness of the results derived is given as well.

Throughout the paper we adopt the customary notation $\sum_{i=k+s}^{k} \mathcal{B}(i) = 0$, where $k$ is an integer, $s$ is a positive integer, and $\mathcal{B}$ denotes the function under consideration regardless of whether it is defined for the arguments indicated or not.

## 2. Auxiliary Results

Let $\mathcal{C} := \mathcal{C}(\mathbb{Z}_{-k}^{0}, \mathbb{R})$ be the space of discrete functions mapping the discrete interval $\mathbb{Z}_{-k}^{0}$ into $\mathbb{R}$. Let $v \in \mathbb{Z}_{n_0}^{\infty}$ be given. The function $y : \mathbb{Z}_{v-k}^{\infty} \to \mathbb{R}$ is said to be a *solution of* (1.1) *on* $\mathbb{Z}_{v-k}^{\infty}$ if it satisfies (1.1) for every $n \in \mathbb{Z}_{v}^{\infty}$. A solution $y$ of (1.1) on $\mathbb{Z}_{v-k}^{\infty}$ is *asymptotically convergent* if the limit $\lim_{n\to\infty} y(n)$ exists and is finite. For a given $v \in \mathbb{Z}_{n_0}^{\infty}$ and $\varphi \in \mathcal{C}$, we say that $y = y_{(v,\varphi)}$ is a *solution of* (1.1) defined by the initial conditions $(v, \varphi)$ if $y_{(v,\varphi)}$ is a solution of (1.1) on $\mathbb{Z}_{v-k}^{\infty}$ and $y_{(v,\varphi)}(v + m) = \varphi(m)$ for $m \in \mathbb{Z}_{-k}^{0}$.

### *2.1. Auxiliary Inequality*

The auxiliary inequality

$$\Delta\omega(n) \geq \beta(n)\left[\omega(n-j) - \omega(n-k)\right] \tag{2.1}$$

will serve as a helpful tool in the analysis of (1.1). Let $v \in \mathbb{Z}_{n_0}^{\infty}$. The function $\omega : \mathbb{Z}_{v-k}^{\infty} \to \mathbb{R}$ is said to be a *solution* of (2.1) on $\mathbb{Z}_{v-k}^{\infty}$ if $\omega$ satisfies inequality (2.1) for $n \in \mathbb{Z}_{v}^{\infty}$. A solution $\omega$ of (2.1) on $\mathbb{Z}_{v-k}^{\infty}$ is *asymptotically convergent* if the limit $\lim_{n\to\infty}\omega(n)$ exists and is finite.

We give some properties of solutions of inequalities of the type (2.1), which will be utilized later on. We will also compare the solutions of (1.1) with the solutions of inequality (2.1).

**Lemma 2.1.** *Let $\varphi \in \mathcal{C}$ be strictly increasing (nondecreasing, strictly decreasing, nonincreasing) on $\mathbb{Z}_{-k}^{0}$. Then the corresponding solution $y_{(n^*,\varphi)}(n)$ of* (1.1) *with $n^* \in \mathbb{Z}_{n_0}^{\infty}$ is strictly increasing (nondecreasing, strictly decreasing, nonincreasing) on $\mathbb{Z}_{n^*-k}^{\infty}$ too.*

*If $\varphi$ is strictly increasing (nondecreasing) and $\omega : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}$ is a solution of inequality* (2.1) *with $\omega(n_0 + m) = \varphi(m)$, $m \in \mathbb{Z}_{n_0-k}^{n_0}$, then $\omega$ is strictly increasing (nondecreasing) on $\mathbb{Z}_{n_0-k}^{\infty}$.*

*Proof.* This follows directly from (1.1), inequality (2.1), and from the properties $\beta(n) > 0$, $n \in \mathbb{Z}_{n_0-k}^{\infty}$, $k > j \geq 0$. □

**Theorem 2.2.** *Let $\omega(n)$ be a solution of inequality* (2.1) *on $\mathbb{Z}_{n_0-k}^{\infty}$. Then there exists a solution $y(n)$ of* (1.1) *on $\mathbb{Z}_{n_0-k}^{\infty}$ such that the inequality*

$$y(n) \leq \omega(n) \tag{2.2}$$

*holds on* $\mathbb{Z}_{n_0-k}^{\infty}$*. In particular, a solution* $y(n_0, \phi)$ *of* (1.1) *with* $\phi \in \mathcal{C}$ *defined by the equation*

$$\phi(m) := \omega(n_0 + m), \quad m \in \mathbb{Z}_{-k}^{0} \tag{2.3}$$

*is such a solution.*

*Proof.* Let $\omega(n)$ be a solution of inequality (2.1) on $\mathbb{Z}_{n_0-k}^{\infty}$. We will show that the solution $y(n) := y_{(n_0,\phi)}(n)$ of (1.1) satisfies inequality (2.2), that is,

$$y_{(n_0,\phi)}(n) \leq \omega(n) \tag{2.4}$$

on $\mathbb{Z}_{n_0-k}^{\infty}$. Let $W : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}$ be defined by $W(n) = \omega(n) - y(n)$. Then $W = 0$ on $\mathbb{Z}_{n_0-k}^{n_0}$, and, in addition, $W$ is a solution of (2.1) on $\mathbb{Z}_{n_0-k}^{\infty}$. Lemma 2.1 implies that $W$ is nondecreasing. Consequently, $\omega(n) - y(n) \geq \omega(n_0) - y(n_0) = 0$ for all $n \geq n_0$. □

## *2.2. Comparison Lemma*

Now we consider an inequality of the type (2.1)

$$\Delta\omega^*(n) \geq \beta_1(n)\left[\omega^*(n-j) - \omega^*(n-k)\right], \tag{2.5}$$

where $\beta_1 : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^+$ is a discrete function satisfying $\beta_1(n) \geq \beta(n)$ on $\mathbb{Z}_{n_0-k}^{\infty}$. The following comparison lemma holds.

**Lemma 2.3.** *Let* $\omega^* : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^+$ *be a nondecreasing positive solution of inequality* (2.5) *on* $\mathbb{Z}_{n_0-k}^{\infty}$*. Then* $\omega^*$ *is a solution of inequality* (2.1) *on* $\mathbb{Z}_{n_0-k}^{\infty}$ *too.*

*Proof.* Let $\omega^*$ be a nondecreasing solution of (2.5) on $\mathbb{Z}_{n_0-k}^{\infty}$. We have

$$\omega^*(n-j) - \omega^*(n-k) \geq 0 \tag{2.6}$$

because $n - k < n - j$. Then

$$\Delta\omega^*(n) \geq \beta_1(n)\left[\omega^*(n-j) - \omega^*(n-k)\right] \geq \beta(n)\left[\omega^*(n-j) - \omega^*(n-k)\right] \tag{2.7}$$

on $\mathbb{Z}_{n_0}^{\infty}$. Consequently, the function $\omega := \omega^*$ solves inequality (2.1) on $\mathbb{Z}_{n_0}^{\infty}$, too. □

## *2.3. A Solution of Inequality* (2.1)

We will construct a solution of inequality (2.1). In the following lemma, we obtain a solution of inequality (2.1) in the form of a sum. This auxiliary result will help us derive sufficient conditions for the existence of a strictly increasing and asymptotically convergent solution of (1.1) (see Theorem 3.2 below).

**Lemma 2.4.** *Let there exist a discrete function* $\varepsilon : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^+$ *such that*

$$\varepsilon(n+1) \geq \sum_{i=n-k+1}^{n-j} \beta(i-1)\varepsilon(i) \tag{2.8}$$

*on* $\mathbb{Z}_{n_0}^{\infty}$. *Then there exists a solution* $\omega(n) = \omega_{\varepsilon}(n)$ *of inequality* (2.1) *defined on* $\mathbb{Z}_{n_0-k}^{\infty}$ *having the form*

$$\omega_{\varepsilon}(n) := \sum_{i=n_0-k+1}^{n} \beta(i-1)\varepsilon(i). \tag{2.9}$$

*Proof.* For $n \in \mathbb{Z}_{n_0}^{\infty}$, we get

$$\begin{aligned} \Delta\omega_{\varepsilon}(n) &= \omega_{\varepsilon}(n+1) - \omega_{\varepsilon}(n) \\ &= \sum_{i=n_0-k+1}^{n+1} \beta(i-1)\varepsilon(i) - \sum_{i=n_0-k+1}^{n} \beta(i-1)\varepsilon(i) \\ &= \beta(n)\varepsilon(n+1), \\ \omega_{\varepsilon}(n-j) - \omega_{\varepsilon}(n-k) &= \sum_{i=n_0-k+1}^{n-j} \beta(i-1)\varepsilon(i) - \sum_{i=n_0-k+1}^{n-k} \beta(i-1)\varepsilon(i) \\ &= \sum_{i=n-k+1}^{n-j} \beta(i-1)\varepsilon(i). \end{aligned} \tag{2.10}$$

We substitute $\omega_{\varepsilon}$ for $\omega$ in (2.1). Using (2.10), we get

$$\beta(n)\varepsilon(n+1) \geq \beta(n) \sum_{n-k+1}^{n-j} \beta(i-1)\varepsilon(i). \tag{2.11}$$

This inequality will be satisfied if inequality (2.8) holds. Indeed, reducing the last inequality by $\beta(n)$, we obtain

$$\varepsilon(n+1) \geq \sum_{n-k+1}^{n-j} \beta(i-1)\varepsilon(i), \tag{2.12}$$

which is inequality (2.8). □

### 2.4. Decomposition of a Function into the Difference of Two Strictly Increasing Functions

It is well known that every absolutely continuous function is representable as the difference of two increasing absolutely continuous functions [16, page 318]. We will need a simple discrete analogue of this result.

**Lemma 2.5.** *Every function $\varphi \in \mathcal{C}$ can be decomposed into the difference of two strictly increasing functions $\varphi_j \in \mathcal{C}$, $j = 1, 2$, that is,*

$$\varphi(n) = \varphi_1(n) - \varphi_2(n), \quad n \in \mathbb{Z}_{-k}^{0}. \tag{2.13}$$

*Proof.* Let constants $M_n > 0$, $n \in \mathbb{Z}_{-k}^{0}$ be such that inequalities

$$M_{n+1} > M_n + \max\{0, \varphi(n) - \varphi(n+1)\} \tag{2.14}$$

are valid for $n \in \mathbb{Z}_{-k}^{-1}$. We set

$$\begin{aligned} \varphi_1(n) &:= \varphi(n) + M_n, \quad n \in \mathbb{Z}_{-k}^{0}, \\ \varphi_2(n) &:= M_n, \quad n \in \mathbb{Z}_{-k}^{0}. \end{aligned} \tag{2.15}$$

It is obvious that (2.13) holds. Now we verify that both functions $\varphi_j$, $j = 1, 2$ are strictly increasing. The first one should satisfy $\varphi_1(n+1) > \varphi_1(n)$ for $n \in \mathbb{Z}_{-k}^{-1}$, which means that

$$\varphi(n+1) + M_{n+1} > \varphi(n) + M_n \tag{2.16}$$

or

$$M_{n+1} > M_n + \varphi(n) - \varphi(n+1). \tag{2.17}$$

We conclude that the last inequality holds because, due to (2.14), we have

$$M_{n+1} > M_n + \max\{0, \varphi(n) - \varphi(n+1)\} \geq M_n + \varphi(n) - \varphi(n+1). \tag{2.18}$$

The inequality $\varphi_2(n+1) > \varphi_2(n)$ obviously holds for $n \in \mathbb{Z}_{-k}^{-1}$ due to (2.14) as well. □

### 2.5. Auxiliary Asymptotic Decomposition

The following lemma can be proved easily by induction. The symbol $\mathcal{O}$ stands for the Landau order symbol.

**Lemma 2.6.** *For fixed* $r, \sigma \in \mathbb{R} \setminus \{0\}$, *the asymptotic representation*

$$(n-r)^{\sigma} = n^{\sigma}\left[1 - \frac{\sigma r}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \tag{2.19}$$

*holds for* $n \to \infty$.

## 3. Convergent Solutions of (1.1)

This part deals with the problem of detecting the existence of asymptotically convergent solutions. The results shown below provide sufficient conditions for the existence of an asymptotically convergent solution of (1.1). First we present two obvious statements concerning asymptotic convergence. From Lemma 2.1 and Theorem 2.2, we immediately get the following.

**Theorem 3.1.** *Let* $\omega(n)$ *be a strictly increasing and bounded solution of* (2.1) *on* $\mathbb{Z}_{n_0-k}^{\infty}$. *Then there exists a strictly increasing and asymptotically convergent solution* $y(n)$ *of* (1.1) *on* $\mathbb{Z}_{n_0-k}^{\infty}$.

From Lemma 2.1, Theorem 2.2, and Lemma 2.4, we get the following.

**Theorem 3.2.** *Let there exist a function* $\varepsilon : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^{+}$ *satisfying*

$$\sum_{i=n_0-k+1}^{\infty} \beta(i-1)\varepsilon(i) < \infty \tag{3.1}$$

*and inequality* (2.8) *on* $\mathbb{Z}_{n_0}^{\infty}$. *Then the initial function*

$$\varphi(n) = \sum_{i=n_0-k+1}^{n_0+n} \beta(i-1)\varepsilon(i), \quad n \in \mathbb{Z}_{-k}^{0} \tag{3.2}$$

*defines a strictly increasing and asymptotically convergent solution* $y_{(n_0,\varphi)}(n)$ *of* (1.1) *on* $\mathbb{Z}_{n_0-k}^{\infty}$ *satisfying the inequality*

$$y(n) \le \sum_{i=n_0-k+1}^{n} \beta(i-1)\varepsilon(i) \tag{3.3}$$

*on* $\mathbb{Z}_{n_0}^{\infty}$.

Assuming that the coefficient $\beta(n)$ in (1.1) can be estimated by a suitable function, we can prove that (1.1) has a convergent solution.

**Theorem 3.3.** *Let there exist a* $p > 1$ *such that the inequality*

$$\beta(n) \le \frac{1}{k-j} - \frac{p(k+j+1)}{2n(k-j)} \tag{3.4}$$

*holds for all* $n \in \mathbb{Z}_{n_0-k}^{\infty}$*. Then there exists a strictly increasing and asymptotically convergent solution* $y(n)$ *of* (1.1) *as* $n \to \infty$.

*Proof.* In the proof, we assume (without loss of generality) that $n_0$ is sufficiently large for asymptotic computations to be valid. Let us verify that inequality (2.8) has a solution $\varepsilon$ such that

$$\sum_{i=n_0-k+1}^{\infty} \beta(i-1)\varepsilon(i) < \infty. \tag{3.5}$$

We put

$$\beta(n) = \beta^*(n) := \frac{1}{k-j} - \frac{p^*}{2n}, \qquad \varepsilon(n) := \frac{1}{n^{\alpha}} \tag{3.6}$$

in inequality (2.8), where $p^* > 0$ and $\alpha > 1$ are constants. Then, for the right-hand side $\mathcal{R}(n)$ of (2.8), we have

$$\begin{aligned}
\mathcal{R}(n) &= \sum_{i=n-k+1}^{n-j} \left[\frac{1}{k-j} - \frac{p^*}{2(i-1)}\right] \frac{1}{i^{\alpha}} \\
&= \frac{1}{k-j} \sum_{i=n-k+1}^{n-j} \frac{1}{i^{\alpha}} - \frac{p^*}{2} \sum_{i=n-k+1}^{n-j} \frac{1}{(i-1)i^{\alpha}} \\
&= \frac{1}{k-j} \left[\frac{1}{(n-k+1)^{\alpha}} + \frac{1}{(n-k+2)^{\alpha}} + \cdots + \frac{1}{(n-j)^{\alpha}}\right] \\
&\quad - \frac{p^*}{2} \left[\frac{1}{(n-k)(n-k+1)^{\alpha}} + \frac{1}{(n-k+1)(n-k+2)^{\alpha}} + \cdots + \frac{1}{(n-j-1)(n-j)^{\alpha}}\right].
\end{aligned} \tag{3.7}$$

We asymptotically decompose $\mathcal{R}(n)$ as $n \to \infty$ using decomposition formula (2.19) in Lemma 2.6. We apply this formula to each term in the first square bracket with $\sigma = -\alpha$ and with $r = k-1$ for the first term, $r = k-2$ for the second term, and so forth, and, finally, $r = j$ for the last term. To estimate the terms in the second square bracket, we need only the first terms of the decomposition and the order of accuracy, which can be computed easily without using Lemma 2.6. We get

$$\begin{aligned}
\mathcal{R}(n) &= \frac{1}{(k-j)n^{\alpha}} \left[1 + \frac{\alpha(k-1)}{n} + 1 + \frac{\alpha(k-2)}{n} + \cdots + 1 + \frac{\alpha j}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&\quad - \frac{p^*}{2n^{\alpha+1}} \left[1 + 1 + \cdots + 1 + \mathcal{O}\left(\frac{1}{n}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(k-j)n^{\alpha+1}} \left[ (k-j)n + \alpha(k-1) + \alpha(k-2) + \cdots + \alpha j + \mathcal{O}\left(\frac{1}{n}\right) \right] \\
&\quad - \frac{p^*}{2n^{\alpha+1}} \left[ (k-j) + \mathcal{O}\left(\frac{1}{n}\right) \right] \\
&= \frac{1}{n^\alpha} + \frac{\alpha}{(k-j)n^{\alpha+1}} \frac{(k+j-1)(k-j)}{2} - \frac{p^*}{2n^{\alpha+1}}(k-j) + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right),
\end{aligned}
\tag{3.8}
$$

and, finally,

$$
\mathcal{R}(n) = \frac{1}{n^\alpha} + \frac{\alpha}{2n^{\alpha+1}}(k+j-1) - \frac{p^*}{2n^{\alpha+1}}(k-j) + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right). \tag{3.9}
$$

A similar decomposition of the left-hand side $\mathcal{L}(n) = \varepsilon(n+1) = (n+1)^{-\alpha}$ in inequality (2.8) leads to

$$
\mathcal{L}(n) \equiv \frac{1}{(n+1)^\alpha} = \frac{1}{n^\alpha}\left[1 - \frac{\alpha}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] = \frac{1}{n^\alpha} - \frac{\alpha}{n^{\alpha+1}} + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right) \tag{3.10}
$$

(we use decomposition formula (2.19) in Lemma 2.6 with $\sigma = -\alpha$ and $r = -1$).

Comparing $\mathcal{L}(n)$ and $\mathcal{R}(n)$, we see that, for $\mathcal{L}(n) \geq \mathcal{R}(n)$, it is necessary to match the coefficients of the terms $n^{-\alpha-1}$ because the coefficients of the terms $n^{-\alpha}$ are equal. It means that we need the inequality

$$
-\alpha > \frac{\alpha(k+j-1)}{2} - \frac{p^*}{2}(k-j). \tag{3.11}
$$

Simplifying this inequality, we get

$$
\begin{gathered}
\frac{p^*}{2}(k-j) > \alpha + \frac{\alpha(k+j-1)}{2}, \\
p^*(k-j) > \alpha(k+j+1),
\end{gathered}
\tag{3.12}
$$

and, finally,

$$
p^* > \frac{\alpha(k+j+1)}{k-j}. \tag{3.13}
$$

We set

$$
p^* := p\,\frac{k+j+1}{k-j}, \tag{3.14}
$$

where $p = \text{const}$. Then the previous inequality holds for $p > \alpha$, that is, for $p > 1$. Consequently, the function $\beta^*$ defined by (3.6) has the form

$$\beta^*(n) = \frac{1}{k-j} - \frac{p(k+j+1)}{2(k-j)n} \tag{3.15}$$

with $p > 1$, and, for the function $\omega_\varepsilon$ defined by formula (2.9), we have

$$\omega_\varepsilon(n) = \sum_{i=n_0-k+1}^{n} \left( \frac{1}{k-j} - \frac{p(k+j+1)}{2(k-j)(i-1)} \right) \frac{1}{i^\alpha}. \tag{3.16}$$

Function $\omega_\varepsilon(n)$ is a positive solution of inequality (2.1), and, moreover, it is easy to verify that $\omega_\varepsilon(\infty) < \infty$ since $\alpha > 1$. This is a solution to every inequality of the type (2.1) if the function $\beta^*$ fixed by formula (3.15) is changed by an arbitrary function $\beta$ satisfying inequality (3.4). This is a straightforward consequence of Lemma 2.3 if, in its formulation, we set

$$\beta_1(n) := \beta^*(n) = \frac{1}{k-j} - \frac{p(k+j+1)}{2(k-j)n} \tag{3.17}$$

with $p > 1$ since $\omega^* \equiv \omega_\varepsilon$ is the desired solution of inequality (2.5). Finally, by Theorem 3.1 with $\omega := \omega_\varepsilon$ as defined by (3.16), we conclude that there exists a strictly increasing and convergent solution $y(n)$ of (1.1) as $n \to \infty$ satisfying the inequality

$$y(n) < \omega_\varepsilon(n) \tag{3.18}$$

on $\mathbb{Z}_{n_0-k}^{\infty}$. □

## 4. Convergence of All Solutions

In this part we present results concerning the convergence of all solutions of (1.1). First we use inequality (3.4) to state the convergence of all the solutions.

**Theorem 4.1.** *Let there exist a $p > 1$ such that inequality* (3.4) *holds for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$. Then all solutions of* (1.1) *are convergent as $n \to \infty$.*

*Proof.* First we prove that every solution defined by a monotone initial function is convergent. We will assume that a strictly monotone initial function $\varphi \in \mathcal{C}$ is given. For definiteness, let $\varphi$ be strictly increasing or nondecreasing (the case when it is strictly decreasing or nonincreasing can be considered in much the same way). By Lemma 2.1, the solution $y_{(n_0,\varphi)}$ is monotone; that is, it is either strictly increasing or nondecreasing. We prove that $y_{(n_0,\varphi)}$ is convergent.

By Theorem 3.3 there exists a strictly increasing and asymptotically convergent solution $y = Y(n)$ of (1.1) on $\mathbb{Z}_{n_0-k}^{\infty}$. Without loss of generality we assume $y_{(n_0,\varphi)} \not\equiv Y(n)$ on

$\mathbb{Z}_{n_0-k}^{\infty}$ since, in the opposite case, we can choose another initial function. Similarly, without loss of generality, we can assume

$$\Delta Y(n) > 0, \quad n \in \mathbb{Z}_{n_0-k}^{n_0-1}. \tag{4.1}$$

Hence, there is a constant $\gamma > 0$ such that

$$\Delta Y(n) - \gamma \Delta y(n) > 0, \quad n \in \mathbb{Z}_{n_0-k}^{n_0-1} \tag{4.2}$$

or

$$\Delta\big(Y(n) - \gamma y(n)\big) > 0, \quad n \in \mathbb{Z}_{n_0-k}^{n_0-1}, \tag{4.3}$$

and the function $Y(n) - \gamma y(n)$ is strictly increasing on $\mathbb{Z}_{n_0-k}^{n_0-1}$. Then Lemma 2.1 implies that $Y(n) - \gamma y(n)$ is strictly increasing on $\mathbb{Z}_{n_0-k}^{\infty}$. Thus

$$Y(n) - \gamma y(n) > Y(n_0) - \gamma y(n_0), \quad n \in \mathbb{Z}_{n_0}^{\infty} \tag{4.4}$$

or

$$y(n) < \frac{1}{\gamma}(Y(n) - Y(n_0)) + y(n_0), \quad n \in \mathbb{Z}_{n_0}^{\infty}, \tag{4.5}$$

and, consequently, $y(n)$ is a bounded function on $\mathbb{Z}_{n_0-k}^{\infty}$ because of the boundedness of $Y(n)$. Obviously, in such a case, $y(n)$ is asymptotically convergent and has a finite limit.

Summarizing the previous section, we state that every monotone solution is convergent. It remains to consider a class of all nonmonotone initial functions. For the behavior of a solution $y_{(n_0,\varphi)}$ generated by a nonmonotone initial function $\varphi \in \mathcal{C}$, there are two possibilities: $y_{(n_0,\varphi)}$ is either eventually monotone and, consequently, convergent, or $y_{(n_0,\varphi)}$ is eventually nonmonotone.

Now we use the statement of Lemma 2.5 that every discrete function $\varphi \in \mathcal{C}$ can be decomposed into the difference of two strictly increasing discrete functions $\varphi_j \in \mathcal{C}$, $j = 1, 2$. In accordance with the previous part of the proof, every function $\varphi_j \in \mathcal{C}$, $j = 1, 2$ defines a strictly increasing and asymptotically convergent solution $y_{(n_0,\varphi_j)}$. Now it is clear that the solution $y_{(n_0,\varphi)}$ is asymptotically convergent. □

We will finish the paper with two obvious results. Inequality (3.4) in Theorem 3.3 was necessary only for the proof of the existence of an asymptotically convergent solution. If we assume the existence of an asymptotically convergent solution rather than inequality (3.4), we can formulate the following result, the proof of which is an elementary modification of the proof of Theorem 4.1.

**Theorem 4.2.** *If* (1.1) *has a strictly monotone and asymptotically convergent solution on* $\mathbb{Z}_{n_0-k}^{\infty}$*, then all the solutions of* (1.1) *defined on* $\mathbb{Z}_{n_0-k}^{\infty}$ *are asymptotically convergent.*

Combining the statements of Theorems 2.2, 3.1, and 4.2, we get a series of equivalent statements below.

**Theorem 4.3.** *The following three statements are equivalent.*

(a) *Equation* (1.1) *has a strictly monotone and asymptotically convergent solution on* $\mathbb{Z}^{\infty}_{n_0-k}$.

(b) *All solutions of* (1.1) *defined on* $\mathbb{Z}^{\infty}_{n_0-k}$ *are asymptotically convergent.*

(c) *Inequality* (2.1) *has a strictly monotone and asymptotically convergent solution on* $\mathbb{Z}^{\infty}_{n_0-k}$.

*Example 4.4.* We will demonstrate the sharpness of the criterion (3.4) by the following example. Let $k = 1$, $j = 0$, $\beta(n) = 1 - 1/n$, $n \in \mathbb{Z}^{\infty}_{n_0-1}$, $n_0 = 2$ in (1.1); that is, we consider the equation

$$\Delta y(n) = \left(1 - \frac{1}{n}\right)\left[y(n) - y(n-1)\right]. \tag{4.6}$$

By Theorems 3.3 and 4.3, all solutions are asymptotically convergent if

$$\beta(n) \leq \frac{1}{k-j} - \frac{p(k+j+1)}{2n(k-j)} = 1 - \frac{p}{n}, \tag{4.7}$$

where a constant $p > 1$. In our case the inequality (4.7) does not hold since inequality

$$\beta(n) = 1 - \frac{1}{n} \leq 1 - \frac{p}{n} \tag{4.8}$$

is valid only for $p \leq 1$. Inequality (4.7) is sharp because there exists a solution $y = y^*(n)$ of (4.6) having the form of an $n$th partial sum of harmonic series, that is,

$$y^*(n) = \sum_{i=1}^{n} \frac{1}{i} \tag{4.9}$$

with the obvious property $\lim_{n\to\infty} y^*(n) = +\infty$. Then (by Theorem 4.3), all strictly monotone (increasing or decreasing) solutions of (4.6) tend to infinity.

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