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Research Article

Stability and Bifurcation Analysis in a Class of Two-Neuron Networks with Resonant Bilinear Terms

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A class of two-neuron networks with resonant bilinear terms is considered. The stability of the zero equilibrium and existence of Hopf bifurcation is studied. It is shown that the zero equilibrium is locally asymptotically stable when the time delay is small enough, while change of stability of the zero equilibrium will cause a bifurcating periodic solution as the time delay passes through a sequence of critical values. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by using the normal form theory and center manifold theory. Finally, numerical simulations supporting the theoretical analysis are carried out.

1. Introduction

Based on the assumption that the elements in the network can respond to and communicate with each other instantaneously without time delays, Hopfield proposed Hopfield neural networks (HNNs) model in 1980s [1, 2]. During the past several years, the dynamical phenomena of neural networks have been extensively studied because of the widely application in various information processing, optimization problems, and so forth. In particular, the appearance of a cycle bifurcating from an equilibrium of an ordinary or a delayed neural network with a single parameter, which is known as a Hopf bifurcation, has attracted much attention (see [3–13]).

In 2008, Yang et al. [14] investigated the Bautin bifurcation of the two-neuron networks with resonant bilinear terms and without delay:

$$\begin{aligned} \dot{x}_1(t) &= (\alpha_1 + a) f(x_1) + (\alpha_2 + b) f(x_2) + c x_1 x_2, \\ \dot{x}_2(t) &= (\alpha_2 - b) f(x_1) + (\alpha_1 - a) f(x_2) + d x_1 x_2, \end{aligned} \tag{1.1}$$

where $x_i(t)$ $(i = 1,2)$ represents the state of the ith neuron at time t, $f(x_i)$ $(i = 1,2)$ is the connection function between two neurons, and $\alpha_1, \alpha_2, a, b, c, d$ are real parameters, and obtained a sufficient condition for a Bautin bifurcation to occur for system (1.1) by using the standard normal form theory and with Maple software. It is well known that in the implementation of networks, time delays are inevitably encountered because of the finite switching speed of signal transmission. Motivated by the viewpoint, in the following, we assume that the time delay from the first neuron to the second neuron is τ_2 and back to the first neuron is τ_1, then we have the following neural networks whose delays are introduced:

$$\begin{aligned} \dot{x}_1(t) &= (\alpha_1 + a) f(x_1) + (\alpha_2 + b) f(x_2(t - \tau_1)) + c x_1 x_2, \\ \dot{x}_2(t) &= (\alpha_2 - b) f(x_1(t - \tau_2)) + (\alpha_1 - a) f(x_2) + d x_1 x_2, \end{aligned} \tag{1.2}$$

where $x_i(t)$ $(i = 1,2)$ represents the state of the i-th neuron at time t, $f(x_i)$ $(i = 1,2)$ is the connection function between two neurons, $\alpha_1, \alpha_2, a, b, c, d$ are real parameters, and τ_1, τ_2 are positive constants. We all know that time delays that occurred in the interaction between neurons will affect the stability of a network by creating instability, oscillation, and chaos phenomena.

The purpose of this paper is to discuss the stability and the properties of Hopf bifurcation of model (1.2). To the best of our knowledge, it is the first to deal with the stability and Hopf bifurcation of the system (1.2).

This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are studied. In Section 3, the direction of Hopf bifurcation and the stability and periods of bifurcating periodic solutions on the center manifold are determined. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

2. Stability of the Equilibrium and Local Hopf Bifurcations

Throughout this paper, we assume that the function f satisfies the following conditions:

(H1) $f \in C^3(R)$, $f(0) = 0$, and $uf(u) > 0$, for $u \neq 0$.

Hypothesis (H1) implies that $E_*(0,0)$ is an equilibrium of the system (1.2) and linearized system of (1.2) takes the form

$$\begin{aligned} \dot{x}_1(t) &= (\alpha_1 + a) f'(0) x_1 + (\alpha_2 + b) f'(0) x_2(t - \tau_1), \\ \dot{x}_2(t) &= (\alpha_2 - b) f'(0) x_1(t - \tau_2) + (\alpha_1 - a) f'(0) x_2. \end{aligned} \tag{2.1}$$

The associated characteristic equation of (2.1) is

$$\lambda^2 - 2\alpha_1 f'(0)\lambda + \left(\alpha_1^2 - a^2\right) f'^2(0) - \left(\alpha_2^2 - b^2\right) f'^2(0) e^{-\lambda\tau} = 0, \tag{2.2}$$

where $\tau = \tau_1 + \tau_2$.

In the section, we consider the sum of two delays as the parameter to give some conditions that separate the first quadrant of the (τ_1, τ_2) plane into two parts, one is the stable region another is the unstable region, and the boundary is the Hopf bifurcation curve.

In order to investigate the distribution of roots of the transcendental equation (2.2), the following Lemma that is stated in [15] is useful.

Lemma 2.1 (see [15]). *For the transcendental equation*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}\right) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots \\ &\quad + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0, \end{aligned} \tag{2.3}$$

as $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ in the open right half plane can change and only a zero appears on or crosses the imaginary axis.

Now we make the following assumptions:
(H2) $\alpha_1 f'(0) < 0$ and $\alpha_1^2 - \alpha_2^2 - a^2 + b^2 > 0$;
(H3) $|\alpha_1^2 - a^2| < |\alpha_2^2 - b^2|$.

Lemma 2.2. *If (H1)–(H3) hold, then one has the following.*

(i) *When*

$$\tau = \tau^j \stackrel{\text{def}}{=} \frac{1}{\omega_0}\left[\arccos\frac{2\alpha_1\omega_0}{(\alpha_2^2 - b^2)f'^2(0)} + 2j\pi\right], \quad j = 0, 1, 2, \ldots. \tag{2.4}$$

Equation (2.2) *has a simple pair of imaginary roots $\pm i\omega_0$, where*

$$\omega_0 = \sqrt{\sqrt{(\alpha_1^2 + a^2)^2 - \left[(\alpha_1^2 - a^2)^2 - (\alpha_2^2 - b^2)^2\right]f'^4(0)} - (\alpha_1^2 + a^2)}. \tag{2.5}$$

(ii) *For $\tau \in [0, \tau^0)$, all roots of* (2.2) *have strictly negative real parts.*

(iii) *When $\tau = \tau^0$,* (2.2) *has a pair of imaginary roots $\pm i\omega_0$ and all other roots have strictly negative real parts.*

Proof. Obviously, by assumption (H2), $\lambda = 0$ is not the root of (2.2). When $\tau = 0$, then (2.2) becomes

$$\lambda^2 - 2\alpha_1 f'(0)\lambda + \left[\alpha_1^2 - \alpha_2^2 - a^2 + b^2\right]f'^2(0) = 0. \tag{2.6}$$

It is easy to see that all roots of (2.6) have negative real parts.

$\pm i\omega(\omega > 0)$ is a pair of purely imaginary roots of (2.2) if and only if ω satisfies

$$-\omega^2 - 2\alpha_1 f'(0)\omega i + \left(\alpha_1^2 - a^2\right)f'^2(0) - \left(\alpha_2 - b^2\right)f'^2(0)(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{2.7}$$

Separating the real and imaginary parts, we get

$$\begin{aligned}\left(\alpha_2^2 - b^2\right) f'^2(0)\cos\omega\tau &= \left(\alpha_1^2 - a^2\right) f'^2(0) - \omega^2,\\ \left(\alpha_2^2 - b^2\right) f'^2(0)\sin\omega\tau &= 2\alpha_1 f'(0)\omega.\end{aligned} \tag{2.8}$$

It follows from (2.8) that

$$\omega^4 + 2\left(\alpha_1^2 + a^2\right) f'^2(0)\omega^2 + \left[\left(\alpha_1^2 - a^2\right)^2 - \left(\alpha_2^2 - b^2\right)^2\right] f'^4(0) = 0. \tag{2.9}$$

Thus, we obtain

$$\omega = \sqrt{\sqrt{\left(\alpha_1^2 + a^2\right)^2 - \left[\left(\alpha_1^2 - a^2\right)^2 - \left(\alpha_2^2 - b^2\right)^2\right] f'^4(0)} - \left(\alpha_1^2 + a^2\right)}. \tag{2.10}$$

It is clear that ω is well defined if condition (H3) holds.

Denote

$$\omega_0 = \sqrt{\sqrt{\left(\alpha_1^2 + a^2\right)^2 - \left[\left(\alpha_1^2 - a^2\right)^2 - \left(\alpha_2^2 - b^2\right)^2\right] f'^4(0)} - \left(\alpha_1^2 + a^2\right)}. \tag{2.11}$$

Let

$$\tau^j = \frac{1}{\omega_0}\left[\arccos\frac{2\alpha_1\omega_0}{\left(\alpha_2^2 - b^2\right) f'^2(0)} + 2j\pi\right], \quad j = 0, 1, 2, \ldots. \tag{2.12}$$

From (2.8), we know that (2.2) with $\tau = \tau^j (j = 0, 1, 2, \ldots)$ has a pair of imaginary roots $\pm i\omega_0$, which are simple.

According, the discussion and applying the Lemma 2.1 and Cooke and Grossman [16], we obtain the conclusion (ii) and (iii). This completes the proof. □

Let $\lambda_j(\tau) = \alpha_j(\tau) + i\omega_j(\tau)$ be a root of (2.2) near $\tau = \tau^j$, and $\alpha_j(\tau^j) = 0$, $\omega_j(\tau^j) = \omega_0, (j = 0, 1, 2\ldots)$. Due to functional differential equation theory, for every $\tau^j, k = 0, 1, 2\ldots$, there exists $\varepsilon > 0$ such that $\lambda_j(\tau)$ is continuously differentiable in τ for $|\tau - \tau^j| < \varepsilon$. Substituting $\lambda(\tau)$ into the left-hand side of (2.2) and taking derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\sigma}\right]^{-1} = -\frac{2\lambda - 2\alpha_1 f'(0)}{\left(\alpha_2^2 - b^2\right) f'^2(0) e^{-\lambda\tau}\lambda} - \frac{\tau}{\lambda}, \tag{2.13}$$

which leads to

$$\operatorname{Re}\left[\frac{d\lambda}{d\sigma}\right]^{-1}_{\tau=\tau^j} = \frac{2\alpha_1 f'(0)\left(\alpha_2^2-b^2\right)f'^2(0)\omega_0\sin\omega_0\tau^j - 2\omega_0^2\left(\alpha_2^2-b^2\right)f'^2(0)\cos\omega_0\tau^j}{\left[\left(\alpha_2^2-b^2\right)f'^2(0)\omega_0\sin\omega_0\tau^j\right]^2+\left[\omega_0\left(\alpha_2^2-b^2\right)f'^2(0)\cos\omega_0\tau^j\right]^2}. \tag{2.14}$$

By (2.8), we get

$$\operatorname{Re}\left[\frac{d\lambda}{d\sigma}\right]^{-1}_{\tau=\tau^j} = \frac{2\omega_0^2+\left(3\alpha_1^2+a^2\right)f'^2(0)}{\left[2\omega_0\alpha_1 f'(0)\right]^2+\left[\left(\alpha_1^2-a^2\right)f'^2(0)-\omega_0^2\right]^2} > 0. \tag{2.15}$$

So we have

$$\operatorname{signRe}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau^j} = \operatorname{signRe}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau^j} > 0. \tag{2.16}$$

From the above analysis, we have the following results.

Lemma 2.3. *Let $\tau=\tau^j$, then the following transversality condition:*

$$\frac{d}{d\tau}\operatorname{Re}\left[\lambda_j(\tau)\right]\Big|_{\tau=\tau^j} > 0 \tag{2.17}$$

is satisfied.

From Lemma 2.3, we can obtain the following lemma.

Lemma 2.4. *Assume that (H3) holds. If $\tau>\tau^0$, then* (2.2) *has at least one root with strictly positive real part.*

Remark 2.5. In fact, Applying the lemma in Cooke and Grossman [16] and Lemma 2.3, we can easily see that if $\tau\in(\tau^j,\tau^{j+1})$, (2.2) has $2(j+1)$ $(j=0,1,2,\ldots)$ roots with positive real parts.

From Lemma 2.2–2.4, we have the following results on the local stability and Hopf bifurcation for system (1.2).

Theorem 2.6. *For system* (1.2), *let τ^0 be defined by* (2.4) *and assume that (H1)–(H3) hold.*

(i) *If $\tau\in[0,\tau^0)$, then the equilibrium point of system* (1.2) *is asymptotically stable.*

(ii) *If $\tau>\tau^0$, then the equilibrium point of system* (1.2) *is unstable.*

(iii) *$\tau=\tau^j$ $(j=0,1,2,\ldots)$ are Hopf bifurcation values for system* (1.2).

3. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtained some conditions which guarantee that the two-neuron networks with resonant bilinear terms undergo the Hopf bifurcation at some values of $\tau=\tau_1+\tau_2$. In this section, we shall derived the explicit formulae determining the direction,

stability, and period of these periodic solutions bifurcating from the equilibrium $E_*(0,0)$ at this critical value of τ, by using techniques from normal form and center manifold theory [17], Throughout this section, we always assume that system (2.1) undergoes Hopf bifurcation at the equilibrium $E_*(0,0)$ for $\tau = \tau^0$ and then $\pm i\omega_0$ is corresponding purely imaginary roots of the characteristic equation at the equilibrium $E_*(0,0)$.

For convenience, let $\tau = \tau^0 + \mu, \mu \in R$. Then $\mu = 0$ is the Hopf bifurcation value of (1.2). Thus, we shall study Hopf bifurcation of small amplitude periodic solutions of (1.2) from the equilibrium point for μ close to 0. Without loss of generality, we assume that $\tau_1^0 > \tau_2^0$ and let $|\mu| \leq \tau_1^0 - \tau_2^0$. Since our analysis is local, where $\tau^0 = \tau_1^0 + \tau_2^0$ and $\tau = \tau_1^0 + (\tau_2^0 + \mu)$. We can consider the fixed phase space $C = C([-\tau_1^0, 0], R^2)$.

For $(\phi_1, \phi_2) \in C$, define

$$L_\mu \phi = A_1 \phi(0) + B\phi(-\tau_2) + C\phi(-\tau_1), \tag{3.1}$$

where

$$A_1 = \begin{pmatrix} (\alpha_1+a) f'(0) & 0 \\ 0 & (\alpha_1-a) f'(0) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ (\alpha_2-b) f'(0) & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & (\alpha_2+b) f'(0) \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

We expand the nonlinear part of the system (1.2) and derive the following expression:

$$f(\mu, \phi) = \begin{pmatrix} f_1(\mu, \phi) \\ f_2(\mu, \phi) \end{pmatrix}, \tag{3.3}$$

where

$$\begin{aligned} f_1(\mu, \phi) &= (\alpha_1 + a)\left[\frac{f''(0)}{2}\phi_1^2(0) + \frac{f'''(0)}{3!}\phi_1^3(0)\right] \\ &\quad + (\alpha_2 + b)\left[\frac{f''(0)}{2}\phi_2^2(-\tau_1) + \frac{f'''(0)}{3!}\phi_2^3(-\tau_1)\right] + c\phi_1(0)\phi_2(0) + \text{h.o.t.}, \\ f_2(\mu, \phi) &= (\alpha_2 - b)\left[\frac{f''(0)}{2}\phi_1^2(-\tau_2) + \frac{f'''(0)}{3!}\phi_1^3(-\tau_2)\right] \\ &\quad + (\alpha_1 - a)\left[\frac{f''(0)}{2}\phi_2^2(0) + \frac{f'''(0)}{3!}\phi_2^3(0)\right] + d\phi_1(0)\phi_2(0) + \text{h.o.t.}. \end{aligned} \tag{3.4}$$

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta, \mu), \theta \in [-\tau_1^0, 0]$ such that

$$L_\mu \phi = \int_{-\tau_1^0}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \text{for } \phi \in C. \tag{3.5}$$

In fact, we can choose

$$\eta(\theta,\mu)=\begin{cases}A_1, & \theta=0,\\ B\delta(\theta+\tau_2), & \theta\in[-\tau_2,0),\\ -C\delta\left(\theta+\tau_1^0\right), & \theta\in\left[-\tau_1^0,-\tau_2\right),\end{cases}\tag{3.6}$$

where δ is the Dirac delta function.

For $\phi\in C([-\tau_1^0,0],R^2)$, define

$$\begin{aligned}A(\mu)\phi&=\begin{cases}\dfrac{d\phi(\theta)}{d\theta}, & -\tau_1^0\le\theta<0,\\ \displaystyle\int_{-\tau_1^0}^{0}d\eta(s,\mu)\phi(s), & \theta=0,\end{cases}\\ R(\mu)\phi&=\begin{cases}0, & -\tau_1^0\le\theta<0,\\ f(\mu,\phi), & \theta=0.\end{cases}\end{aligned}\tag{3.7}$$

Then (1.2) is equivalent to the abstract differential equation

$$\dot{x}_t=A(\mu)x_t+R(\mu)x_t,\tag{3.8}$$

where $x=(x_1,x_2)^T$, $x_t(\theta)=x(t+\theta)$, $\theta\in[-\tau_1^0,0]$.

For $\psi\in C([0,\tau_1^0],(R^2)^*)$, define

$$A^*\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & \mathrm{s}\in(0,\tau_1^0],\\ \displaystyle\int_{-\tau_1^0}^{0}d\eta^T(t,0)\psi(-t), & \mathrm{s}=0.\end{cases}\tag{3.9}$$

For $\phi\in C([-\tau_1^0,0],R^2)$ and $\psi\in C([0,\tau_1^0],(R^2)^*)$, define the bilinear form

$$\langle\psi,\phi\rangle=\overline{\psi}(0)\phi(0)-\int_{-\tau_1^0}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi,\tag{3.10}$$

where $\eta(\theta)=\eta(\theta,0)$. We have the following result on the relation between the operators $A=A(0)$ and A^*.

Lemma 3.1. *$A=A(0)$ and A^* are adjoint operators.*

Proof. Let $\phi \in C^1([-\tau_1^0, 0], R^2)$ and $\psi \in C^1([0, \tau_1^0], (R^2)^*)$. It follows from (3.10) and the definitions of $A = A(0)$ and A^* that

$$\begin{aligned}
\langle \psi(s), A(0)\phi(\theta)\rangle &= \overline{\psi}(0)A(0)\phi(0) - \int_{-\tau_1^0}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&= \overline{\psi}(0)\int_{-\tau_1^0}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tau_1^0}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&= \overline{\psi}(0)\int_{-\tau_1^0}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tau_1^0}^{0}\left[\overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)\right]_{\xi=0}^{\theta} \\
&\quad + \int_{-\tau_1^0}^{0}\int_{\xi=0}^{\theta}\frac{d\overline{\psi}(\xi-\theta)}{d\xi}d\eta(\theta)\phi(\xi)d\xi \\
&= \int_{-\tau_1^0}^{0}\overline{\psi}(-\theta)d\eta(\theta)\phi(0) - \int_{-\tau_1^0}^{0}\int_{\xi=0}^{\theta}\left[-\frac{d\overline{\psi}(\xi-\theta)}{d\xi}\right]d\eta(\theta)\phi(\xi)d\xi \\
&= A*\overline{\psi}(0)\phi(0) - \int_{-\tau_1^0}^{0}\int_{\xi=0}^{\theta}A^*\overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi \\
&= \langle A^*\psi(s), \phi(\theta)\rangle.
\end{aligned} \tag{3.11}$$

This shows that $A = A(0)$ and A^* are adjoint operators and the proof is complete. □

By the discussions in the Section 2, we know that $\pm i\omega_0$ are eigenvalues of $A(0)$ and they are also eigenvalues of A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively. We have the following result.

Lemma 3.2. *The vector*

$$q(\theta) = (1, \gamma)^T e^{i\omega_0\theta}, \quad \theta \in \left[-\tau_1^0, 0\right], \tag{3.12}$$

where

$$\gamma = \frac{i\omega_0 - (\alpha_1 + a)f'(0)}{(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0}}, \tag{3.13}$$

is the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$, and

$$q^*(s) = D(1, \gamma^*)e^{i\omega s}, \quad s \in \left[0, \tau_1^0\right], \tag{3.14}$$

where

$$\gamma^* = -\frac{i\omega_0 + (\alpha_1 + a)f'(0)}{(\alpha_2 - b)f'(0)e^{-i\omega\tau_2^0}}, \tag{3.15}$$

is the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0$, moreover, $\langle q^*(s), q(\theta)\rangle = 1$, where

$$D = 1 + \overline{\gamma}\overline{\gamma^*} + \gamma^*\tau_2^0(\alpha_2 - b)f'(0)e^{-i\omega_0\tau_2^0} + \overline{\gamma}\tau_1^0(\alpha_2 + b)f'(0)e^{i\omega_0\tau_1^0}. \tag{3.16}$$

Proof. Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$ and $q^*(s)$ be the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0$, namely, $A(0)q(\theta) = i\omega_0 q(\theta)$ and $A^*q^{*T}(s) = -i\omega_0 q^{*T}(s)$. From the definitions of $A(0)$ and A^*, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^*q^{*T}(s) = -dq^{*T}(s)/ds$. Thus, $q(\theta) = q(0)e^{i\omega_0\theta}$ and $q^*(s) = q^*(0)e^{i\omega_0 s}$. In addition,

$$\int_{-\tau_1^0}^{0} d\eta(\theta)q(\theta) = A_1 q(0) + Bq(-\tau_2) + Cq(-\tau_1) = A(0)q(0) = i\omega_0 q(0). \tag{3.17}$$

That is,

$$\begin{pmatrix} i\omega_0 - (\alpha_1 + a)f'(0) & -(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0} \\ -(\alpha_2 - b)f'(0)e^{-i\omega_0\tau_2^0} & i\omega_0 - (\alpha_1 - a)f'(0) \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.18}$$

Therefore, we can easily obtain

$$\gamma = \frac{i\omega_0 - (\alpha_1 + a)f'(0)}{(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0}} \tag{3.19}$$

and so

$$q(0) = \left(1, \frac{i\omega_0 - (\alpha_1 + a)f'(0)}{(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0}}\right)^T \tag{3.20}$$

hence

$$q(\theta) = \left(1, \frac{i\omega_0 - (\alpha_1 + a)f'(0)}{(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0}}\right)^T e^{i\omega_0\theta}. \tag{3.21}$$

On the other hand,

$$\int_{-\tau_1^0}^{0} q^*(-t)d\eta(t) = A_1^T q^{*T}(0) + B^T q^{*T}\left(-\tau_2^0\right) + C^T q^{*T}\left(-\tau_1^0\right) = A^*q^{*T}(0) = -i\omega_0 q^{*T}(0). \tag{3.22}$$

Namely,

$$\begin{pmatrix} i\omega_0 + (\alpha_1 + a)f'(0) & (\alpha_2 - b)f'(0)e^{-i\omega_0\tau_2^0} \\ (\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0} & i\omega_0 + (\alpha_1 - a)f'(0) \end{pmatrix} q^*(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.23}$$

Therefore, we can easily obtain

$$\gamma^* = -\frac{i\omega_0 + (\alpha_1 + a)f'(0)}{(\alpha_2 - b)f'(0)e^{-i\omega_0\tau_2^0}} \tag{3.24}$$

and so

$$q^*(0) = \left(1, -\frac{i\omega_0 + (\alpha_1 + a)f'(0)}{(\alpha_2 - b)f'(0)e^{-i\omega_0\tau_2^0}}\right) \tag{3.25}$$

hence

$$q^*(s) = \left(1, -\frac{i\omega_0 + (\alpha_1 + a)f'(0)}{(\alpha_2 - b)f'(0)e^{-i\omega_0\tau_2^0}}\right)e^{i\omega_0 s}. \tag{3.26}$$

In the sequel, we will verify that $\langle q^*(s), q(\theta)\rangle = 1$. In fact, from (3.10), we have

$$\begin{aligned}
\langle q^*(s), q(\theta)\rangle &= \overline{D}\left(1, \overline{\gamma^*}\right)(1,\gamma)^T \\
&\quad - \int_{-\tau_1^0}^{0}\int_{\xi=0}^{\theta} \overline{D}\left(1\overline{\gamma^*}\right)e^{-i\omega_0(\xi-\theta)}d\eta(\theta)(1,\gamma)^T e^{i\omega_0\xi}d\xi \\
&= \overline{D}\left[1 + \gamma\overline{\gamma^*} - \int_{-\tau_1^0}^{0}\left(1, \overline{\gamma^*}\right)\theta e^{i\omega_0\theta}d\eta(\theta)(1,\gamma)^T\right] \\
&= \overline{D}\left\{1 + \gamma\overline{\gamma^*} - \left(1, \overline{\gamma^*}\right)\left[B\left(-\tau_2^0\right)e^{-i\omega_0\tau_2^0} + C\left(-\tau_1^0\right)e^{-i\omega_0\tau_1^0}\right](1,\gamma)^T\right\} \\
&= \overline{D}\left[1 + \gamma\overline{\gamma^*} + \overline{\gamma}^*\tau_2^0(\alpha_2 - b)f'(0)e^{i\omega_0\tau_2^0} + \gamma\tau_1^0(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0}\right] \\
&= 1.
\end{aligned} \tag{3.27}$$

□

Next, we use the same notations as those in Hassard et al. [17] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of (1.2) when $\mu = 0$.

Define

$$z(t) = \langle q^*, x_t\rangle, \qquad W(t,\theta) = x_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\} \tag{3.28}$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t), \overline{z}(t), \theta), \tag{3.29}$$

where

$$W(z(t), \overline{z}(t), \theta) = W(z, \overline{z}) = W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots \tag{3.30}$$

and z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. Noting that W is also real if x_t is real, we consider only real solutions. For solutions $x_t \in C_0$ of (1.2),

$$\begin{aligned}
\dot{z}(t) &= \langle q^*(s), \dot{x}_t \rangle = \langle q^*(s), A(0)x_t + R(0)x_t \rangle \\
&= \langle q^*(s), A(0)x_t \rangle + \langle q^*(\mathrm{s}), R(0)x_t \rangle \\
&= \langle A^*q^*(s), x_t \rangle + \overline{q^*}(0)R(0)x_t - \int_{-\tau_1^0}^{0} \int_{\xi=0}^{\theta} \overline{q^*}(\xi-\theta)d\eta(\theta)A(0)R(0)x_t(\xi)d\xi \\
&= \langle i\omega_0 q^*(s), x_t \rangle + \overline{q^*}(0)f(0, x_t(\theta)) \\
&\stackrel{\text{def}}{=} i\omega_0 z(t) + \overline{q^*}(0)f_0(z(t), \overline{z}(t)).
\end{aligned} \tag{3.31}$$

That is,

$$\dot{z}(t) = i\omega_0 z + g(z, \overline{z}), \tag{3.32}$$

where

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.33}$$

Hence, we have

$$g(z, \overline{z}) = \overline{q}^*(0)f_0(z, \overline{z}) = f(0, x_t) = \overline{D}(1, \gamma^*)\left(f_1(0, x_t), f_2(0, x_t)\right)^T, \tag{3.34}$$

where

$$\begin{aligned}
f_1(0, x_t) &= (\alpha_1 + a)\left[\frac{f''(0)}{2}x_{1t}^2(0) + \frac{f'''(0)}{3!}x_{1t}^3(0)\right] \\
&\quad + (\alpha_2 + b)\left[\frac{f''(0)}{2}x_{2t}^2\left(-\tau_1^0\right) + \frac{f'''(0)}{3!}x_{2t}^3\left(-\tau_1^0\right)\right] + cx_{1t}(0)x_{2t}(0) + \text{h.o.t.}, \\
f_2(0, x_t) &= (\alpha_2 - b)\left[\frac{f''(0)}{2}x_{1t}^2\left(-\tau_2^0\right) + \frac{f'''(0)}{3!}x_{1t}^3\left(-\tau_2^0\right)\right] \\
&\quad + (\alpha_1 - a)\left[\frac{f''(0)}{2}x_{2t}^2(0) + \frac{f'''(0)}{3!}x_{2t}^3(0)\right] + dx_{1t}(0)x_{2t}(0) + \text{h.o.t.}.
\end{aligned} \tag{3.35}$$

Noticing $x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta))^T = W(t,\theta) + zq(\theta) + \overline{z}\overline{q}(\theta)$ and $q(\theta) = (1,\gamma)^T e^{i\omega_0\theta}$, we have

$$\begin{aligned}
x_{1t}(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots,\\
x_{2t}(0) &= \gamma z + \overline{\gamma z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots,\\
x_{1t}\left(-\tau_2^0\right) &= e^{-i\omega_0\tau_2^0}z + e^{i\omega_0\tau_2^0}\overline{z} + W_{20}^{(1)}\left(-\tau_2^0\right)\frac{z^2}{2} + W_{11}^{(1)}\left(-\tau_2^0\right)z\overline{z} + W_{02}^{(1)}\left(-\tau_2^0\right)\frac{\overline{z}^2}{2} + \cdots,\\
x_{2t}\left(-\tau_1^0\right) &= \gamma e^{-i\omega_0\tau_1^0}z + \overline{\gamma}e^{i\omega_0\tau_1^0}\overline{z} + W_{20}^{(2)}\left(-\tau_1^0\right)\frac{z^2}{2} + W_{11}^{(2)}\left(-\tau_1^0\right)z\overline{z} + W_{02}^{(2)}\left(-\tau_1^0\right)\frac{\overline{z}^2}{2} + \cdots.
\end{aligned} \tag{3.36}$$

From (3.33) and (3.34), we have

$$\begin{aligned}
g(z,\overline{z}) &= \overline{q}^*(0)f_0(z,\overline{z}) = \overline{D}\left[f_1(0,x_t) + \overline{\gamma^*}f_2(0,x_t)\right]\\
&= \left\{\overline{D}\left[(\alpha_1+a)\frac{f''(0)}{2} + (\alpha_2+b)\frac{f''(0)}{2}\gamma^2 + c\gamma\right]\right.\\
&\quad \left.+\overline{D}\,\overline{\gamma^*}\left[(\alpha_2-b)\frac{f''(0)}{2}e^{-2i\omega_0\tau_2^0} + (\alpha_1-a)\frac{f''(0)}{2}\gamma^2 + d\gamma\right]\right\}z^2\\
&\quad + \left\{\overline{D}\left[(\alpha_1+a)f''(0) + (\alpha_2+b)f''(0)2\gamma\overline{\gamma} + 2c\,\mathrm{Re}\{\gamma\}\right]\right.\\
&\quad \left.+\overline{D}\,\overline{\gamma^*}\left[(\alpha_2-b)f''(0) + (\alpha_1-a)f''(0)\gamma\overline{\gamma} + 2d\,\mathrm{Re}\{\gamma\}\right]\right\}z\overline{z}\\
&\quad + \left\{\overline{D}\left[(\alpha_1+a)\frac{f''(0)}{2} + (\alpha_2+b)\frac{f''(0)}{2}\overline{\gamma}^2 + c\overline{\gamma}\right]\right.\\
&\quad \left.+\overline{D}\,\overline{\gamma^*}\left[(\alpha_2-b)\frac{f''(0)}{2}e^{2i\omega_0\tau_2^0} + (\alpha_1-a)\frac{f''(0)}{2}\overline{\gamma}^2 + d\overline{\gamma}\right]\right\}\overline{z}^2\\
&\quad + \left\{\overline{D}\left[(\alpha_1+a)\frac{f''(0)}{2}\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) + (\alpha_1+a)\frac{f'''(0)}{2}\right.\right.\\
&\qquad + (\alpha_2+b)f''(0)\gamma e^{-i\omega_0\tau_1^0}W_{11}^{(2)}\left(-\tau_1^0\right) + (\alpha_2+b)\frac{f'''(0)}{2}\gamma^2\overline{\gamma}e^{-i\omega_0\tau_1^0}\\
&\qquad \left.\left.+\frac{1}{2}c\left(W_{20}^{(2)}(0) + \overline{\gamma}W_{20}^{(1)}(0)\right)\right]\right.\\
&\quad \left.+\overline{D}\,\overline{\gamma^*}\left[(\alpha_2-b)\frac{f''(0)}{2}\left(W_{20}^{(1)}\left(-\tau_2^0\right)\right)e^{i\omega_0\tau_2^0}\right.\right.\\
&\qquad +2e^{-i\omega_0\tau_2^0}W_{11}^{(1)}\left(-\tau_2^0\right) + (\alpha_2-b)\frac{f'''(0)}{2}e^{-i\omega_0\tau_2^0} + (\alpha_1-a)\frac{f''(0)}{2}\left(2\gamma W_{11}^{(2)}(0)\right)\\
&\qquad \left.\left.+W_{20}^{(2)}(0)\overline{\gamma} + (\alpha_1-a)\frac{f''(0)}{2}\gamma^2\overline{\gamma} + \frac{1}{2}d\left(W_{20}^{(2)}(0) + \overline{\gamma}W_{20}^{(1)}(0)\right)\right]\right\}z^2\overline{z} + \text{h.o.t.}
\end{aligned} \tag{3.37}$$

and we obtain

$$\begin{aligned}
g_{20} &= \overline{D}\left[(\alpha_1+a)f''(0)+(\alpha_2+b)f''(0)\gamma^2+c\gamma\right]\\
&\quad+\overline{D}\overline{\gamma^*}\left[(\alpha_2-b)f''(0)e^{-2i\omega_0\tau_2^0}+(\alpha_1-a)f''(0)\gamma^2+d\gamma\right],\\
g_{11} &= \overline{D}\left[(\alpha_1+a)f''(0)+(\alpha_2+b)f''(0)a\overline{a}+2c\,\mathrm{Re}\{\gamma\}\right]\\
&\quad+\overline{D}\overline{\gamma^*}\left[(\alpha_2-b)f''(0)+(\alpha_1-a)f''(0)\gamma\overline{\gamma}+2d\,\mathrm{Re}\{\gamma\}\right],\\
g_{02} &= \overline{D}\left[(\alpha_1+a)f''(0)+(\alpha_2+b)f''(0)\overline{\gamma}^2+2c\overline{\gamma}\right]\\
&\quad+\overline{D}\overline{\gamma^*}\left[(\alpha_2-b)f''(0)e^{2i\omega_0\tau_2^0}+(\alpha_1-a)f''(0)\overline{\gamma}^2+2d\overline{\gamma}\right],\\
g_{21} &= \overline{D}\left[(\alpha_1+a)f''(0)\left(2W_{11}^{(1)}(0)+W_{20}^{(1)}(0)\right)+(\alpha_1+a)f'''(0)\right.\\
&\qquad\left.+2(\alpha_2+b)f''\gamma e^{-i\omega_0\tau_1^0}W_{11}^{(2)}\left(-\tau_1^0\right)+(\alpha_2+b)f'''(0)\gamma^2\overline{\gamma}e^{-i\omega_0\tau_1^0}\right]\\
&\quad+c\left(W_{20}^{(2)}(0)+\overline{\gamma}W_{20}^{(1)}(0)\right)\\
&\quad+\overline{D}\overline{\gamma^*}\left[(\alpha_2-b)f''(0)\left(W_{20}^{(1)}\left(-\tau_2^0\right)\right)e^{i\omega_0\tau_2^0}\right.\\
&\qquad+2e^{-i\omega_0\tau_2^0}W_{11}^{(1)}\left(-\tau_2^0\right)+(\alpha_2-b)f'''(0)e^{-i\omega_0\tau_2^0}+(\alpha_1-a)f''(0)\left(2\gamma W_{11}^{(2)}(0)\right)\\
&\qquad\left.+W_{20}^{(2)}(0)\overline{\gamma}+(\alpha_1-a)f''(0)\gamma^2\overline{\gamma}+d\left(W_{20}^{(2)}(0)+\overline{\gamma}W_{20}^{(1)}(0)\right)\right].
\end{aligned}\tag{3.38}$$

For unknown

$$W_{20}^{(1)}(0),\quad W_{20}^{(1)}\left(-\tau_2^0\right),\quad W_{11}^{(1)}(0),\quad W_{11}^{(2)}(0),\quad W_{11}^{(2)}\left(-\tau_1^0\right),\quad W_{11}^{(2)}\left(-\tau_2^0\right)\tag{3.39}$$

in g_{21}, we still need to compute them.

Form (3.8), (3.32), we have

$$\begin{aligned}
W' &= \begin{cases} AW-2\,\mathrm{Re}\{\overline{q}^*(0)fq(\theta)\}, & -\tau_1^0\le\theta<0,\\ AW-2\,\mathrm{Re}\{\overline{q}^*(0)fq(\theta)\}+f, & \theta=0 \end{cases}\\
&\overset{\mathrm{def}}{=} AW+H(z,\overline{z},\theta),
\end{aligned}\tag{3.40}$$

where

$$H(z,\overline{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\overline{z}+H_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots.\tag{3.41}$$

Comparing the coefficients, we obtain

$$(A - 2i\omega_0)W_{20} = -H_{20}(\theta), \tag{3.42}$$

$$\begin{aligned} AW_{11}(\theta) &= -H_{11}(\theta), \\ &\cdots. \end{aligned} \tag{3.43}$$

And we know that, for $\theta \in [-\tau_1^0, 0)$,

$$H(z,\overline{z},\theta) = -\overline{q}^*(0)f_0q(\theta) - q^*(0)\overline{f}_0\overline{q}(\theta) = -g(z,\overline{z})q(\theta) - \overline{g}(z,\overline{z})\overline{q}(\theta). \tag{3.44}$$

Comparing the coefficients of (3.41) with (3.44) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \tag{3.45}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{3.46}$$

From (3.42), (3.45), and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2i\omega_0W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta). \tag{3.47}$$

Noting that $q(\theta) = q(0)e^{i\omega_0\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\overline{g}_{02}}{3\omega_0}\overline{q}(0)e^{-i\omega_0\theta} + E_1e^{2i\omega_0\theta}, \tag{3.48}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T$ is a constant vector.

Similarly, from (3.43), (3.46), and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta), \tag{3.49}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\overline{g}_{11}}{\omega_0}\overline{q}(0)e^{-i\omega_0\theta} + E_2, \tag{3.50}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})^T$ is a constant vector.

In what follows, we will seek appropriate E_1, E_2 in (3.48), (3.50), respectively. It follows from the definition of A and (3.45), (3.46) that

$$\int_{-\tau_1^0}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0W_{20}(0) - H_{20}(0) \tag{3.51}$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.52}$$

where $\eta(\theta) = \eta(0,\theta)$.

From (3.42), we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + (H_1, H_2)^T, \tag{3.53}$$

where

$$\begin{aligned} H_1 &= (\alpha_1 + a)\frac{f''(0)}{2} + (\alpha_2 + b)\frac{f''(0)}{2}\gamma^2, \\ H_2 &= (\alpha_2 - b)\frac{f''(0)}{2}e^{-2i\omega_0\tau_2^0} + (\alpha_1 - a)\frac{f''(0)}{2}\gamma^2. \end{aligned} \tag{3.54}$$

From (3.43), we have

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}(0)\overline{q}(0) + (P_1, P_2)^T, \tag{3.55}$$

where

$$\begin{aligned} P_1 &= (\alpha_1 + a)f''(0) + (\alpha_2 + b)f''(0)\gamma\overline{\gamma}, \\ P_2 &= (\alpha_2 - b)f''(0) + (\alpha_1 - a)f''(0)\gamma\overline{\gamma}. \end{aligned} \tag{3.56}$$

Noting that

$$\begin{aligned} \left(i\omega_0 I - \int_{-\tau_1^0}^{0} e^{i\omega_0\theta}d\eta(\theta)\right)q(0) &= 0, \\ \left(-i\omega_0 I - \int_{-\tau_1^0}^{0} e^{-i\omega_0\theta}d\eta(\theta)\right)\overline{q}(0) &= 0 \end{aligned} \tag{3.57}$$

and substituting (3.48) and (3.53) into (3.51), we have

$$\left(2i\omega_0 I - \int_{-\tau_1^0}^{0} e^{2i\omega_0\theta}d\eta(\theta)\right)E_1 = (H_1, H_2)^T. \tag{3.58}$$

That is,

$$\left(2i\omega_0 I - A_1 - Be^{-2i\omega_0\tau_2^0} - Ce^{-2i\omega_0\tau_1^0}\right)E_1 = (H_1, H_2)^T, \tag{3.59}$$

then

$$\begin{pmatrix} 2i\omega_0 - (\alpha_1 + a)f'(0) & -(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0} \\ -(\alpha_2 - b)f'(0)e^{-i\omega_0\tau_2^0} & 2i\omega_0 - (\alpha_1 - a)f'(0) \end{pmatrix} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}. \tag{3.60}$$

Hence,

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, \qquad E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, \tag{3.61}$$

where

$$\begin{aligned}
\Delta_1 &= \det\begin{pmatrix} 2i\omega_0 - (\alpha_1 + a)f'(0) & -(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0} \\ -(\alpha_2 - b)f'(0)e^{-i\omega_0\tau_2^0} & 2i\omega_0 - (\alpha_1 - a)f'(0) \end{pmatrix}, \\
\Delta_{11} &= \det\begin{pmatrix} H_1 & -(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0} \\ H_2 & 2i\omega_0 - (\alpha_1 - a)f'(0) \end{pmatrix}, \\
\Delta_{12} &= \det\begin{pmatrix} 2i\omega_0 - (\alpha_1 + a)f'(0) & H_1 \\ -(\alpha_2 - b)f'(0)e^{-i\omega_0\tau_2^0} & H_2 \end{pmatrix}.
\end{aligned} \tag{3.62}$$

Similarly, substituting (3.49) and (3.55) into (3.52), we have

$$\left(\int_{-\tau_1^0}^{0} d\eta(\theta)\right) E_2 = (P_1, P_2)^T. \tag{3.63}$$

Then,

$$(A_1 + B + C)E_2 = (-P_1, -P_2)^T. \tag{3.64}$$

That is,

$$\begin{pmatrix} (\alpha_1 + a)f'(0) & (\alpha_2 + b)f'(0) \\ (\alpha_2 - b)f'(0) & (\alpha_1 - a)f'(0) \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \end{pmatrix} = \begin{pmatrix} -P_1 \\ -P_2 \end{pmatrix}. \tag{3.65}$$

Hence,

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, \qquad E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \tag{3.66}$$

where

$$\begin{aligned}
\Delta_2 &= \det\begin{pmatrix} (\alpha_1 + a)f'(0) & (\alpha_2 + b)f'(0) \\ (\alpha_2 - b)f'(0) & (\alpha_1 - a)f'(0) \end{pmatrix}, \\
\Delta_{21} &= \det\begin{pmatrix} -P_1 & (\alpha_2 + b)f'(0) \\ -P_2 & (\alpha_1 - a)f'(0) \end{pmatrix}, \\
\Delta_{22} &= \det\begin{pmatrix} (\alpha_1 + a)f'(0) & -P_1 \\ (\alpha_2 - b)f'(0) & -P_2 \end{pmatrix}.
\end{aligned} \tag{3.67}$$

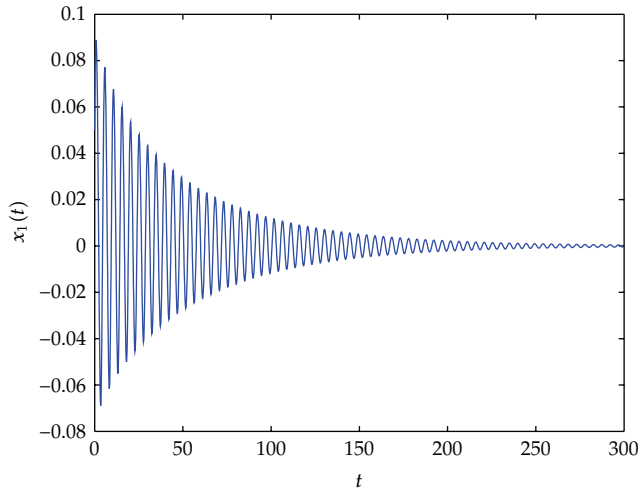

Figure 1: Trajectories graphs of the system (4.1) with $\tau_1 = 0.003, \tau_2 = 0.004$, and $\tau_1 + \tau_2 = 0.007 < \tau^0 \approx 0.01$. The equilibrium $E_*(0,0)$ is asymptotically stable. The initial value is (0.05, 0.01).

From (3.48), (3.50), we can calculate g_{21} and derive the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau^0}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau^0)\}}, \\ \beta_2 &= 2\operatorname{Re}(c_1(0)), \\ T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau^0)\}}{\omega_0\tau^0}. \end{aligned} \tag{3.68}$$

These formulae give a description of the Hopf bifurcation periodic solutions of (1.2) at $\tau = \tau^0$ on the center manifold. From the discussion above, we have the following result.

Theorem 3.3. *The periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$); The bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); the periods of bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*

4. Numerical Examples

In this section, we present some numerical results to verify the analytical predictions obtained in the previous section. As an example, we consider the following special case of the system (1.2) with the parameters $\alpha_1 = -0.03, \alpha_2 = -0.1, a = 1.5, b = 2, c = 0, d = -4$, and $f(x) = \tanh(x)$. Then, the system (1.2) becomes

$$\begin{aligned} \dot{x}_1(t) &= 1.47\tanh(x_1(t)) + 1.9\tanh(x_2(t-\tau_1)), \\ \dot{x}_2(t) &= -2.1\tanh(x_1(t-\tau_2)) - 1.53\tanh(x_2(t)) - 4x_1(t)x_2(t). \end{aligned} \tag{4.1}$$

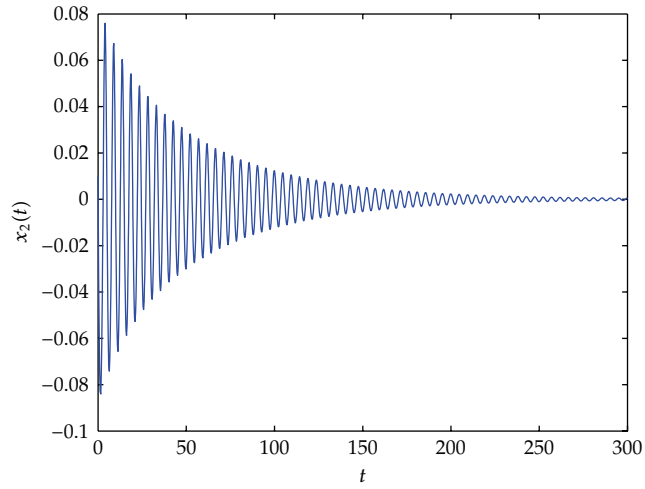

Figure 2: Trajectories graphs of the system (4.1) with $\tau_1 = 0.003, \tau_2 = 0.004$, and $\tau_1 + \tau_2 = 0.007 < \tau^0 \approx 0.01$. The equilibrium $E_*(0,0)$ is asymptotically stable. The initial value is (0.05, 0.01).

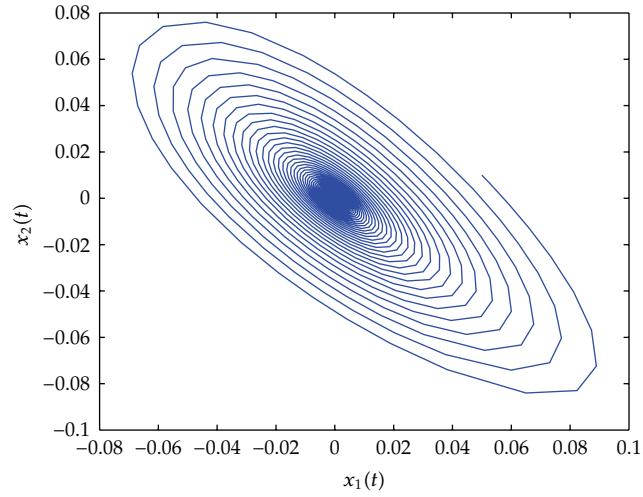

Figure 3: Trajectories graphs of the system (4.1) with $\tau_1 = 0.003, \tau_2 = 0.004$, and $\tau_1 + \tau_2 = 0.007 < \tau^0 \approx 0.01$. The equilibrium $E_*(0,0)$ is asymptotically stable. The initial value is (0.05, 0.01).

By some complicated computation by means of Matlab 7.0, we get $\omega_0 \approx 1.3211$, $\tau_0 \approx 0.01$, $\lambda'(\tau_0) \approx 0.0140 - 1.4926i$. We can easily obtain $g_{20} = -0.2501 + 2.3128i$, $g_{11} = 1.2377 + 0.3484i$, $g_{02} = 0.4533 - 0.5693i$, $g_{21} = -2.3022 + 4.3015i$. Thus, we can calculate the following values: $c_1(0) = -1.0617 - 1.7138i$, $\mu_2 = 75.8357$, $\beta_2 = -2.1234$, $T_2 = 86.9780$. We obtain that the conditions indicated in Theorem 2.6 are satisfied. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the equilibrium $E_*(0,0)$ is stable when $\tau < \tau^0$ as illustrated by the computer simulations (see Figures 1, 2, and 3). When τ passes through the critical value τ^0, the equilibrium $E_*(0,0)$ loses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcations from the equilibrium $E_*(0,0)$. Since $\mu_2 > 0$ and $\beta_2 < 0$,

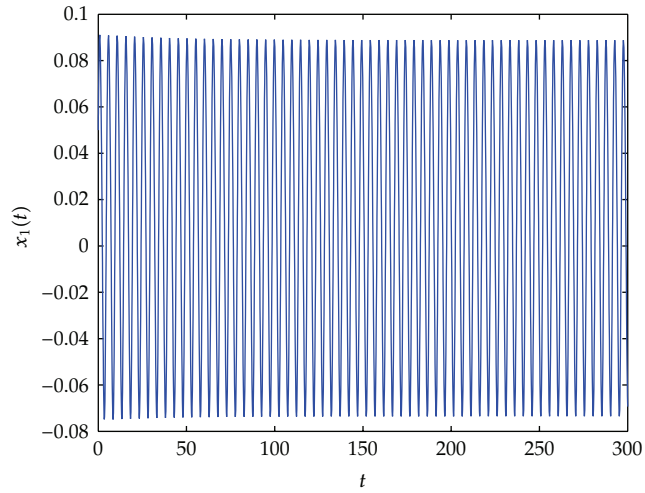

Figure 4: Trajectories graphs of thesystem (4.1) with $\tau_1 = 0.01, \tau_2 = 0.04$, and $\tau_1 + \tau_2 = 0.05 > \tau^0 \approx 0.01$. Hopf bifurcation occurs from the equilibrium $E_*(0,0)$. The initial value is (0.05, 0.01).

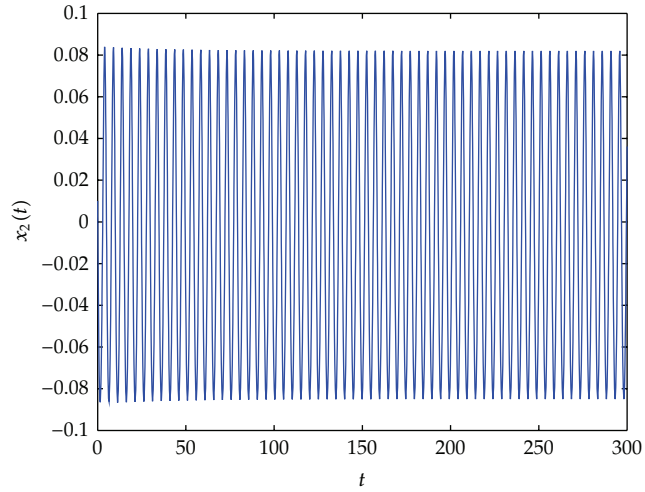

Figure 5: Trajectories graphs of the system (4.1) with $\tau_1 = 0.01, \tau_2 = 0.04$, and $\tau_1 + \tau_2 = 0.05 > \tau^0 \approx 0.01$. Hopf bifurcation occurs from the equilibrium $E_*(0,0)$. The initial value is (0.05, 0.01).

the direction of the Hopf bifurcation is $\tau > \tau^0$ and these bifurcating periodic solutions from $E_*(0,0)$ at τ^0 are stable, which are depicted in Figures 4, 5, and 6.

5. Conclusions

In this paper, we have analyzed a two-neuron networks with resonant bilinear terms. Firstly, we obtained the sufficient conditions to ensure local stability of the equilibrium $E_*(0,0)$ and the existence of local Hopf bifurcation. Moreover, we note also that, if the two-neuron networks with resonant bilinear terms begin with a stable equilibrium, but then become

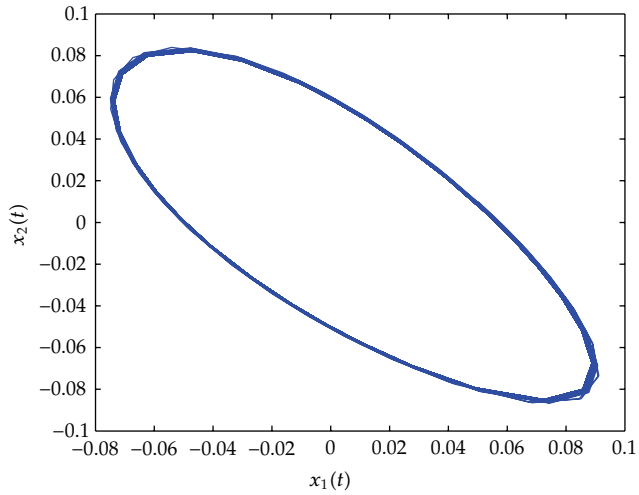

Figure 6: Trajectories graphs of the system (4.1) with $\tau_1 = 0.01, \tau_2 = 0.04$, and $\tau_1 + \tau_2 = 0.05 > \tau^0 \approx 0.01$. Hopf bifurcation occurs from the equilibrium $E_*(0,0)$. The initial value is (0.05, 0.01).

unstable due to delay, then it will likely be destabilized by means of a Hopf bifurcation which leads to periodic solutions with small amplitudes. Finally, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying normal form theory and center manifold theorem.

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