Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 626254, 26 pages
doi:10.1155/2011/626254

Research Article

Inner Functions in Lipschitz, Besov, and Sobolev Spaces

Daniel Girela,[1] Cristóbal González,[1] and Miroljub Jevtić[2]

[1] *Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain*
[2] *Matematički Fakultet, University of Belgrade, PP. 550, 11000 Belgrade, Serbia*

Correspondence should be addressed to Daniel Girela, girela@uma.es

Received 23 February 2011; Accepted 12 April 2011

Academic Editor: Wolfgang Ruess

We study the membership of inner functions in Besov, Lipschitz, and Hardy-Sobolev spaces, finding conditions that enable an inner function to be in one of these spaces. Several results in this direction are given that complement or extend previous works on the subject from different authors. In particular, we prove that the only inner functions in either any of the Hardy-Sobolev spaces H^p_α with $1/p \le \alpha < \infty$ or any of the Besov spaces $B^{p,q}_\alpha$ with $0 < p,\ q \le \infty$ and $\alpha \ge 1/p$, except when $p = \infty$, $\alpha = 0$, and $2 < q \le \infty$ or when $0 < p < \infty$, $q = \infty$, and $\alpha = 1/p$ are finite Blaschke products. Our assertion for the spaces $B^{\infty,q}_0$, $0 < q \le 2$, follows from the fact that they are included in the space *VMOA*. We prove also that for $2 < q < \infty$, *VMOA* is not contained in $B^{\infty,q}_0$ and that this space contains infinite Blaschke products. Furthermore, we obtain distinct results for other values of α relating the membership of an inner function I in the spaces under consideration with the distribution of the sequences of preimages $\{I^{-1}(a)\}$, $|a| < 1$. In addition, we include a section devoted to Blaschke products with zeros in a Stolz angle.

1. Introduction

One of the central questions about inner functions is that of their membership in some classical function spaces. This problem was studied in a number of papers in the 70's and 80's (see, e. g.,[1–10]) and also recently (see, e. g., [11–18]). In this paper, we shall be mainly concerned in studying the membership of inner functions in Besov, Lipschitz, and Hardy-Sobolev spaces.

Denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in the complex plane $\mathbb{C}$, and byHol($\mathbb{D}$) the class of analytic functions on $\mathbb{D}$. The classical Hardy space H^p, $0 < p \le \infty$, consists of those functions $f \in \mathrm{Hol}(\mathbb{D})$ for which

$$\|f\|_{H^p} := \sup_{0<r<1} M_p(r,f) < \infty, \tag{1.1}$$

where $M_p(r,f) = ((1/2\pi)\int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{1/p}$ if $0 < p < \infty$, and $M_\infty(r,f) = \max_{|z|=r}|f(z)|$.

We mention [19, 20] as references for the theory of Hardy spaces. The H^p spaces form a decreasing chain as p increases, and any function in H^p has nontangential limits almost everywhere, with the additional property that the boundary function thus formed defines an isometry between H^p and $L^p(\partial\mathbb{D})$. The sequence $\{z_k\}$ of zeros of an H^p function, counting multiplicities, satisfies the so-called *Blaschke condition*: $\sum_k(1-|z_k|) < \infty$. This condition characterizes the zero sequences of H^p functions. By writing $b_a(z) = (|a|/a)((a-z)/(1-\overline{a}z))$ when $a \in \mathbb{D}$, $a \neq 0$, and $b_0(z) = z$, the Blaschke condition implies that the product $\prod_k b_{z_k}(z)$ converges absolutely and uniformly in each compact subset of $\mathbb{D}$, hence defining a function $B(z)$, called *the Blaschke product associated to the sequence* $\{z_k\}$, which is analytic in the unit disk $\mathbb{D}$, and whose exact sequence of zeros, counting multiplicities, is $\{z_k\}$. Also, $\|B\|_{H^\infty} \leq 1$, and its boundary function has modulus 1 almost everywhere.

An *inner function* in $\mathbb{D}$ is an H^∞ function whose nontangential boundary function has modulus 1 almost everywhere. Thus, Blaschke products are inner functions. Any inner function I admits a factorization of the type $I(z) = e^{i\gamma}B(z)S(z)$, where γ is a real constant, B is a Blaschke product (carrying all the zeros of I), and S is what it is called a *singular inner function*, having the form

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right), \tag{1.2}$$

where μ is a positive Borel measure on $[0,2\pi)$, singular with respect to Lebesgue measure. From now on, a Blaschke product times a unimodular constant (that may be 1) will also be called a Blaschke product, just to simplify the language. Continuing with the terminology that may appear in the paper, a Blaschke product with a finite number of zeros will be called a *finite Blaschke product*, while that with an infinite number of zeros will be called an *infinite Blaschke product*.

The different classes of analytic functions that will be treated in this paper are presented now. Before, let us say a word about the notational conventions used in this paper. Constants will usually be denoted by the letter C. Their dependence on other quantities, if specified, will appear as subindexes. In the same expression, the constant C may change from one occurrence to the other. Two quantities or expressions, A and B, are said to be comparable (written $A \asymp B$) if there exists a positive constant C such that $C^{-1}B \leq A \leq CB$. If functions are involved in the quantities that are being compared, the constants relating them do not usually depend on those functions, neither on their variables.

The *weighted Bergman space* $A^{p,\alpha}$, $0 < p < \infty$, $-1 < \alpha < \infty$ consists of those functions $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{A^{p,\alpha}} := \left(\frac{\alpha+1}{\pi}\int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^\alpha dA(z)\right)^{1/p} < \infty, \tag{1.3}$$

where $dA(z) = dxdy$ denotes the Lebesgue area measure in $\mathbb{D}$. We mention the books [21, 22] as general references for the theory of Bergman spaces.

If $f(z) = \sum_{k=0}^{\infty} \widehat{f}(k)z^k$ is analytic in $\mathbb{D}$ and $\alpha > 0$, *the fractional derivative of order* α *of* f, $D^\alpha f$ is defined as

$$D^\alpha f(z) = \sum_{k=0}^{\infty}(k+1)^\alpha \widehat{f}(k)z^k. \tag{1.4}$$

(This definition also makes sense for $\alpha \le 0$, only that the name "derivative" would look a bit awkward.) For a positive integer n, it is actually an "n-th derivative": $D^n f(z) = ((d/dz)z)^n f(z)$. Besides, the integral means $M_p(r, D^n f)$ and $M_p(r, f^{(n)})$ are usually interchangeable, in the sense that their quotient is bounded away from 0 and ∞ (see, e.g., [23] for this fact and the formulas that appear below). In general, there is a formula to recover the function from its fractional derivative, which can be easily verified,

$$f(z) = \frac{1}{\Gamma(\alpha)}\int_0^1 D^\alpha f(sz)\log^{\alpha-1}\frac{1}{s}ds. \tag{1.5}$$

This yields the estimate

$$\left|f(z)\right| \le C\int_0^1 \left|D^\alpha f(sz)\right|(1-s)^{\alpha-1}ds. \tag{1.6}$$

If $0 < p \le \infty$ and $0 \le \alpha < \infty$, then a function $f \in \mathrm{Hol}(\mathbb{D})$ is said to belong to the *Lipschitz space* $\Lambda^{p,\alpha}$ if

$$\|f\|_{\Lambda^{p,\alpha}} := \sup_{0<r<1}(1-r)M_p\left(r, D^{1+\alpha}f\right) < \infty. \tag{1.7}$$

The subspace $\lambda^{p,\alpha}$ consists of those $f \in \mathrm{Hol}(\mathbb{D})$ for which

$$\lim_{r\to 1}(1-r)M_p\left(r, D^{1+\alpha}f\right) = 0. \tag{1.8}$$

Notice that $\Lambda^{\infty,0}$ is another name for the usual Bloch space $\mathcal{B}$, of functions $f \in \mathrm{Hol}(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$. Analogously, $\lambda^{\infty,0}$ is the little Bloch space $\mathcal{B}_0$, of functions $f \in \mathrm{Hol}(\mathbb{D})$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$.

Replacing the sup-norm with an L^q-norm in the above definition gives way to the *Besov space* $B_\alpha^{p,q}$, $0 < p \le \infty$, $0 < q < \infty$, $0 \le \alpha < \infty$, consisting of those functions $f \in \mathrm{Hol}(\mathbb{D})$ for which

$$\|f\|_{B_\alpha^{p,q}} := \left(\int_0^1 (1-r)^{q-1}M_p^q\left(r, D^{1+\alpha}f\right)dr\right)^{1/q} < \infty. \tag{1.9}$$

The space obtained when $q = \infty$ is precisely $B^{p,\infty}_\alpha \equiv \Lambda^{p,\alpha}$. Other classes treated in this paper are the *Hardy-Sobolev spaces* H^p_α, $0 < p \le \infty$, $0 \le \alpha < \infty$, consisting of those functions $f \in \mathrm{Hol}(\mathbb{D})$ for which

$$\|f\|_{H^p_\alpha} := \|D^\alpha f\|_{H^p} < \infty. \tag{1.10}$$

The paper of Flett [23] gives many relations between the different types of integrals just mentioned above. Since they will occur recurrently along this paper, it may be convenient to state once for all some of them. Starting from the estimate (1.6), and working it out into the integral means of order p gives

$$M^p_p(r,f) \le C\int_0^1 (1-s)^{p\alpha-1} M^p_p(rs, D^\alpha f)\,ds, \quad (0 < p < 1,\ 0 < \alpha), \tag{1.11}$$

$$M_p(r,f) \le C\int_0^1 (1-s)^{\alpha-1} M_p(rs, D^\alpha f)\,ds, \quad (1 \le p \le \infty,\ 0 < \alpha). \tag{1.12}$$

The same estimate (1.6) combined with variants of Hardy's inequality gives

$$\begin{gathered}\sup_{0<\rho\le r}(1-\rho)^\alpha M_p\left(\rho, D^\beta f\right) \asymp \sup_{0<\rho\le r}(1-\rho)^{\alpha-\beta+\gamma} M_p(\rho, D^\gamma f), \\ (0 < p \le \infty, \quad \alpha > \max\{0, \beta-\gamma\}),\end{gathered} \tag{1.13}$$

$$\begin{gathered}\int_0^1 (1-r)^{\alpha-1} M^q_p\left(r, D^\beta f\right)dr \asymp \int_0^1 (1-r)^{\alpha-q(\beta-\gamma)-1} M^q_p(r, D^\gamma f)\,dr, \\ (0 < q < \infty, \quad 0 < p \le \infty, \quad \alpha > \max\{0, q(\beta-\gamma)\}).\end{gathered} \tag{1.14}$$

We continue displaying more estimates. The following one is due essentially to Hardy and Littlewood (see, e.g., [19, Theorem 5.9], or [24, Lemma 3.4]):

$$M_q(r,f) \le C_{p,q}(1-r)^{1/q-1/p} M_p\left(\frac{1+r}{2}, f\right), \quad (0 < p < q \le \infty). \tag{1.15}$$

Finally, we provide also the following estimates, due to Littlewood and Paley [25, Theorems 5 and 6], for $p \ge 1$, and to Vinogradov [26, Lemma 1.4], for $0 < p < 1$.

$$\sup_{0\le r<1} M_p(r,f) \le C_p \int_0^1 (1-r)^{p-1} M^p_p\left(r, D^1 f\right)dr, \quad (0 < p \le 2), \tag{1.16}$$

$$\int_0^1 (1-r)^{p-1} M^p_p\left(r, D^1 f\right)dr \le C_p \sup_{0\le r<1} M_p(r,f), \quad (p \ge 2), \tag{1.17}$$

which can be restated as $B^{p,p}_0 \subseteq H^p$, for $0 < p \le 2$, and $H^p \subseteq B^{p,p}_0$, for $p \ge 2$.

With the above estimates, we can easily obtain some more relations of inclusion between the different spaces considered in this paper. We enumerate some of these relations, with the purpose of having them at hand.

($P1$) If $q < \infty$, $B_\alpha^{p,q} \subseteq \lambda^{p,\alpha}$: If $f \in B_\alpha^{p,q}$, then $\int_r^1 (1-\rho)^{q-1} M_p^q(\rho, D^{1+\alpha} f) d\rho \to 0$ as $r \to 1$, so by using the increasing behavior of the means $M_p(r, D^{1+\alpha} f)$, obtain that $(1-r) M_p(r, D^{1+\alpha} f) \to 0$, as $r \to 1$, that is, that $f \in \lambda^{p,\alpha}$.

($P2$) If $q_1 \leq q_2$, $B_\alpha^{p,q_1} \subseteq B_\alpha^{p,q_2}$, because $(1-r) M_p(r, D^{1+\alpha} f) \to 0$, as $r \to 1$, for all $f \in B_\alpha^{p,q_1}$.

($P3$) If $p_1 \leq p_2$, $B_\alpha^{p_2,q} \subseteq B_\alpha^{p_1,q}$, and $H_\alpha^{p_2} \subseteq H_\alpha^{p_1}$, because $M_{p_1}(r, g) \leq M_{p_2}(r, g)$ for all $r \in (0,1)$ and all $g \in \mathrm{Hol}(\mathbb{D})$.

($P4$) If $\alpha_1 \leq \alpha_2$, $B_{\alpha_2}^{p,q} \subseteq B_{\alpha_1}^{p,q}$. Indeed, for $f \in B_{\alpha_2}^{p,q}$, (1.14) gives

$$\|f\|_{B_{\alpha_1}^{p,q}}^q \asymp \int_0^1 (1-r)^{q-1+q(\alpha_2-\alpha_1)} M_p^q\left(r, D^{1+\alpha_2} f\right) dr < \|f\|_{B_{\alpha_2}^{p,q}}^q < \infty. \tag{1.18}$$

($P5$) If $\alpha_1 \leq \alpha_2$, $H_{\alpha_2}^p \subseteq H_{\alpha_1}^p$, because of (1.11) and (1.12).

($P6$) If $0 < p_1 \leq p_2$, $B_{1/p_1}^{p_1,q} \subseteq B_{1/p_2}^{p_2,q}$. (When $q = \infty$, this says that $\Lambda^{p,1/p}$ increases with p). A proof of this lies in an application of (1.15) and (1.14),

$$\|f\|_{B_{1/p_2}^{p_2,q}}^q \asymp \int_0^1 (1-r)^{q-1+q(1/p_1-1/p_2)} M_{p_2}^q\left(r, D^{1+1/p_1} f\right) dr \leq C \|f\|_{B_{1/p_1}^{p_1,q}}^q. \tag{1.19}$$

($P7$) $\Lambda^{p,\alpha} \subseteq \cap \{ B_\beta^{p,q} : \quad \beta < \alpha, \quad 0 < q \}$. This is again an application of (1.14),

$$\|f\|_{B_\beta^{p,q}}^q \asymp \int_0^1 (1-r)^{q(\alpha-\beta)-1+q} M_p^q\left(r, D^{1+\alpha} f\right) dr \leq C \|f\|_{\Lambda^{p,\alpha}}, \tag{1.20}$$

where $C = \int_0^1 (1-r)^{q(\alpha-\beta)-1} dr < \infty$.

($P8$) $H_\alpha^p \subseteq \cap \{ B_\beta^{p,q} : \quad \beta < \alpha, \quad 0 < q \}$. Again an application of (1.14).

($P9$) If $0 < p \leq 2$, $B_\alpha^{p,p} \subseteq H_\alpha^p$. This is (1.16).

($P10$) If $2 \leq p$, $H_\alpha^p \subseteq B_\alpha^{p,p}$. And this is (1.17).

In the following, we introduce some notation related to inner functions. Given an inner function I and a point $a \in \mathbb{D}$, its *Frostman shift* I_a is defined as

$$I_a(z) = \frac{I(z) - a}{1 - \overline{a} I(z)}, \quad z \in \mathbb{D}. \tag{1.21}$$

A classical result of Frostman (see, e.g., [20, Section 2.6] asserts that if I is an inner function in $\mathbb{D}$ then the Frostman shifts I_a are Blaschke products for all $a \in \mathbb{D}$ except for those in a set E (depending on I) of logarithmic capacity zero. Even more, if I cannot be analytically continued across one boundary point, that is, if I is not a finite Blaschke product, then for all $a \in \mathbb{D} \setminus E$, with E a set of logarithmic capacity zero, the Frostman shift I_a is an infinite Blaschke product.

The fact that mixed norms of derivatives of an inner function are comparable to those of its Frostman shifts must be a well known result, which we have not found in the literature. Since it plays a key role to obtain some of our results, we include a proof just for the sake of completeness.

Lemma 1.1. *Let $0 < p, q \leq \infty$, $0 \leq \alpha$, $0 < \delta < 1/2$, and $a \in K_\delta = \{z \in \mathbb{D} : \quad \delta < |z| < 1 - \delta\}$. Then $\|I_a\|_{B^{p,q}_\alpha} \asymp \|I\|_{B^{p,q}_\alpha}$, with constants depending only on p, q, α, and δ, but not on I or a. In the case $q = \infty$, the formulation is*

$$\sup_{0 \leq \rho < r} (1-\rho) M_p\left(\rho, D^{1+\alpha} I_a\right) \asymp \sup_{0 \leq \rho < r} (1-\rho) M_p\left(\rho, D^{1+\alpha} I\right). \tag{1.22}$$

Proof. Put $1+\alpha = n+\beta$, with n a positive integer and $\beta \in [0,1)$. We shall only consider the case $q < \infty$. The procedure for $q = \infty$ is rather similar. Also, it is enough to estimate the $B^{p,q}_\alpha$-norm of I_a in terms of that of I, for $I = (I_a)_{-a}$.

Writing $\psi_a(z) = \overline{a}/(1 - \overline{a}z)$, $z \in \mathbb{D}$ observe that the first derivative of I_a can be written as $I_a' = ((1-|a|^2)/\overline{a}^2)(\psi_a \circ I)'$, and, in general, for a positive integer n, the nth derivative is given by $I_a^{(n)} = ((1-|a|^2)/\overline{a}^2)(\psi_a \circ I)^{(n)}$. This, together with the Faà di Bruno's formula for the nth derivative of a composition, gives

$$I_a^{(n)} = \frac{1-|a|^2}{\overline{a}^2}(\psi_a \circ I)^{(n)} = \frac{1-|a|^2}{\overline{a}^2} \sum \frac{n!}{k_1! \cdots k_n!} \psi_a^{(k)} \circ I \prod_{j=1}^{n} \left(\frac{I^{(j)}}{j!}\right)^{k_j}, \tag{1.23}$$

where $k = \sum_{j=1}^n k_j$, and the sum runs over all n-tuples $\vec{k} = (k_1, \ldots, k_n)$ of nonnegative integers such that $\sum_{j=1}^n j k_j = n$. Observe that if $a \in K_\delta$, the quantities $(1-|a|^2)/|\overline{a}|^2$ and $|\psi_a^{(k)} \circ I| = |\psi \circ I|^{k+1}$ are bounded away from 0 and ∞ by constants depending only on k and δ, but not on a.

Now, to estimate $\|I_a\|^q_{B^{p,q}_\alpha}$, we use, in order, (1.14), (1.23), twice Hölder's inequality (Hö) with indices $\{n/(jk_j) : k_j \neq 0\}$, again (1.14), and finally we appeal to the fact (I) that $|I^{(n)}(z)| \leq C_n (1-|z|)^{-n}$ (obtained as a result of using Cauchy's integral formula for the nth derivative and Lemma 3 in Section 5.5 of Duren's book [19]),

$$\begin{aligned}
\|I_a\|^q_{B^{p,q}_\alpha} &= \int_0^1 (1-r)^{q-1} M_p^q\left(r, D^{n+\beta} I_a\right) dr \overset{(1.14)}{\leq} C \int_0^1 (1-r)^{q(1-\beta)-1} M_p^q\left(r, I_a^{(n)}\right) dr \\
&\overset{(1.23)}{\leq} C \sum_{\vec{k}} \int_0^1 (1-r)^{q(1-\beta)-1} M_p^q\left(r, \prod_{j=1}^n \left(I^{(j)}\right)^{k_j}\right) dr
\end{aligned}$$

$$
\begin{aligned}
&\overset{(\text{Hö})}{\leq} C\sum_{\vec{k}}\int_0^1 (1-r)^{q(1-\beta)-1}\prod_{j=1}^{n} M_{pn/j}^{qk_j}\left(r, I^{(j)}\right)dr \\
&\overset{(\text{Hö})}{\leq} C\sum_{\vec{k}}\prod_{j=1}^{n}\left(\int_0^1 (1-r)^{q(1-\beta)-1} M_{pn/j}^{qn/j}\left(r, I^{(j)}\right)dr\right)^{jk_j/n} \\
&\overset{(1.14)}{\leq} C\sum_{\vec{k}}\prod_{j=1}^{n}\left(\int_0^1 (1-r)^{q(1-\beta)-1+(qn(n-j))/j} M_{pn/j}^{qn/j}\left(r, I^{(n)}\right)dr\right)^{jk_j/n} \\
&\overset{(\text{I})}{\leq} C\sum_{\vec{k}}\prod_{j=1}^{n}\left(\int_0^1 (1-r)^{q(1-\beta)-1} M_{p}^{q}\left(r, I^{(n)}\right)dr\right)^{jk_j/n} \overset{(1.14)}{\leq} C\|I\|_{B_\alpha^{p,q}}^{q}.
\end{aligned}
\tag{1.24}
$$

□

2. Inner Functions in the Spaces $B_\alpha^{p,q}$ and H_α^p, $\alpha \geq 1/p$

Ahern and Jevtić [5] proved that a Blaschke product lies in the space $B_{1/p}^{p,\infty} \equiv \Lambda^{p,1/p}$ ($0 < p < \infty$) if and only if its sequence of zeros is a finite union of exponential sequences, (see also Verbitskiĭ [27] for the case $1 \leq p < \infty$). We refer the reader to the recent work of Jevtić [15] on this subject where references to previous works are given. In particular, we have

$$
\text{for } 0 < p < \infty, \quad \text{the space } B_{1/p}^{p,\infty} \text{ contains infinite Blaschke products.} \tag{2.1}
$$

Our results in this section imply that the opposite is true for all the spaces $B_\alpha^{p,q}$ with $0 < p, q \leq \infty$ and $\alpha \geq 1/p$, except for the mentioned case, $0 < p < \infty$, $q = \infty$ and $\alpha = 1/p$, and when $p = \infty$, $\alpha = 0$ and $2 < q \leq \infty$.

Theorem 2.1. (a) *Let $0 < p, q \leq \infty$ and $\alpha > 1/p$. Then the only inner functions in $B_\alpha^{p,q}$ are finite Blaschke products.*

(b) *If $0 < p, q < \infty$ then the only inner functions in $B_{1/p}^{p,q}$ are finite Blaschke products.*

Proof. To prove (a) observe that $B_\alpha^{p,q} \subseteq B_\alpha^{p,\infty} \equiv \Lambda^{p,\alpha} \subseteq \Lambda^{\infty,\alpha-1/p}$. The first inclusion comes from the properties above and the last inclusion may be found in [28, Corollary 2.3], or directly using (1.13). Now, it is well known (see, e.g., [19, Theorem 5.1] that any function in $\Lambda^{\infty,\beta}$, with $\beta > 0$, (even if it is forced to be a constant), belongs to the disk algebra $\mathcal{A}$ (that is, it admits a continuous extension to $\overline{\mathbb{D}}$). Thus if we are in the conditions of part (a) and I is an inner function in $B_\alpha^{p,q}$ then $I \in \mathcal{A}$. Then it follows easily that I is a finite Blaschke product. Indeed, write $I(z) = S(z)B(z)$, where B is a Blaschke product and S is a singular inner function. The fact that $I \in \mathcal{A}$ readily implies that B is a finite Blaschke product and then it follows that S also belongs to $\mathcal{A}$. Then S is a function in the disk algebra without zeros and with $|S(\xi)| = 1$, for all $\xi \in \partial\mathbb{D}$. A simple application of the maximum-minimum principle readily yields that S is a unimodular constant. Thus, I is a finite Blaschke product as asserted.

Let us now turn to prove part (b). The following results come in our aid.

Theorem A (see [8, Corollary 1.6]). *Let $0 < p < \infty$. If B is a Blaschke product in $H^p_{1/p}$, then B is a finite Blaschke product.*

Theorem B (see [5, Theorem 3.2]). *For each $0 < p < \infty$ there exists $\varepsilon_p > 0$ such that if B is a Blaschke product and*

$$\limsup_{r\to 1}(1-r)M_p\left(r, D^{1+1/p}B\right) < \varepsilon_p, \tag{2.2}$$

then B is a finite Blaschke product. In particular, the only Blaschke products in $\lambda^{p,1/p}$ are the finite ones.

Using Lemma 1.1, and the fact that the Frostman shifts of a finite Blaschke product are again finite Blaschke products: we deduce that Theorem A and Theorem B yield the following.

Proposition 2.2. *For $0 < p < \infty$, the only inner functions in either $H^p_{1/p}$ or $\lambda^{p,1/p}$ are the finite Blaschke products.*

Now Theorem 2.1(b) follows from this result and the fact that $B^{p,q}_{1/p} \subseteq \lambda^{p,1/p}$. □

The fact that $H^p_\alpha \subseteq \cap\{B^{p,q}_\beta : \beta < \alpha, 0 < q\}$ immediately implies the following.

Corollary 2.3. *Let $0 < p \le \infty$ and $\alpha > 1/p$. If $f \in H^p_\alpha$ then f admits a continuous extension to $\overline{\mathbb{D}}$. Consequently, the only inner functions in H^p_α, with $\alpha > 1/p$, are finite Blaschke products.*

It remains to consider the case $p = \infty$ and $\alpha = 0$. Of course, $H^\infty_0 \equiv H^\infty$ contains the whole class of inner functions.

Let us deal now with the spaces $B^{\infty,q}_0$. First of all, $B^{\infty,\infty}_0$ is the Bloch space, and, hence, it contains all inner functions.

Bishop [29] proved that the little Bloch space, $\lambda^{\infty,0}$, contains infinite Blaschke products. Since, for $q < \infty$, $B^{\infty,q}_0$ is a subspace of $\lambda^{\infty,0}$, the natural question rises as whether $B^{\infty,q}_0$ contains or not infinite Blaschke products. The answer depends on the result of intersecting $B^{\infty,q}_0$ with the subspace $VMOA$ of $\lambda^{\infty,0}$, consisting of those H^1 functions whose boundary values have vanishing mean oscillation. The space $VMOA$ was introduced by Sarason [30] and admits a number of equivalent definitions. Among them, we mention that a function $f \in H^1$ is said to belong to $VMOA$ if

$$\lim_{|a|\to 1}\left\| f \circ \varphi_a - f(a)\right\|_{H^p} = 0, \tag{2.3}$$

for some (or, equivalently, for all) finite positive p. Here, $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ is the typical involutive automorphism of $\mathbb{D}$ interchanging the points 0 and $a \in \mathbb{D}$. Using this definition and the fact that nonconstant inner functions take values as close to 0 as desired, Anderson [31] proved that $VMOA$ contains no inner functions other than finite Blaschke products. (See also

[32] for an extensive survey on $BMOA$ and $VMOA$.) In the following result, we use another characterization of $VMOA$; it is the space of functions $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\frac{1}{|J|}\int_{S(J)}\left(1-|z|^2\right)\left|f'(z)\right|dA(z) \longrightarrow 0, \quad \text{as} |J| \longrightarrow 0, \tag{2.4}$$

where J is an interval in $\partial\mathbb{D}$, $|J|$ is its length, and $S(J)$ is the Carleson square defined by $S(J) = \{re^{i\theta} : \ e^{i\theta} \in J, \ 1-|J| \leq r < 1\}$.

Theorem 2.4. (a), *If* $0 < q \leq 2$, *then* $B_0^{\infty,q} \subseteq VMOA$. *Consequently, any inner function in* $B_0^{\infty,q}$ *is a finite Blaschke product.*

(b) *There are infinite Blaschke products in* $\cap_{2<q<\infty} B_0^{\infty,q}$.

Proof. To prove (a) observe that, since $B_0^{\infty,q_1} \subseteq B_0^{\infty,q_2}$ for $0 < q_1 < q_2 < \infty$, it suffices to settle the result for $q = 2$.

Thus, take $f \in B_0^{\infty,2}$ and take an interval J in $\partial\mathbb{D}$ with $|J| < 1/2$, then

$$\begin{aligned}\frac{1}{|J|}\int_{S(J)}\left(1-|z|^2\right)\left|f'(z)\right|^2 dA(z) &\leq 2\frac{1}{|J|}\int_{1-|J|}^{1}(1-r)\int_J \left|f'\left(re^{i\theta}\right)\right|^2 d\theta dr \\ &\leq 2\int_{1-|J|}^{1}(1-r)M_\infty^2(r,f')dr,\end{aligned} \tag{2.5}$$

and observe that, since $f \in B_0^{\infty,2}$, the right hand side tends to 0 as $|J| \to 0$.

To prove (b), observe that, by Theorem 5.2 of [33], there is a (singular) inner function I such that

$$\left(1-|z|^2\right)\frac{|I'(z)|}{1-|I(z)|^2} \leq \log^{-1/2}\frac{e}{1-|z|}. \tag{2.6}$$

This implies that $I \in B_0^{\infty,q}$ for all $q > 2$. Also, as explained also in [33, after (1.1)], such inner function cannot be analytically continued across any boundary point of $\mathbb{D}$. Therefore, we may choose $a \in \mathbb{D}$ such that the Frostman shift I_a is an infinite Blaschke product (actually, this is true for all $a \in \mathbb{D}$ except for those in a set of zero logarithmic capacity). Now, Lemma 1.1 shows that I_a is an infinite Blaschke product in $\cap_{2<q<\infty} B_0^{\infty,q}$. □

Once Theorem 2.4 is proved, it is natural to ask whether or not the inclusion $VMOA \subseteq B_0^{\infty,q}$ holds for $2 < q < \infty$. An argument based on duality shows that this is not so.

Theorem 2.5. *If* $2 < q < \infty$, *then the class* $VMOA \setminus B_0^{\infty,q}$ *is nonempty.*

Proof. Observe that the dual of $VMOA$ is H^1 under the usual pairing: $\langle f,g\rangle = \lim_{r\to 1}\sum_k \widehat{f}(k)\overline{\widehat{g}(k)}r^k$, (see [32, 34]). Also, using the same techniques as in [4], we get that the dual of $B_0^{\infty,q}$ is $B_0^{1,q'}$, $1/q + 1/q' = 1$, under the same pairing as before. Thus, the problem reduces to show that $B_0^{1,q'} \setminus H^1$ is nonempty.

It is shown in Theorem 3 of [35] that the function $f(z) = ((1-z)\log(2e/(1-z)))^{-1}$, $z \in \mathbb{D}$ is univalent in $\mathbb{D}$ and $f \notin H^1$. Also, an argument given in [24, page 61] shows that there exist $c > 0$ and $r_0 \in (0,1)$ such that

$$M_1(r, f') \leq \frac{c}{(1-r)\left(\log(2e/(1-r))\right)}, \quad r_0 < r < 1. \tag{2.7}$$

It then follows that $f \in B_0^{1,q'}$, whenever $1 < q' < \infty$. This finishes the proof. □

3. The Case Max $\{0, 1/p - 1\} < \alpha < 1/p$

For this range of values, we shall obtain a number of results relating the membership of an inner function I in Besov or Hardy-Sobolev spaces with the distribution of the preimages $\{I^{-1}(a)\}$, $a \in \mathbb{D}$. We start introducing certain counting functions.

If I is an inner function and $a \in \mathbb{D}$, denote by $\{z_k(a)\}$ the exact sequence of zeros, multiplicities included, of I_a, placed in increasing modulus as the subindex k increases (in other words, $\{z_k(a)\}$ is the ordered sequence of preimages of a). Writing $d_k(a) = 1 - |z_k(a)|$, the distribution of zeros in each annulus may be studied with the sequences $\{k_n(a)\}_{n=0}^{\infty}$ and $\{\nu_n(a)\}_{n=0}^{\infty}$:

$$\begin{aligned} k_n(a) &= \operatorname{Card}\{k : 2^{-n} < d_k(a)\} = \max\{k : 2^{-n} < d_k(a)\}, \\ \nu_n(a) &= \operatorname{Card}\left\{k : 2^{-n-1} < d_k(a) \leq 2^{-n}\right\} = k_{n+1}(a) - k_n(a). \end{aligned} \tag{3.1}$$

Observe that $k_0(a) = 0$ always. When $a = 0$, just write $\{z_k\}$, $\{d_k\}$, $\{k_n\}$, and $\{\nu_n\}$. The following relations may be used in the text without further notice.

Lemma 3.1. *Under the previous settings, let $\alpha, \beta > 0$. Then*

(a) $\{2^{-n\alpha}\nu_n^{\beta}(a)\} \in \ell^{\infty}$ *if and only* $\{d_k^{\alpha}(a)k^{\beta}\} \in \ell^{\infty}$, *and, in either case, their ℓ^{∞}-norms are comparable.*

(b) $\sum_{n\geq 0} 2^{-n\alpha}\nu_n^{\beta}(a) \asymp \sum_{k\geq 1} d_k^{\alpha}(a)k^{\beta-1}$.

Proof. In order to keep up with readability, it is better to omit the letter value a in what follows, that is, assume $a = 0$.

To prove (a), assume first that $d_k^{\alpha}k^{\beta} \leq C$ for all k, then for each $n = 0, 1 \ldots$,

$$2^{-n\alpha}\nu_n^{\beta} = 2^{-n\alpha}(k_{n+1} - k_n)^{\beta} \leq 2^{\alpha} d_{k_{n+1}}^{\alpha} k_{n+1}^{\beta} \leq C. \tag{3.2}$$

In the other direction, assume that $2^{-n\alpha}v_n^\beta \le C$ for all n. Given k find the unique $n = n(k)$ such that $2^{-n-1} < d_k \le 2^{-n}$. This implies that $k \le k_{n+1}$, and thus,

$$\begin{aligned} d_k^\alpha k^\beta \le 2^{-n\alpha}k_{n+1}^\beta &= \left(\sum_{j=0}^{n} 2^{-\alpha j/\beta} v_j 2^{-\alpha(n-j)/\beta}\right)^\beta, \\ &\le C\left(\sum_{j=0}^{n} 2^{-\alpha(n-j)/\beta}\right)^\beta \le C\left(1-2^{-\alpha/\beta}\right)^{-\beta}. \end{aligned} \tag{3.3}$$

To prove (b), assume first that $\sum_{n\ge0} 2^{-n\alpha} v_n^\beta < \infty$. In the case $\beta \le 1$, use an easy integral estimate and the fact that $k_{n+1}^\beta - k_n^\beta \le (k_{n+1}-k_n)^\beta$, to obtain the desired result,

$$\begin{aligned} \sum_{k\ge1} d_k^\alpha k^{\beta-1} &= \sum_{n\ge0}\sum_{k=k_n+1}^{k_{n+1}} d_k^\alpha k^{\beta-1} \le \sum_{n\ge0} 2^{-n\alpha} \sum_{k=k_n+1}^{k_{n+1}} k^{\beta-1} \\ &\le \sum_{n\ge0} 2^{-n\alpha} \int_{k_n}^{k_{n+1}} x^{\beta-1}dx = \frac{1}{\beta}\sum_{n\ge0} 2^{-n\alpha}\left(k_{n+1}^\beta - k_n^\beta\right) \\ &\le \frac{1}{\beta}\sum_{n\ge0} 2^{-n\alpha}(k_{n+1}-k_n)^\beta = \frac{1}{\beta}\sum_{n\ge0} 2^{-n\alpha} v_n^\beta. \end{aligned} \tag{3.4}$$

In the case $\beta > 1$, it is easy to arrive at $\sum_{k\ge1} d_k^\alpha k^{\beta-1} \le \sum_{n\ge0} 2^{-n\alpha} k_{n+1}^\beta$. Now we imitate the proof of Hardy's inequality given in [36, Theorem 326 in page 239]. Write $h_n = 2^{-n\alpha/\beta}k_{n+1} = 2^{-n\alpha/\beta}\sum_{j=0}^{n} v_j$, for $n \ge 0$, and $h_{-1} = 0$. Then $v_n = 2^{n\alpha/\beta}(h_n - 2^{-\alpha/\beta}h_{n-1})$, for $n \ge 0$, and by the inequality between the geometric and arithmetic means [36, Theorem 9 in page 17],

$$\begin{aligned} h_n^\beta - \frac{1}{1-2^{-\alpha/\beta}} h_n^{\beta-1} 2^{-n\alpha/\beta} v_n &= \frac{1}{2^{\alpha/\beta}-1}\left(h_n^{\beta-1}h_{\mathrm{n}-1} - h_n^\beta\right) \\ &\le \frac{1}{2^{\alpha/\beta}-1}\left(\frac{(\beta-1)h_n^\beta + h_{n-1}^\beta}{\beta} - h_n^\beta\right) \\ &= \frac{1}{\beta(2^{\alpha/\beta}-1)}\left(h_{n-1}^\beta - h_n^\beta\right). \end{aligned} \tag{3.5}$$

So the sum on n of the left hand side is negative, and, therefore, using Hölder's inequality with exponents $\beta/(\beta-1)$ and β,

$$\begin{aligned} \sum_{n\ge0} 2^{-n\alpha}k_{n+1}^\beta = \sum_{n\ge0} h_n^\beta &\le \frac{1}{1-2^{-\alpha/\beta}}\sum_{n\ge0} h_n^{\beta-1} 2^{-n\alpha/\beta} v_n \\ &\le \frac{1}{1-2^{-\alpha/\beta}}\left(\sum_{n\ge0} h_n^\beta\right)^{\beta-1/\beta}\left(\sum_{n\ge0} 2^{-n\alpha} v_n^\beta\right)^{1/\beta}, \end{aligned} \tag{3.6}$$

giving as a result that $\sum_{n\ge0} 2^{-n\alpha}k_{n+1}^\beta \le (1-2^{-\alpha/\beta})^{-\beta}\sum_{n\ge0} 2^{-n\alpha} v_n^\beta$, as desired.

In the other direction, assume that $\sum_{k\geq 1} d_k^\alpha k^{\beta-1} < \infty$. In the case $\beta < 1$, it is easily verified that $\sum_{n\geq 0} 2^{-n\alpha}\nu_n^\beta \leq \sum_{n\geq 0} 2^{-n\alpha}k_{n+1}^\beta$. To continue, use that $k_0 = 0$ and that the function $x \mapsto x^{\beta-1}$ is decreasing in $(0,\infty)$,

$$\begin{aligned}\sum_{n\geq 0} 2^{-n\alpha}k_{n+1}^\beta &= \sum_{n\geq 0} 2^{-n\alpha}k_{n+1}^{\beta-1}\sum_{k=1}^{k_{n+1}} 1 \leq \sum_{n\geq 0} 2^{-n\alpha}\sum_{k=k_n+1}^{k_{n+1}} k^{\beta-1} + \sum_{n\geq 1} 2^{-n\alpha}k_n^\beta \\ &\leq 2^\alpha \sum_{n\geq 0}\sum_{k=k_n+1}^{k_{n+1}} d_k^\alpha k^{\beta-1} + 2^{-\alpha}\sum_{n\geq 0} 2^{-n\alpha}k_{n+1}^\beta,\end{aligned} \tag{3.7}$$

from where it follows that $\sum_{n\geq 0} 2^{-n\alpha}k_{n+1}^\beta \leq 2^\alpha(1-2^{-\alpha})^{-1}\sum_{k\geq 1} d_k^\alpha k^{\beta-1} < \infty$.

It remains to deal with the case $\beta \geq 1$ under the assumption $\sum_{k\geq 1} d_k^\alpha k^{\beta-1} < \infty$. Here, we use that, when $0 \leq a \leq b$, $(b-a)^\beta \leq b^\beta - a^\beta$ (because $((b-a)^\beta + a^\beta)^{1/\beta} \leq b$), and use also the Mean Value Theorem,

$$\begin{aligned}\sum_{n\geq 0} 2^{-n\alpha}\nu_n^\beta &= \sum_{n\geq 0} 2^{-n\alpha}(k_{n+1}-k_n)^\beta \\ &\leq \sum_{n\geq 0} 2^{-n\alpha}\left(k_{n+1}^\beta - k_n^\beta\right) = 2^\alpha \sum_{n\geq 0} 2^{-(n+1)\alpha}\sum_{k=k_n+1}^{k_{n+1}}\left(k^\beta - (k-1)^\beta\right) \\ &\leq 2^\alpha\beta\sum_{n\geq 0}\sum_{k=k_n+1}^{k_{n+1}} d_k^\alpha k^{\beta-1} = 2^\alpha\beta\sum_{k\geq 1} d_k^\alpha k^{\beta-1}.\end{aligned} \tag{3.8}$$

□

Now we recall the following characterization, due to Ahern [37, Theorem 6].

Theorem C (see [37, Theorem 6]). *Assume that $0 < p, q < \infty$, that $0 < \alpha < 1$, and that I is an inner function. Then the following quantities are comparable,*

$$\|I\|_{B_\alpha^{p,q}}^q = \int_0^1 (1-r)^{q-1} M_p^q\left(r, D^{1+\alpha}I\right)dr, \tag{3.9}$$

$$\int_0^1 (1-r)^{(1-\alpha)q-1} M_p^q(r, I')dr, \tag{3.10}$$

$$\int_0^1 (1-r)^{-\alpha q-1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left(1-\left|I\left(re^{i\theta}\right)\right|\right)^p d\theta\right)^{q/p} dr, \tag{3.11}$$

$$\int_0^1 (1-r)^{-\alpha q-1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|I\left(e^{i\theta}\right) - I\left(re^{i\theta}\right)\right|^p d\theta\right)^{q/p} dr. \tag{3.12}$$

Remark 3.2. An examination of the proof in which the quantity (3.12) is controlled by that of (3.10), shows that it does not really require the function I to be inner. Any bounded function would just work fine.

In [15], the corresponding characterization for $q = \infty$ is mentioned without proof ((3.2) of [15]). Its verification is done by following the same steps of the previous result (even easier, Hardy's inequality is not needed).

Theorem D. *If* $0 < p < \infty$, $0 < \alpha < 1$, *and* I *is an inner function, then the following quantities are comparable,*

$$\|I\|_{\Lambda^{p,\alpha}} = \sup_{0\le r<1}(1-r)M_p\left(r, D^{1+\alpha}I\right), \tag{3.13}$$

$$\sup_{0\le r<1}(1-r)^{1-\alpha}M_p(r, I'), \tag{3.14}$$

$$\sup_{0\le r<1}(1-r)^{-\alpha}\left(\frac{1}{2\pi}\int_0^{2\pi}\left(1-\left|I\left(re^{i\theta}\right)\right|\right)^p d\theta\right)^{1/p}, \tag{3.15}$$

$$\sup_{0\le r<1}(1-r)^{-\alpha}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|I\left(e^{i\theta}\right)-I\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p}. \tag{3.16}$$

For Blaschke products, the third author [15, 38, Theorems 1.2 and 1.4] gave sufficient (and in special cases, necessary) conditions for their membership in $B_\alpha^{p,q}$ in terms of the distribution of their zeros. Recall that a *Carleson-Newman sequence* is a finite union of *interpolating sequences*, and a sequence $\{z_k\}$ in the unit disk is called *interpolating* if it is *uniformly separated*, that is,

$$\inf_n \prod_{\{k:\ z_k\neq z_n\}}\left|\frac{z_k - z_n}{1-\overline{z_k}z_n}\right| > 0. \tag{3.17}$$

Theorem E (see [15, 38, Theorems 1.2 and 1.4]). *Let* $0 < p, \alpha < \infty$ *be such that* $\max\{0, 1/p-1\} < \alpha < 1/p$. *Assume that* $0 < q \le \infty$, *and that* B *is a Blaschke product. If* $\{(2^{-n(1-\alpha p)}\nu_n)^{1/p}\} \in \ell^q$, *then* $B \in B_\alpha^{p,q}$ *and*

$$\|B\|_{B_\alpha^{p,q}} \le C\left\|\left\{\left(2^{-n(1-\alpha p)}\nu_n\right)^{1/p}\right\}\right\|_{\ell^q}. \tag{3.18}$$

On the other hand, if the zero sequence $\{z_k\}$ *of* B *is Carleson-Newman and* $B \in B_\alpha^{p,q}$, *then* $\{(2^{-n(1-\alpha p)}\nu_n)^{1/p}\} \in \ell^q$ *and their respective norms are equivalent.*

As a consequence of this result, Lemma 1.1, and the fact that the Frostman shifts of inner functions are Blaschke products almost always, we have that if $0 < p, \alpha < \infty$ with $\max\{0, 1/p - 1\} < \alpha < 1/p$, $0 < q \le \infty$, and I is an inner function satisfying

$$\left(\int_{K_\delta} \left\| \left\{ \left(2^{-n(1-\alpha p)} \nu_n(a)\right)^{1/p} \right\} \right\|_{\ell^q}^p dA(a)\right)^{1/p} < \infty, \tag{3.19}$$

for $K_\delta = \{a \in \mathbb{D} : \quad \delta \le |a| \le 1 - \delta\}$, and $0 < \delta < 1/2$, then $I \in B_\alpha^{p,q}$, and the norm $\|I\|_{B_\alpha^{p,q}}$ is controlled by the integral in (3.19).

The crux of the matter here is that the above condition is also necessary for I to belong to $B_\alpha^{p,q}$.

Theorem 3.3. *Let $0 < p, \alpha < \infty$ be such that $\max\{0, 1/p - 1\} < \alpha < 1/p$. Assume that $0 < q \le \infty$, and that I is an inner function. Then $I \in B_\alpha^{p,q}$ if and only if* (3.19) *holds for some $\delta \in (0, 1/2)$. In that case, both quantities, $\|I\|_{B_\alpha^{p,q}}$ and the integral in* (3.19)*, are comparable.*

In order to prove this theorem, certain homogeneity property is needed. See [38, Lemma 4.4], [7, Lemma 2.2] for similar statements on H^∞-functions, and also [15, Proposition 3.1] for the case $q = \infty$.

Lemma 3.4. *If $0 < p < \infty$, $0 < q \le \infty$, $0 \le \alpha < \infty$, and $1 \le t < \infty$, then $\Lambda^{\infty,0} \cap B_\alpha^{p,q} \subseteq B_{\alpha/t}^{pt,qt}$, that is, Bloch functions in $B_\alpha^{p,q}$ are also in $B_{\alpha/t}^{pt,qt}$ for any $t \ge 1$. Furthermore, the following relation holds:*

$$\|f\|_{B_{\alpha/t}^{pt,qt}} \le C \|f\|_{\Lambda^{\infty,0}}^{1-1/t} \|f\|_{B_\alpha^{p,q}}^{1/t}. \tag{3.20}$$

Proof of Lemma 3.4. The case $q = \infty$ will not be treated due to its similarity with the other cases. Take $0 < p, q < \infty$, $0 \le \alpha < \infty$, $1 \le t < \infty$, and $f \in \Lambda^{\infty,0} \cap B_\alpha^{p,q}$. We need to show that $f \in B_{\alpha/t}^{pt,qt}$. For that, use (1.14) to find an equivalent quantity to $\|f\|_{B_{\alpha/t}^{pt,qt}}$, and then separate it into two factors, the first will be controlled by $\|f\|_{\Lambda^{\infty,0}}$, and the second by $\|f\|_{B_\alpha^{p,q}}$.

$$\begin{aligned}
\|f\|_{B_{\alpha/t}^{pt,qt}} &= \left(\int_0^1 (1-r)^{qt-1} M_{pt}^{qt}\left(r, D^{1+\alpha/t} f\right) dr\right)^{1/qt} \\
&\overset{(1.14)}{\le} C\left(\int_0^1 (1-r)^{qt-1-q\alpha+q\alpha t} M_{pt}^{qt}\left(r, D^{1+\alpha} f\right) dr\right)^{1/qt} \\
&= C\left(\int_0^1 (1-r)^{q(t-1)(1+\alpha)+(q-1)} \left(\int_0^{2\pi} \left|D^{1+\alpha} f\left(re^{i\theta}\right)\right|^{pt} d\theta\right)^{q/p} dr\right)^{1/qt} \\
&\le C \sup_{0 \le r < 1} \left((1-r)^{1+\alpha} M_\infty\left(r, D^{1+\alpha} f\right)\right)^{1-1/t} \left(\int_0^1 (1-r)^{q-1} M_p^q\left(r, D^{1+\alpha} f\right) dr\right)^{1/qt} \\
&\le C \|f\|_{\Lambda^{\infty,0}}^{1-1/t} \|f\|_{B_\alpha^{p,q}}^{1/t}.
\end{aligned} \tag{3.21}$$

□

Two more lemmas are needed.

Lemma F (see [15, Corollary 4.5]). *If $p \geq 1$, I is an inner function, and if $1-2^{-n} < r \leq 1-2^{-(n+1)}$, then*

$$2^{-n}\nu_n \leq C_p \int_0^{2\pi} \log^p \frac{1}{\left|I(re^{i\theta})\right|} d\theta. \tag{3.22}$$

Lemma G (see [15, Corollary 4.7]). *If $0 < p < \infty$, $0 < \delta < 1/2$, I is an inner function, and if $z \in \mathbb{D}$, then*

$$\int_{K_\delta} \log^p \frac{1}{|I_a(z)|} dA(a) \leq C_{p,\delta}(1-|I(z)|)^p. \tag{3.23}$$

Proof of Theorem 3.3. Again, we deal only with the case $q < \infty$. The sufficiency of condition (3.19) has already been established. To prove its necessity, assume that $I \in B_\alpha^{p,q}$ and, rather than imposing the whole restriction $\max\{0, 1/p-1\} < \alpha < 1/p$, just assume $0 < \alpha, p, q < \infty$. Observe that the integral in (3.19) (without the power $1/p$) remains unchanged if we replace p, q, α with $pt, qt, \alpha/t$. Now choose $t \geq 1$ such that $\alpha/t < 1$, $pt > 1$ and $qt > 1$. If the result holds in this situation, then, by the homogeneity property of Lemma 3.4, we have

$$\begin{aligned}\left(\int_{K_\delta} \left\| \left\{ \left(2^{-n(1-\alpha p)}\nu_n(a)\right)^{1/p} \right\} \right\|_{\ell^q}^p dA(a)\right)^{1/p} &= \left(\int_{K_\delta} \left\| \left\{ \left(2^{-n(1-\alpha p)}\nu_n(a)\right)^{1/pt} \right\} \right\|_{\ell^{qt}}^{pt} dA(a)\right)^{1/p} \\ &\leq C\|I\|_{B_{\alpha/t}^{pt,qt}}^t \leq C\|I\|_{\Lambda^{\infty,0}}^{t-1} \|I\|_{B_\alpha^{p,q}}.\end{aligned} \tag{3.24}$$

So it suffices to prove the result for $0 < \alpha < 1$ and $1 < p, q < \infty$. In what follows $r_n = 1-2^{-n}$. First assume that $p > q$. Then use, in order, Minkowski's inequality for $p/q > 1$, the fact that $\int_{r_n}^{r_{n+1}} r^{-\alpha q-1} dr \asymp 2^{n\alpha q}$, and finally Lemmas F and G together with Theorem C to arrive at the desired estimate for (3.19),

$$\begin{aligned}&\left(\int_{K_\delta} \left\| \left\{ \left(2^{-n(1-\alpha p)}\nu_n(a)\right)^{1/p} \right\} \right\|_{\ell^q}^p dA(a)\right)^{1/p} \\ &\quad = \left(\int_{K_\delta} \left(\sum_{n\geq 0} 2^{n\alpha q}\left(2^{-n}\nu_n(a)\right)^{q/p}\right)^{p/q} dA(a)\right)^{(q/p)(1/q)}\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sum_{n\geq 0}2^{n\alpha q}\left(\int_{K_\delta}2^{-n}\nu_n(a)\,dA(a)\right)^{q/p}\right)^{1/q}\\
&\leq C\left(\sum_{n\geq 0}\int_{\mathrm{r}_n}^{r_{n+1}}(1-r)^{-\alpha q-1}\left(\int_{K_\delta}\int_0^{2\pi}\log^p\frac{1}{\left|I_a\left(re^{i\theta}\right)\right|}d\theta\, dA(a)\right)^{q/p}dr\right)^{1/q}\\
&\leq C\left(\int_0^1(1-r)^{-\alpha q-1}\left(\int_0^{2\pi}\int_{K_\delta}\log^p\frac{1}{\left|I_a\left(re^{i\theta}\right)\right|}dA(a)\,d\theta\right)^{q/p}dr\right)^{1/q}\\
&\leq C\left(\int_0^1(1-r)^{-\alpha q-1}\left(\int_0^{2\pi}\left(1-\left|I\left(re^{i\theta}\right)\right|\right)^p d\theta\right)^{q/p}dr\right)^{1/q}\\
&\leq C\|I\|_{B_\alpha^{p,q}}.
\end{aligned}
\tag{3.25}
$$

The case $p < q$ follows the same procedure, only that instead of using Minkowski's inequality, we use Hölder's with exponents $q/(q-p)$ and q/p, and then, after applying Lemma F and before Lemma G, use again Minkowski's inequality with $q/p > 1$,

$$
\begin{aligned}
&\left(\int_{K_\delta}\left\|\left\{\left(2^{-n(1-\alpha p)}\nu_n(a)\right)^{1/p}\right\}\right\|_{\ell^q}^p dA(a)\right)^{1/p}\\
&\quad=\left(\int_{K_\delta}\left(\sum_{n\geq 0}2^{n\alpha q}\left(2^{-n}\nu_n(a)\right)^{q/p}\right)^{p/q}dA(a)\right)^{1/p}\\
&\quad\leq\left(\int_{K_\delta}dA(a)\right)^{1/p-1/q}\left(\int_{K_\delta}\sum_{n\geq 0}2^{n\alpha q}\left(2^{-n}\nu_n(a)\right)^{q/p}dA(a)\right)^{1/q}\\
&\quad=C\left(\sum_{n\geq 0}2^{n\alpha q}\int_{K_\delta}\left(2^{-n}\nu_n(a)\right)^{q/p}dA(a)\right)^{1/q}\\
&\quad\leq C\left(\sum_{n\geq 0}\int_{r_n}^{r_{n+1}}(1-r)^{-\alpha q-1}\left(\int_{K_\delta}\left(\int_0^{2\pi}\log^p\frac{1}{\left|I_a\left(re^{i\theta}\right)\right|}d\theta\right)^{q/p}dA(a)\right)^{(p/q)(q/p)}dr\right)^{1/q}\\
&\quad\leq C\left(\int_0^1(1-r)^{-\alpha q-1}\left(\int_0^{2\pi}\left(\int_{K_\delta}\log^q\frac{1}{\left|I_a\left(re^{i\theta}\right)\right|}dA(a)\right)^{p/q}d\theta\right)^{q/p}d\right)^{1/q}
\end{aligned}
$$

$$\leq C\left(\int_0^1 r^{-\alpha q-1}\left(\int_0^{2\pi}\left(1-\left|I\left(re^{i\theta}\right)\right|\right)^p d\theta\right)^{q/p} dr\right)^{1/q}$$
$$\leq C\|I\|_{B_\alpha^{p,q}}. \tag{3.26}$$

□

Remark 3.5. Along the proof of this theorem, we have actually proved that if I is an inner function in $B_\alpha^{p,q}$, with $0 < \alpha, p < \infty$ and $0 < q \leq \infty$, then (3.19) holds and

$$\left(\int_{K_\delta}\left\|\left\{\left(2^{-n(1-\alpha p)}\nu_n(a)\right)^{1/p}\right\}\right\|_{\ell^q}^p dA(a)\right)^{1/p} \leq C\|I\|_{B_\alpha^{p,q}}. \tag{3.27}$$

Also, as we observed before, the integral in (3.19), or (3.27), is unchanged if p, q, α is replaced with $pt, qt, \alpha/t$, $(t > 0)$. This allows us to extend the homogeneity property of Lemma 3.4 to other values of t, provided that we can apply Theorem 3.3, that is, that we work with inner functions and that $\max\{0, 1/(pt) - 1\} < \alpha/t < 1/(pt)$.

Corollary 3.6. *Let $0 < p, \alpha < \infty$ be such that $0 < \alpha < 1/p$. Assume that $0 < q \leq \infty$, and that I is an inner function in $B_\alpha^{p,q}$. Then $I \in B_{\alpha/t}^{pt,qt}$ for all $t > 1/p - \alpha$.*

Remark 3.7. Of course, when $\alpha > 1/p$, the class of inner functions in $B_\alpha^{p,q}$ coincides with that of $B_{\alpha/t}^{pt,qt}$ for any $t > 0$ (or any $B_{\alpha_1}^{p_1,q_1}$ with $\alpha_1 > 1/p_1$, whatsoever), because they only contain the finite Blaschke products. The same reasoning applies when $\alpha = 1/p$ and $0 < q < \infty$. Is it the same for $q = \infty$? that is, is the class of inner functions in $\Lambda^{p,1/p}$ the same for all $p > 0$? The answer is affirmative for the class of Blaschke products [5, Theorem 3.1] and then, using once more Lemma 1.1 and the fact that the Frostman shifts of inner functions are almost always Blaschke products, we arrive at an affirmative answer for the whole class of inner functions in $\Lambda^{p,1/p}$. We should mention here that we shall prove later (see Remark 4.4 below) that the only inner functions in $\Lambda^{p,1/p}$ are Blaschke products.

Remark 3.8. In view of the previous remark, we could ask whether the result of the corollary remains true for the whole range of $t > 0$. The answer is negative. Ahern and Clark [3, Lemma 2] have constructed a Blaschke product B in $B_{1/2}^{1,1}$ but not in $H_1^{1/2}$. By property $(P9)$, we deduce that $B \notin B_1^{1/2,1/2}$, and this is the space that would be obtained from $B_{1/2}^{1,1}$ by taking $t = 1/2$, which coincides with $1/p - \alpha$ with the usual notation.

Proof of Corollary 3.6. Notice first that Remark 3.5 implies that (3.27) holds for p, q, α, and thus the integral on the left hand side is unchanged for $pt, qt, \alpha/t$. Now, take $t > 1/p - \alpha > 0$. Then $1/(pt) - 1 < \alpha/t < 1/(pt)$. If now $t \leq 1/p$, then $0 \leq 1/(pt) - 1$, and so $\max\{0, 1/(pt) - 1\} < \alpha/t < 1/(pt)$, proving that $I \in B_{\alpha/t}^{pt,qt}$ by Theorem 3.3. If, on the contrary, $t > 1/p$, then $1/(pt) - 1 < 0$, and we still have $\max\{0, 1/(pt) - 1\} = 0 < \alpha/t < 1/(pt)$, and again, $I \in B_{\alpha/t}^{pt,qt}$ by Theorem 3.3. □

Remark 3.9. As an application of these results, we show how to recover a known result by Protas [9, Theorem 1]. Assume that B is a Blaschke product satisfying $\sum_k d_k^{2-p} < \infty$ for some

$p \in (1,2)$. By Lemma 3.1(b), this condition is equivalent to $\sum_{n\geq 0} 2^{-n(2-p)}\nu_n < \infty$, which implies that $B \in B^{p,p}_{1-1/p}$ by Theorem E. (Notice that this is equivalent to $B' \in A^p$ by (1.14).) Observe now that we can apply Corollary 3.6 with $t = 1/p$, obtaining that $B \in B^{1,1}_{p-1}$, which is the aforementioned result by Protas, with our notation. On the other hand, notice also that if $B' \in A^p$, for some $1 < p < 2$, and the zero sequence of B is Carleson-Newman then, by Theorem E, $\sum_k d_k^{2-p} < \infty$.

Next we will turn to study the membership of inner functions in the spaces H^p_α. Properties ($P9$) and ($P10$), that is, the Littlewood-Paley inequalities, relate quite well H^p_α with $B^{p,p}_\alpha$. Our main result in this direction is that these relations of inclusion become equalities when the spaces are cut with the class of inner functions.

Theorem 3.10. *Let $0 < p \leq \infty$ and $\alpha > \max\{0, 1/p - 1\}$. Then the class of inner functions in H^p_α coincides with that of inner functions in $B^{p,p}_\alpha$.*

The proof of this result requires again an homogeneity property.

Proposition 3.11. *Let $0 < p, \alpha < \infty$ and let f be a Bloch function in H^p_α. Then $f \in H^{pt}_{\alpha/t}$ for all $t \geq 1$.*

This is a straightforward consequence of the following result and the complex maximal theorem.

Lemma H (see [5, Lemma 2.1]). *For each $0 < \alpha < \beta < \infty$, there is a constant $C = C_{\alpha,\beta}$ such that if $f \in \Lambda^{\infty,0}$ then*

$$\left|D^\alpha f(z)\right| \leq C\left(\max_{0\leq t\leq 1}\left|D^\beta f(tz)\right|\right)^{\alpha/\beta}. \tag{3.28}$$

The original proof of this lemma runs with H^∞ functions instead of Bloch ($\Lambda^{\infty,0}$) functions. However, the lemma can be proved for Bloch functions by just noticing the validity of the estimate $|D^\beta f(z)| \leq C(1-|z|)^{-\beta}$ for Bloch functions.

Proof of Theorem 3.10. First notice that when $\alpha \geq 1/p$, the only inner functions in H^p_α and $B^{p,p}_\alpha$ are the finite Blaschke products. So we may assume without loss of generality the additional hypothesis $\alpha < 1/p$. (This already implies $p < \infty$.)

Now consider the case $0 < p \leq 2$. Then $B^{p,p}_\alpha \subseteq H^p_\alpha$ by ($P9$). To go in the other direction, take an inner function I in H^p_α. Then, by Proposition 3.11, $I \in H^{pt}_{\alpha/t}$ for all $t \geq 1$. For $t \geq 2/p \geq 1$, we have $pt \geq 2$, and hence $H^{pt}_{\alpha/t} \subseteq B^{pt,pt}_{\alpha/t}$ by ($P10$). Using now that $\max\{0, 1/p-1\} < \alpha < 1/p$, we get $1/t > 1/(pt) - \alpha/t$, and we conclude that $I \in B^{p,p}_\alpha$ by Corollary 3.6.

Next consider the case $2 \leq p < \infty$. (Then $\max\{0, 1/p-1\} = 0$.) By ($P10$), $H^p_\alpha \subseteq B^{p,p}_\alpha$. To go in the other direction, take an inner function I in $B^{p,p}_\alpha$. Since $0 < \alpha < 1/p$ then, by Corollary 3.6, $I \in B^{pt,pt}_{\alpha/t}$ for all $t > 1/p - \alpha$. Choose $t \in (1/p - \alpha, 2/p]$. Thus, as $pt \leq 2$, ($P9$) gives $B^{pt,pt}_{\alpha/t} \subseteq H^{pt}_{\alpha/t}$. Finally, as $1/t \geq p/2 \geq 1$, Proposition 3.11 gives $I \in H^p_\alpha$. □

Remark 3.12. A careful reading of this proof shows that, in fact, any inner function I in $B^{p,p}_\alpha$ is also in H^p_α, whenever $0 < p \leq \infty$ and $0 < \alpha$.

The analogous to Corollary 3.6 is the following.

Corollary 3.13. *Let $0 < p, \alpha < \infty$ be such that $0 < \alpha < 1/p$. Assume that I is an inner function in H^p_α. Then $I \in H^{pt}_{\alpha/t}$ for all $t > 1/p - \alpha$.*

Proof. Take $t > 1/p-\alpha$. If $t \geq 1$ then $I \in H^{pt}_{\alpha/t}$ by Proposition 3.11. Otherwise we have $1/p-\alpha < t < 1$, which implies $1/p-1 < \alpha$, and, together with $0 < \alpha < 1/p$, it gives $\max\{0, 1/p-1\} < \alpha < 1/p$. So, by Theorem 3.10, $I \in B^{p,p}_\alpha$ and, by Corollary 3.6, $I \in B^{pt,pt}_{\alpha/t}$ and, finally, by Remark 3.12, $I \in H^{pt}_{\alpha/t}$. □

Remark 3.14. Again, since it only contains finite Blaschke products, the class of inner functions in H^p_α, with $\alpha \geq 1/p$, coincides with that of $H^{pt}_{\alpha/t}$ for all $t > 0$, or for the case, with the class of inner functions of any $H^{p_1}_{\alpha_1}$, with $\alpha_1 \geq 1/p_1$. As for the accuracy of the corollary with regards to whether there is a possibility to establish the result for the whole range of $t > 0$, the same example given in Remark 3.8 shows that it is impossible. The Blaschke product B constructed in [3, Lemma 2] is in $B^{1,1}_{1/2}$ but not in $H^{1/2}_1$, which is the space that would be obtained from $H^1_{1/2}$ by taking $t = 1/2 (= 1/p - \alpha$ with the usual notation). By Theorem 3.10, $B \in B^{1,1}_{1/2}$ implies that $B \in H^1_{1/2}$.

Remark 3.15. As an application, we again regain a known result of Protas [9, Theorem 2], namely, *if B is a Blaschke product such that $\sum_k d_k^{1-p} < \infty$, for some $1/2 < p < 1$, then $B \in H^p_1$.* Indeed, by Lemma 3.1(b), $\sum_k d_k^{1-p} \asymp \sum_{n\geq 0} 2^{-n(1-p)} \nu_n$, and so $B \in B^{p,p}_1$ by Theorem E, and we are done with $B \in H^p_1$ by Theorem 3.10. On the other hand, Theorem E also says that if the zero sequence of B is Carleson-Newman then the condition $\sum_k d_k^{1-p} < \infty$ is also necessary for $B \in H^p_1$.

4. The Case $1/(2p) < \alpha < 1/p$

Ahern and Clark [2, Theorem 3] proved that the only inner functions in $H^{1/2}_1$ are Blaschke products. Later on, Ahern and Jevtic' obtained the following generalization:

Theorem I (see [5, Theorem 2.1]). *If I is an inner function and*

$$M_p^p\left(r, D^{1/(2p)} I\right) = o\left(\log \frac{1}{1-r}\right), \quad \text{as } r \longrightarrow 1, \tag{4.1}$$

for some $0 < p < \infty$, then I is a Blaschke product.

Now, all functions in $H^p_{1/(2p)}$ satisfy condition (4.1) and, by ($P5$), the same is true for all H^p_α-functions with $\alpha \geq 1/(2p)$. So the following is immediate:

Corollary 4.1. *Let $0 < p, \alpha < \infty$ with $\alpha \geq 1/(2p)$. Then the only inner functions in H^p_α are Blaschke products, finite ones if $\alpha \geq 1/p$.*

This result finds its analogue for Besov spaces. Its essence may be traced back to the last corollary in [37].

Proposition J (see [37]). *Let $0 < p, q, \alpha < \infty$ with $\alpha \geq 1/(2p)$. Then the only inner functions in $B_\alpha^{p,q}$ are Blaschke products, finite ones if $\alpha \geq 1/p$.*

Remark 4.2. These results are again accurate, for the "atomic" singular inner function $S(z) = \exp(-(1+z)/(1-z))$ is in $B_\alpha^{p,q} \cap H_\alpha^p$ for all $0 < p, q, \alpha < \infty$ with $\alpha < 1/(2p)$. Indeed, in [39], it is shown that, for any positive integer n,

$$M_{1/(2n)}\left(r, S^{(n)}\right) \asymp \log \frac{1}{1-r}. \tag{4.2}$$

This implies that $S \in H_\beta^{1/(2n)}$ for all $\beta < n$, because by (1.11),

$$\begin{aligned} M_{1/(2n)}^{1/(2n)}\left(r, D^\beta S\right) &\leq C \int_0^1 (1-s)^{(n-\beta)/(2n)-1} M_{1/(2n)}^{1/(2n)}(rs, D^n S) ds \\ &\leq C \int_0^1 (1-s)^{(n-\beta)/(2n)-1} \log^{1/(2n)}\left(\frac{1}{1-rs}\right) ds < \infty, \end{aligned} \tag{4.3}$$

and also $S \in B_\beta^{1/(2n),\tilde{q}}$ for all $\beta < n$ and all $0 < \tilde{q} < \infty$, because by (1.14),

$$\begin{aligned} \int_0^1 (1-r)^{\tilde{q}-1} M_{1/(2n)}^{\tilde{q}}\left(r, D^{1+\beta} S\right) dr &\leq C \int_0^1 (1-r)^{\tilde{q}(n-\beta)-1} M_{1/(2n)}^{\tilde{q}}(r, D^n S) dr \\ &\leq C \int_0^1 (1-r)^{\tilde{q}(n-\beta)-1} \log^{\tilde{q}}\left(\frac{1}{1-r}\right) dr < \infty. \end{aligned} \tag{4.4}$$

Hence, given $0 < p, q, \alpha < \infty$ with $\alpha < 1/(2p)$, take the smallest integer n such that $1/(2p) \leq n$ and then take $\beta = 2np\alpha < n$ and $\tilde{q} = q/(2np)$. In this way, $S \in H_\beta^{1/(2n)} \cap B_\beta^{1/(2n),\tilde{q}}$, so by the homogeneity properties of Lemma 3.4 and Proposition 3.11 with $t = 2np \geq 1$, get that $S \in H_{\beta/t}^{t/(2n)} \cap B_{\beta/t}^{t/(2n),\tilde{q}t} = H_\alpha^p \cap B_\alpha^{p,q}$, as desired.

Remark 4.3. The case $p = \infty$ would deal with the question of whether the inner functions in $B_0^{\infty,q}$ are just Blaschke products. From Theorem 2.4, we know that if $q \leq 2$, the inner functions in $B_0^{\infty,q}$ are finite Blaschke products, while the example considered to prove its part (b) is initially a singular inner function in $\cap_{q>2} B_0^{\infty,q}$.

Remark 4.4. The case $q = \infty$ corresponds to the Lipschitz spaces $\Lambda^{p,\alpha}$. By property $(P7)$, $\Lambda^{p,\alpha} \subseteq \cap\{B_\beta^{p,q} : \ \beta < \alpha, \ 0 < q\}$. Thus, if $\alpha > 1/(2p)$, the only inner functions in $\Lambda^{p,\alpha}$ are Blaschke products (finite ones if $\alpha > 1/p$). Combining now the property $(P1)$ and the fact that the atomic singular inner function S is in $B_\alpha^{p,q}$ for all $0 < p, q, \alpha < \infty$ with $\alpha < 1/(2p)$, we obtain that $S \in \lambda^{p,\alpha}$ for all $0 < p, \alpha < \infty$ with $\alpha < 1/(2p)$. In fact, we claim that $S \in \lambda^{p,1/(2p)}$ for all $p > 0$: using the corresponding homogeneity property for $\lambda^{p,\alpha}$ (i.e., Bloch functions in $\lambda^{p,\alpha}$ are also in $\lambda^{pt,\alpha/t}$ for all $t \geq 1$, see [15, Proposition 4.1]), we obtain that the sequence of spaces $\Lambda^{\infty,0} \cap \lambda^{p,1/(2p)}$ increases with p. Therefore, to prove our claim, it suffices to see that $S \in \lambda^{1/(2n),n}$ for all positive integer n, and this is so by the result in [39],

$$(1-r) M_{1/(2n)}\left(r, D^{1+n} S\right) \asymp (1-r)(1-r)^{1/2-(n+1)/(2n)} = (1-r)^{1-1/(2n)} \longrightarrow 0. \tag{4.5}$$

Now we come to results relating the membership of a Blaschke product in the spaces under consideration for the range $1/(2p) < \alpha < 1/p$ with summability properties of the associated counting sequence $\{\nu_n\}$. We start mentioning the following results of Verbitskiĭ.

Theorem K (see [27, Theorem3] and [10, Theorem 5]). *Let B be a Blaschke product with zeros $\{z_k\}$ in a fixed Stolz angle, and let p, q, α satisfy the relations $1 \le p < \infty$, $1/(2p) < \alpha < 1/p$, and $0 < q \le \infty$. Then*

(a) $B \in \Lambda^{p,\alpha} = B_\alpha^{p,\infty} \Leftrightarrow \{2^{-n(1/p-\alpha)}\nu_n^\alpha\} \in \ell^\infty \Leftrightarrow \{d_k^{1/p-\alpha} k^\alpha\} \in \ell^\infty$,

(b) $B \in B_\alpha^{p,q} \Leftrightarrow \{2^{-n(1/p-\alpha)}\nu_n^\alpha\} \in \ell^q \Leftrightarrow \{d_k^{1/p-\alpha} k^{\alpha-1/q}\} \in \ell^q$.

We will come back to Blaschke products with zeros in a Stolz angle in Section 5, but we shall prove next that the implication $B \in B_\alpha^{p,q} \Rightarrow \{2^{-n(1/p-\alpha)}\nu_n^\alpha\} \in \ell^q$ is true for general Blaschke products whenever $1/(2p) < \alpha < 1/p$ (even for $0 < p < 1$).

Theorem 4.5. *Let $0 < p, \alpha < \infty$ with $1/(2p) < \alpha < 1/p$, and let $0 < q \le \infty$. Assume that I is an inner function in $B_\alpha^{p,q}$. Then (by Proposition J and Remark 4.4) I is a Blaschke product B and for its sequence of zeros $\{z_k\}$, we have*

(a) *If $q = \infty$, that is, if $B \in \Lambda^{p,\alpha}$, then $\{2^{-n(1/p-\alpha)}\nu_n^\alpha\} \in \ell^\infty$ or, equivalently, $\{d_k^{1/p-\alpha} k^\alpha\} \in \ell^\infty$.*

(b) *If $q < \infty$, then $\{2^{-n(1/p-\alpha)}\nu_n^\alpha\} \in \ell^q$ or, equivalently, $\{d_k^{1/p-\alpha} k^{\alpha-1/q}\} \in \ell^q$.*

Proof. We will make use of the following inequality of Goldberg [40]:

$$\int_0^{2\pi} \left|B\left(re^{i\theta}\right)\right| d\theta \le \int_0^{2\pi} \left|\widehat{B}\left(re^{i\theta}\right)\right| d\theta, \quad 0 < r < 1, \tag{4.6}$$

where $\widehat{B}$ is the Blaschke product whose zeros $\widehat{z}_k$ are the moduli $|z_k|$ of the corresponding zeros of B. Notice then that the sequence $\{\widehat{d}_k\}$ coincides with $\{d_k\}$. Assume first that $q = \infty$. Since $1/p > 1/p - \alpha$, Corollary 3.6 implies that $B \in \Lambda^{1,\alpha p}$. Now, since $\alpha p < 1$, Theorem D gives the following growth order:

$$\int_0^{2\pi} \left(1 - \left|B\left(re^{i\theta}\right)\right|\right) d\theta = \mathrm{O}\left((1-r)^{\alpha p}\right), \quad \text{as } r \longrightarrow 1. \tag{4.7}$$

Combining this with (4.6), we obtain that $\widehat{B} \in \Lambda^{1,\alpha p}$. So, by Theorem K(a), we arrive at $\{2^{-n(1-\alpha p)}\nu_n^{\alpha p}\} \in \ell^\infty$, which is Theorem 4.5(a).

To prove part (b), we proceed analogously. Assume that $q < \infty$. Since $1/p > 1/p - \alpha$, Corollary 3.6 implies that $B \in B_{\alpha p}^{1,q/p}$. Since $\alpha p < 1$, Theorem C gives

$$\int_0^1 (1-r)^{-\alpha q-1} \left(\frac{1}{2\pi}\int_0^{2\pi} \left(1 - \left|B\left(re^{i\theta}\right)\right|\right) d\theta\right)^{q/p} dr < \infty, \tag{4.8}$$

which combined with (4.6), gives the same estimate for $\widehat{B}$ in place of B, implying that $\widehat{B} \in B_{\alpha p}^{1,q/p}$. So, by Theorem K(b), $\{2^{-n(1-\alpha p)}\nu_n^{\alpha p}\} \in \ell^{q/p}$, and this is equivalent to the desired conclusion of Theorem 4.5(b). □

Theorem 4.5 has some obvious consequences regarding the spaces H^p_α.

Corollary 4.6. *Let $0 < p, \alpha < \infty$ with $1/(2p) < \alpha < 1/p$. Assume that I is an inner function in H^p_α. Then (by Corollary 4.1) I is a Blaschke product B whose sequence of zeros, $\{z_k\}$, satisfies,*

(a) $\sum_k d_k^{1-\alpha p} k^{\alpha p-1} < \infty$ *if* $1/p - 1 < \alpha$,

(b) $\sum_k d_k^{t-\alpha p} k^{\alpha p-1} < \infty$ *for all* $t > 1$ *if* $1/(2p) < \alpha \leq 1/p - 1$.

Proof. Part (a) follows directly from the fact that $B \in B^{p,p}_\alpha$ by Corollary 4.6. As for part (b), it is because, by Property $(P8)$, $B \in B^{p,q}_\beta$ for all $\beta < \alpha$ and all $q > 0$. □

Results of this kind can be also obtained using Theorem 8 of [2], where it is proved that $B \in H^p_1$, $1/2 < p < 1$ implies $\sum_k d_k^{(1-p)/p} < \infty$.

Theorem 4.7. *Let $0 < p, \alpha < \infty$ with $1/(2p) < \alpha < 1/p$. Assume that I is an inner function in H^p_α. Then (by Corollary 4.1) I is a Blaschke product B whose sequence of zeros, $\{z_k\}$, satisfies $\sum_k d_k^{(1-\alpha p)/(\alpha p)} < \infty$.*

Proof. Since $\alpha > 1/p - \alpha$, then $B \in H^{\alpha p}_1$ by Corollary 3.13. Now, by Theorem 8 of [2], we conclude that $\sum_{k=1}^\infty d_k^{(1-\alpha p)/(\alpha p)} < \infty$. □

5. Blaschke Products with Zeros in a Nontangential Region

This section will be mainly focussed on Blaschke products whose zeros lie in a fixed nontangential region, also called Stolz angle.

Given $\xi \in \partial\mathbb{D}$, a *Stolz angle with vertex at ξ* is one of the following regions $\Omega_\sigma(\xi) \subset \mathbb{D}$, with $\sigma \in [1, \infty)$,

$$\Omega_\sigma(\xi) = \left\{ z \in \mathbb{D} : \quad \left| 1 - \overline{\xi} z \right| \leq \sigma(1 - |z|) \right\}. \tag{5.1}$$

The Stolz angles with vertex at $\xi = 1$ will be simply denoted by Ω_σ.

Let us start with the following result.

Theorem L (see [7, Theorem 2.3]). *If B is a Blaschke product whose zeros lie in a fixed Stolz angle, then $B \in \Lambda^{p,1/(2p)}$ for all $p > 0$.*

An improvement of this result for $1/2 < p < \infty$ is given in [15, Theorem 1.6], where it is actually proved that $B \in \lambda^{p,1/(2p)}$. Anyhow, Theorem L provides the following result almost immediately.

Theorem 5.1. *If $0 < p, \alpha < \infty$ with $\alpha < 1/(2p)$, $0 < q \leq \infty$, and B is a Blaschke product whose zeros lie in a fixed Stolz angle, then $B \in B^{p,q}_\alpha \cap H^p_\alpha$.*

Proof. That $B \in B^{p,q}_\alpha$ is a direct application of Theorem L and property $(P7)$. That $B \in H^p_\alpha$ follows directly from the fact that $B \in B^{p,p}_\alpha$ and Remark 3.12. □

This result is quite accurate, as the following one shows.

Theorem 5.2. *If $0 < p, q < \infty$, then there exists a Blaschke product B whose zeros lie on the radius $(0,1)$ and such that $B \notin B^{p,q}_{1/(2p)}$. Also, there exists a Blaschke product B, with its sequence of zeros lying on the radius $(0,1)$, and such that, independently of p, $B \notin H^p_{1/(2p)}$.*

Proof. By Corollary 3.6, as $1/p > 1/p - 1/(2p)$, any inner function in $B^{p,q}_{1/(2p)}$ is also in $B^{1,q/p}_{1/2}$. So, for the first part of the proposition, it will suffice to give a Blaschke product B (depending on q/p) with its zeros lying on the radius $(0,1)$ and such that $B \notin B^{1,q/p}_{1/2}$. On the other hand, since all inner functions in $H^p_{1/(2p)}$ are also in $H^1_{1/2}$ by Corollary 3.13, and since $H^1_{1/2}$ and $B^{1,1}_{1/2}$ contain the same inner functions by Theorem 3.10, it thus suffices to say that the Blaschke product sought in the second part of the proposition is one of the above that is not in $B^{1,1}_{1/2}$.

Now let B be the Blaschke product whose sequence of zeros is given by $z_k = 1 - (k\log^{\beta} k)^{-1}$, $k = 2, 3\ldots$, where $\beta = 1+2p/q > 1$. This example has been mentioned by Gluchoff [6, Example 2 after Corollary 1.15] to assert that, for all r sufficiently close to 1,

$$\int_0^{2\pi} \left(1 - \left|B\left(re^{i\theta}\right)\right|\right) d\theta \asymp (1-r)^{1/2} \log^{-p/q} \frac{1}{1-r}. \tag{5.2}$$

This implies that

$$\int_0^1 (1-r)^{-q/(2p)-1} \left(\int_0^{2\pi} \left(1 - \left|B\left(re^{i\theta}\right)\right|\right) d\theta\right)^{q/p} dr = \infty. \tag{5.3}$$

which, by Theorem C, yields

$$\int_0^1 (1-r)^{q/p-1} M_1^{q/p}\left(r, D^{1+1/2}B\right) dr = \infty, \tag{5.4}$$

that is, $B \notin B^{1,q/p}_{1/2}$. This finishes the proof of the theorem. □

Remark 5.3. The just mentioned example of a Blaschke product B not in $B^{1,1}_{1/2}$ is that one that places its zeros at $[1-(k\log^3 k)^{-1}]$, $k \geq 2$. This Blaschke product does not belong to any $H^p_{1/(2p)}$, $0 < p < \infty$, neither to any $B^{p,p}_{1/(2p)}$. In particular, B does not belong to $B^{3/2,3/2}_{1/3}$ which, by (1.14), is another way to say that $B' \notin A^{3/2}$. We remark that Girela and Peláez [12] proved that the Blaschke B with zeros $a_k = [1-(k\log^2 k)^{-1}]$, $k \geq 2$, also has this property. Actually, arguing as in the proof of [12], Theorem 2.1, we can deduce the following:

If $\beta \geq 2$ and B is the Blaschke product with zeros $a_k = [1-(k\log^{\beta} k)^{-1}]$, $k \geq 2$, then $B' \notin A^{3/2}$.

Regarding Theorem K, we note that Jevtic' [7, Theorem 2.1] extended part (a) to the case $0 < p < 1$. It would be interesting to know whether a similar extension holds for part (b). Of course, our Theorem 4.5 answers affirmatively the question for the implication "$B \in B^{p,q}_{\alpha} \Rightarrow \{2^{-n(1/p-\alpha)}\nu_n^{\alpha}\} \in \ell^q$". We have a partial result regarding the other implication.

Theorem 5.4. *Let B be a Blaschke product with zeros $\{z_k\}$ in a fixed Stolz angle, and let $0 < p, q, \alpha < \infty$ be such that* $\max\{1/p - 1, 1/(2p)\} < \alpha < 1/p$. *If* $\{d_k^{1/p-\alpha} k^{\alpha-1/q}\} \in \ell^q$ *then* $B \in B_\alpha^{p,q}$.

Proof. That $\{d_k^{1/p-\alpha} k^{\alpha-1/q}\} \in \ell^q$, it means that $\sum_k d_k^{q/p-\alpha q} k^{\alpha q-1} < \infty$. Such sum remains unchanged if p, q, α are replaced, respectively, with $pt, qt, \alpha/t$ for any $t > 0$. Thus, by Theorem K(b), $B \in B_{\alpha p}^{1,q/p}$ because $1/2 < \alpha p < 1$. Now, since $p > 1 - \alpha p$ too, Corollary 3.6 can be applied to obtain that $B \in B_\alpha^{p,q}$. □

Using Theorem 9 of [2], we obtain the following result for Hardy-Sobolev spaces.

Theorem 5.5. *Let* $0 < p, \alpha < \infty$ *with* $\max\{1/p-1, 1/(2p)\} < \alpha < 1/p$. *Assume that B is a Blaschke product whose sequence of zeros $\{z_k\}$ lies in a fixed Stolz angle, and satisfies* $\sum_k d_k^{(1-\alpha p)/(\alpha p)} < \infty$. *Then* $B \in H_\alpha^{\tilde{p}}$, *for all* $0 < \tilde{p} < p$.

Proof. Observe that the condition $\max\{1/p-1, 1/(2p)\} < \alpha < 1/p$ is equivalent to saying that $\max\{1/(2\alpha), 1/(1+\alpha)\} < p < 1/\alpha$. Thus, by property $(P3)$, the result will be proved if we show that $B \in H_\alpha^{\tilde{p}}$, for all $\max\{1/(2\alpha), 1/(1+\alpha)\} < \tilde{p} < p$.

For such $\tilde{p}$, $1/2 < \alpha\tilde{p} < \alpha p$ and, also, $(1-\alpha p)/(\alpha p) < (1-\alpha\tilde{p})/(\alpha\tilde{p})$. So, by Theorem 9 of [2], $B' \in H^{\alpha\tilde{p}}$, that is, $B \in H_1^{\alpha\tilde{p}}$. Now, the fact that $1/(1+\alpha) < \tilde{p}$ implies that $1/\alpha > 1/(\alpha\tilde{p}) - 1$. So, by Corollary 3.13, we conclude that $B \in H_\alpha^{\tilde{p}}$. □

Acknowledgments

Daniel Girela and Cristóbal are partially supported by grants from "el Ministerio de Ciencia e Innovación, Spain" (MTM2007-60854 and "Ingenio Mathematica (i-MATH)" No. CSD2006-00032) and from "La Junta de AndalucÍa", Spain, (FQM210 and P09-FQM4468). Daniel Girela is also supported by the European Networking Programme "HCAA" of the European Science Foundation. Miroljub Jevtić is supported by a grant from MN ON144010, Serbia.

References

[1] P. Ahern, "The mean modulus and the derivative of an inner function," *Indiana University Mathematics Journal*, vol. 28, no. 2, pp. 311–347, 1979.

[2] P. R. Ahern and D. N. Clark, "On inner functions with H^p-derivative," *The Michigan Mathematical Journal*, vol. 21, pp. 115–127, 1974.

[3] P. R. Ahern and D. N. Clark, "On inner functions with B^p derivative," *The Michigan Mathematical Journal*, vol. 23, no. 2, pp. 107–118, 1976.

[4] P. Ahern and M. Jevtić, "Duality and multipliers for mixed norm spaces," *The Michigan Mathematical Journal*, vol. 30, no. 1, pp. 53–64, 1983.

[5] P. Ahern and M. Jevtić, "Mean modulus and the fractional derivative of an inner function," *Complex Variables. Theory and Application*, vol. 3, no. 4, pp. 431–445, 1984.

[6] A. Gluchoff, "The mean modulus of a Blaschke product with zeroes in a nontangential region," *Complex Variables. Theory and Application*, vol. 1, no. 4, pp. 311–326, 1983.

[7] M. Jevtić, "A note on Blaschke products with zeroes in a nontangential region," *Canadian Mathematical Bulletin*, vol. 32, no. 1, pp. 18–23, 1989.

[8] H. O. Kim, "Derivatives of Blaschke products," *Pacific Journal of Mathematics*, vol. 114, no. 1, pp. 175–190, 1984.

[9] D. Protas, "Blaschke products with derivative in H^p and B^p," *The Michigan Mathematical Journal*, vol. 20, pp. 393–396, 1973.

[10] I. È. Verbitskiĭ, "Inner functions, Besov spaces and multipliers," *Doklady Akademii Nauk SSSR*, vol. 276, no. 1, pp. 11–14, 1984 (Russian).
[11] A. Aleman and D. Vukotić, "On Blaschke products with derivatives in Bergman spaces with normal weights," *Journal of Mathematical Analysis and Applications*, vol. 361, no. 2, pp. 492–505, 2010.
[12] D. Girela and J. Peláez, "On the membership in Bergman spaces of the derivative of a Blaschke product with zeros in a Stolz domain," *Canadian Mathematical Bulletin*, vol. 49, no. 3, pp. 381–388, 2006.
[13] D. Girela, J. Peláez, and D. Vukotić, "Integrability of the derivative of a Blaschke product," *Proceedings of the Edinburgh Mathematical Society*, vol. 50, no. 3, pp. 673–687, 2007.
[14] D. Girela, J. Peláez, and D. Vukotić, "Interpolating Blaschke products: Stolz and tangential approach regions," *Constructive Approximation*, vol. 27, no. 2, pp. 203–216, 2008.
[15] M. Jevtić, "Blaschke products in Lipschitz spaces," *Proceedings of the Edinburgh Mathematical Society*, vol. 52, no. 3, pp. 689–705, 2009.
[16] J. Peláez, "Sharp results on the integrability of the derivative of an interpolating Blaschke product," *Forum Mathematicum*, vol. 20, no. 6, pp. 1039–1054, 2008.
[17] F. Pérez-González and J. Rättyä, "Inner functions in the Möbius invariant Besov-type spaces," *Proceedings of the Edinburgh Mathematical Society*, vol. 52, no. 3, pp. 751–770, 2009.
[18] D. Protas, "Mean growth of the derivative of a Blaschke product," *Kodai Mathematical Journal*, vol. 27, no. 3, pp. 354–359, 2004.
[19] P. L. Duren, *Theory of H^p Spaces*, vol. 38 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1970, Reprinted in Dover, Mineola, NY, USA, 2000.
[20] J. B. Garnett, *Bounded Analytic Functions*, vol. 96 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1981.
[21] P. Duren and A. Schuster, *Bergman Spaces*, vol. 100 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2004.
[22] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, vol. 199 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2000.
[23] T. M. Flett, "The dual of an inequality of Hardy and Littlewood and some related inequalities," *Journal of Mathematical Analysis and Applications*, vol. 38, pp. 746–765, 1972.
[24] D. Girela, M. Pavlović, and J. Peláez, "Spaces of analytic functions of Hardy-Bloch type," *Journal d'Analyse Mathématique*, vol. 100, pp. 53–81, 2006.
[25] J. E. Littlewood and R. E. A. C. Paley, "Theorems on Fourier series and power series (II)," *Proceedings London Mathematical Society*, vol. s2–42, no. 1, pp. 52–89, 1937.
[26] S. A. Vinogradov, "Multiplication and division in the space of analytic functions with an area-integrable derivative, and in some related spaces," *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta imeni V. A. Steklova Akademii Nauk SSSR (LOMI)*, vol. 222, no. 23, pp. 45–77, 308, 1995 (Russian), English translation in *Journal of Mathematical Sciences*, vol. 87, no. 5, pp. 3806–3827, 1997.
[27] I. È. Verbitskiĭ, "On Taylor coefficients and L^p-moduli of continuity of Blaschke products," *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta imeni V. A. Steklova Akademii Nauk SSSR (LOMI)*, vol. 107, pp. 27–35, 228, 1982 (Russian), Investigations on linear operators and the theory of functions, X.
[28] P. S. Bourdon, J. H. Shapiro, and W. T. Sledd, "Fourier series, mean Lipschitz spaces, and bounded mean oscillation," in *Analysis at Urbana, Vol. I (Urbana, IL, 1986–1987)*, vol. 137 of *London Math. Soc. Lecture Note Ser.*, pp. 81–110, Cambridge University Press, Cambridge, UK, 1989.
[29] C. J. Bishop, "Bounded functions in the little Bloch space," *Pacific Journal of Mathematics*, vol. 142, no. 2, pp. 209–225, 1990.
[30] D. Sarason, "Functions of vanishing mean oscillation," *Transactions of the American Mathematical Society*, vol. 207, pp. 391–405, 1975.
[31] J. M. Anderson, "On division by inner factors," *Commentarii Mathematici Helvetici*, vol. 54, no. 2, pp. 309–317, 1979.
[32] D. Girela, "Analytic functions of bounded mean oscillation," in *Complex Function Spaces (Mekrijärvi, 1999)*, vol. 4 of *Univ. Joensuu Dept. Math. Rep. Ser.*, pp. 61–170, The University of Joensuu, Joensuu, Finland, 2001.
[33] A. B. Aleksandrov, J. M. Anderson, and A. Nicolau, "Inner functions, Bloch spaces and symmetric measures," *Proceedings of the London Mathematical Society*, vol. 79, no. 2, pp. 318–352, 1999.
[34] K. H. Zhu, *Operator Theory in Function Spaces*, vol. 139 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1990.

[35] A. Baernstein, II, D. Girela, and J. Peláez, "Univalent functions, Hardy spaces and spaces of Dirichlet type," *Illinois Journal of Mathematics*, vol. 48, no. 3, pp. 837–859, 2004.

[36] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, UK, 1988, Reprint of the 1952 edition.

[37] P. Ahern, "The Poisson integral of a singular measure," *Canadian Journal of Mathematics*, vol. 35, no. 4, pp. 735–749, 1983.

[38] M. Jevtić, "On Blaschke products in Besov spaces," *Journal of Mathematical Analysis and Applications*, vol. 149, no. 1, pp. 86–95, 1990.

[39] M. Mateljević and M. Pavlović, "On the integral means of derivatives of the atomic function," *Proceedings of the American Mathematical Society*, vol. 86, no. 3, pp. 455–458, 1982.

[40] A. A. Goldberg, "On an inequality connected with logarithmic convex functions," *Dopovidi Akademii Nauk Ukrainskoi RSR*, vol. 1957, pp. 227–230, 1957.