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Research Article

Existence of Random Attractors for a p-Laplacian-Type Equation with Additive Noise

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We first establish the existence and uniqueness of a solution for a stochastic p-Laplacian-type equation with additive white noise and show that the unique solution generates a stochastic dynamical system. By using the Dirichlet forms of Laplacian and an approximation procedure, the nonlinear obstacle, arising from the additive noise is overcome when we make energy estimate. Then, we obtain a random attractor for this stochastic dynamical system. Finally, under a restrictive assumption on the monotonicity coefficient, we find that the random attractor consists of a single point, and therefore the system possesses a unique stationary solution.

1. Introduction

Let $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded open set with regular boundary ∂D. In this paper, we investigate the existence of a solution and a random attractor for the following quasilinear differential equation influenced by additive white noise

$$du + \left(\Delta\left(\Phi_p(\Delta u)\right) + g(x,u)\right)dt = f(x)dt + \sum_{j=1}^{m}\phi_j dW_j(t), \quad x \in D,\ t \geq 0, \tag{1.1}$$

with the boundary conditions

$$\nabla u(t) = \mathbf{0}, \quad u(t) = 0, \quad x \in \partial D,\ t \geq 0, \tag{1.2}$$

and the initial condition

$$u(0,x) = u_0(x), \quad x \in D. \tag{1.3}$$

In (1.1), $\Phi_p(s) = |s|^{p-2}s$, $p \geq 2$, $W_j(t)$ $(1 \leq j \leq m)$ are mutually independent two-sided real-valued Wiener processes, $\phi_j = \phi_j(x)$ $(1 \leq j \leq m, x \in D)$ are given real-valued functions that will be assumed to satisfy some conditions. The unknown $u(t)$ is a real-valued random process, sometimes denoted by $u(t,x)$ or $u(t,x,w)$. The exterior forced function $g(x,s)$ defined in $D \times \mathbb{R}$ is subjected to the following growth and monotonicity assumptions:

$$g(x,s)s \geq C_1|s|^q - \Lambda_1(x), \quad \Lambda_1 \in L^1(D),\ C_1 \in \mathbb{R}^+, \tag{1.4}$$

$$|g(x,s)| \leq C_2|s|^{q-1} + \Lambda_2(x), \quad \Lambda_2 \in L^{q/(q-1)}(D),\ C_2 \in \mathbb{R}^+, \tag{1.5}$$

$$\left(g(x,s_1) - g(x,s_2)\right)(s_1 - s_2) \geq C_3|s_1 - s_2|^2, \quad C_3 \in \mathbb{R}, \tag{1.6}$$

where $2 \leq q \leq p < \infty$.

In deterministic case (without random perturbed term), if $g(x,u) = ku$, Temam [1] proved the existence and uniqueness of the solution, and then obtained the global attractor. Recently, Yang et al. [2, 3] obtained the global attractors for a general p-Laplacian-type equation on unbounded domain and bounded domain, respectively. Chen and Zhong [4] discussed the nonautonomous case where the uniform attractor was derived.

It is well known that the long-time behavior of random dynamical systems (RDS) is characterized by random attractors, which was first introduced by Crauel and Flandoli [5] as a generalization of the global attractors for deterministic dynamical system. The existence of random attractors for RDS has been extensively investigated by many authors, see [5–12] and references therein. However, most of these researches concentrate on the stochastic partial differential equations of semilinear type, such as reaction-diffusion equation [5–8], Ginzburg-Landau equation [9, 10], Navier-Stokes equation [5, 6], FitzHugh-Nagumo system [11] and so on. To our knowledge, recently, the Ladyzhenskaya model in [12] seems the first study on the random attractors for nonsemilinear type equations. It seems that the quasilinear type or complete nonlinear type evolution equations with additive noise take on severe difficulty when one wants to derive the random attractors.

In this paper, we consider the existence and uniqueness of the solution and random attractor for (1.1) with forced term $g(x,u)$ satisfying (1.4)–(1.6). The additive white noise $\sum_{j=1}^{m} \phi_j dW_j(t)$ characterizes all kinds of stochastic influence in nature or man-made complex system which we must take into consideration in the concrete situation.

In order to deal with (1.1), we usually transform by employing a variable change the stochastic equation with a random term into a deterministic one containing a random parameter. Then the structure of the original equation (1.1) is changed by this transformation. As a result, some extra difficulties are developed in the process of the estimate of the solution, especially in the stronger norm space V, where $V \subset H \subset V'$ is the Gelfand triple; see Section 2. Hence, the methods (see [1–3]) used in unperturbed case are completely unavailable for obtaining the random attractors for (1.1).

Though we also follow the classic approach (based on the compact embedding) widely used in [5, 6, 9, 10, 12] and so on, some techniques have to be developed to overcome the difficulty of estimate of the solution to (1.1) in the Sobolev space V. Fortunately, by introducing a new inner product over the resolvent of Laplacian, we surmount this obstacle and obtain the estimate of the solution in the Sobolev space V_0, which is weaker than V, see Lemma 4.2 in Section 4. Here some basic results about Dirichlet forms of Laplacian are used. For details on the Dirichlet forms of a negative definite and self-adjoint operator please refer to [13]. The existence and uniqueness of solution, which ensure the existence of continuous RDS, are proved by employing the standard in [14]. If a restrictive assumption is imposed on the monotonicity coefficient in (1.6) we obtain a compact attractor consisting of a single point which attracts every deterministic bounded subset of H.

The organization of this paper is as follows. In the next section, we present some notions and results on the theory of RDS and Dirichlet forms which are necessary to our discussion. In Section 3, we prove the existence and uniqueness of the solution to the p-Laplacian-type equation with additive noise and obtain the corresponding RDS. In Section 4, we give some estimates for the solution satisfying (1.1)–(1.6) in given Hilbert space and then prove the existence of a random attractor for this RDS. In the last section, we prove the existence of the single point attractor under the given condition.

2. Preliminaries

In this section, we first recall some notions and results concerning the random attractor and the random flow, which can be found in [5, 6]. For more systematic theory of RDS we refer to [15]. We then list the Sobolev spaces, Laplacian and its semigroup and Dirichlet forms.

The basic notion in RDS is a measurable dynamical system (MSD). The form $(\Omega, \mathcal{F}, \mathrm{P}, \theta_s)$ is called a MSD if $(\Omega, \mathcal{F}, \mathrm{P})$ is a complete probability space and $\{\theta_s : \Omega \to \Omega, s \in \mathbb{R}\}$ is a family of measure-preserving transformations such that $(s, w) \mapsto \theta_s w$ is measurable, $\theta_0 = \mathrm{id}$ and $\theta_{t+s} = \theta_t \theta_s$ for all $s, t \in \mathbb{R}$.

A continuous RDS on a complete separable metric space (X, d) with Borel sigma-algebra $\mathcal{B}(X)$ over MSD $(\Omega, \mathcal{F}, \mathrm{P}, \theta_s)$ is by definition a measurable map

$$\varphi : \mathbb{R}^+ \times \Omega \times X \longrightarrow X, \qquad (t, w, x) \longmapsto \varphi(t, w)x \tag{2.1}$$

such that P-a.s. $w \in \Omega$

(i) $\varphi(0, w) = \mathrm{id}$ on X,

(ii) $\varphi(t + s, w) = \varphi(t, \theta_s w)\varphi(s, w)$ for all $s, t \in \mathbb{R}^+$ (cocycle property),

(iii) $\varphi(t, w) : X \to X$ is continuous for all $t \in \mathbb{R}^+$.

A continuous stochastic flow is by definition a family of measurable mapping $S(t, s; w) : X \to X, -\infty \leq s \leq t \leq \infty$, such that P-a.s. $w \in \Omega$

$$\begin{gathered} S(t, r; w)S(r, s; w)x = S(t, s; w)x, \quad x \in X, \\ S(t, s; w)x = S(t - s, 0.; \theta_s w)x, \quad x \in X, \end{gathered} \tag{2.2}$$

for all $s \leq r \leq t$, and $s \mapsto S(t, s; w)x$ is continuous in X for all $s \leq t$ and $x \in X$.

A random compact set $\{K(w)\}_{w \in \Omega}$ is a family of compact sets indexed by w such that for every $x \in X$ the mapping $w \mapsto d(x, K(w))$ is measurable with respect to $\mathcal{F}$.

Let $\mathcal{A}(w)$ be a random set. One says that $\mathcal{A}(w)$ is attracting if for P-a.s. $w \in \Omega$ and every deterministic bounded subset $B \subset X$

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\theta_{-t}w)B, \mathcal{A}(w)) = 0, \tag{2.3}$$

where $\operatorname{dist}(\cdot,\cdot)$ is defined by $\operatorname{dist}(A,B) = \sup_{x\in A}\inf_{y\in B} d(x,y)$.

We say that $\mathcal{A}(w)$ absorbs $B \subset X$ if P-a.s. $w \in \Omega$, there exists $t_B(w) > 0$ such that for all $t \geq t_B(w)$,

$$\varphi(t,\theta_{-t}w)B \subset \mathcal{A}(w). \tag{2.4}$$

Definition 2.1. Recall that a random compact set $w \mapsto \mathcal{A}(w)$ is called to be a random attractor for the RDS φ if for P-a.s. $w \in \Omega$

(i) $\mathcal{A}(w)$ is invariant, that is, $\varphi(t,w)\mathcal{A}(w) = \mathcal{A}(\theta_t w)$, for all $t \geq 0$;

(ii) $\mathcal{A}(w)$ is attracting.

Theorem 2.2 (see [5]). *Let $\varphi(t,w)$ be a continuous RDS over a MDS $(\Omega,\mathcal{F},P;\theta_t)$ with a separable Banach Space X. If there exists a compact random absorbing set $K(w)$ absorbing every deterministic bounded subset of X, then φ possesses a random attractor $\mathcal{A}(w)$ defined by*

$$\mathcal{A}(w) = \overline{\bigcup_{B\in\mathcal{B}(X)} \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \varphi(t,\theta_{-t}w)B}}, \tag{2.5}$$

where $\mathcal{B}(X)$ denotes all the bounded subsets of X.

Let $L^p(D)$ be the p-times integrable functions space on D with norm denoted by $\|\cdot\|_p$, $\mathcal{U}(D)$ be the space consisting of infinitely continuously differential real-valued-functions with a compact support in D. We use V to denote the norm closure of $\mathcal{U}(D)$ in Sobolev space $W^{2,p}(D)$, that is, $V = W_0^{2,p}(D)$. Since D is a bounded smooth domain in $\mathbb{R}^n$, we can endow the Sobolev space V with equivalent norm (see [1, page 166])

$$\|v\|_V = \|\Delta v\|_p = \left(\int_D |\Delta v|^p dx\right)^{1/p}, \quad v \in V. \tag{2.6}$$

Define V' = the dual of V, that is, $V' = W^{-2,p'}(D)$. Then we have

$$T \in W^{-2,p'}(D) \Longleftrightarrow T = \sum_{|\alpha|\leq 2} D^\alpha f_\alpha, \quad f_\alpha \in L^{p'}(D), \tag{2.7}$$

where $1/p + 1/p' = 1$. Let H denote the closure of $L^2(D)$ in $\mathcal{U}(D)$ with the usual scalar product and norm $\{(\cdot,\cdot), \|\cdot\|_2\}$. Identifying H with its dual space H' by the Riesz isomorphism $i : H \to H'$, we have the following Gelfand triple:

$$V \subset H \equiv H' \subset V', \tag{2.8}$$

or concretely

$$W_0^{2,p}(D) \subset L^2(D) \subset \left(W_0^{2,p}(D)\right)' = W^{-2,p/(p-1)}(D), \tag{2.9}$$

where the injections are continuous and each space is dense in the following one.

We define the linear operator A by $Au = \Delta u$ for $u \in H_0^1(D) \bigcap H^2(D)$. Then A is negative definite and self-adjoint. It is well-known that A (with domain $W_0^{2,p}(D)$) generates a strongly continuous semigroup $M(t)$ on $L^p(D)$ which is contractive and positive. Here "contractive" means $\|M(t)\|_p \leq 1$ and "positive" means $M(t)u \geq 0$ for every $0 \leq u \in L^p(D)$. The resolvent of generator A denoted by $R(\lambda, A)$, $\lambda \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. By Lumer-Phillips Theorem in [16], it follows that $(0, \infty) \subset \rho(A)$ and for $u \in L^p(D)$

$$R(\lambda, A)u = (\lambda - A)^{-1}u = \int_0^\infty e^{-\lambda t} M(t)u \, dt, \quad \lambda > 0, \tag{2.10}$$

$$M(t)u = \lim_{n \to \infty} \left[\frac{n}{tR(n/t, A)}\right]^n u, \quad t > 0. \tag{2.11}$$

Furthermore, by (2.10) for every $u \in L^p(D)$ we have

$$\begin{aligned} &\|\lambda R(\lambda, A)u\|_p \leq \|u\|_p, \quad \lambda > 0, \\ &\lambda R(\lambda, A)u \longrightarrow u, \quad \text{as } \lambda \to \infty, \end{aligned} \tag{2.12}$$

where the convergence is in the L^p-norms. Moreover, for $u \in D(A)$, it follows that $R(\lambda, A)u \in D(A)$, and $AR(\lambda, A)u = R(\lambda, A)Au$.

Since A is negative definite and self-adjoint operator on $H_0^1(D) \bigcap H^2(D)$, we associate A with the Dirichlet forms [13] ε by

$$\varepsilon(u, v) = \left(\sqrt{-A}u, \sqrt{-A}v\right), \quad u, v \in H_0^1(D). \tag{2.13}$$

ε is unique determined by A. For $u, v \in H_0^1(D)$, we define a new inner product by

$$\varepsilon^{(\lambda)}(u, v) = \lambda(u - \lambda R(\lambda, A)u, v), \quad \lambda > 0, \tag{2.14}$$

where $R(\lambda, A)$ is the resolvent of A. Then, we have the basic fact (see [13]) that $\varepsilon^{(\lambda)}(u, v) \uparrow$ as $\lambda \to \infty$, and

$$\lim_{\lambda \to \infty} \varepsilon^{(\lambda)}(u, v) = \varepsilon(u, v), \tag{2.15}$$

for $u, v \in H_0^1(D)$.

3. Existence and Uniqueness of RDS

We introduce an auxiliary Wiener process, which enables us to change (1.1) to a deterministic evolution equation depending on a random parameter. Here, we assume that $W(t)$ is a two-sided Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$, where $\Omega = \{w \in C(\mathbb{R}, \mathbb{R}^m) : w(0) = 0\}$, $\mathcal{F}$ is the Borel sigma-algebra induced by the compact-open topology of Ω and P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we have

$$w(t) = W(t) = (W_1(t), W_2(t), \dots, W_m(t)), \quad t \in \mathbb{R}. \tag{3.1}$$

Define the time shift by

$$\theta_t w(s) = w(s+t) - w(t), \quad w \in \Omega,\ t, s \in \mathbb{R}. \tag{3.2}$$

Then $(\Omega, \mathcal{F}, P, \theta_t)$ is a ergodic measurable dynamical system.

In order to obtain the random attractor, in our following discussion, we always assume that $\phi_j\ (1 \le j \le m)$ belong to $W_0^{4,p}(D)$ and $\nabla \phi_j\ (1 \le j \le m) = \mathbf{0}$.

We now employ the approach similar to [5] to translate (1.1) by one classical change of variables

$$v(t) = u(t) - z(t), \tag{3.3}$$

where, for short, $z(t) = z(t, w) = \sum_{j=1}^{m} \phi_j W_j(t)$. Then, formally, $v(t)$ satisfies the following equation parameterized by $w \in \Omega$:

$$\frac{dv}{dt} + \Delta(\Phi(\Delta v + \Delta z)) + g(x, v+z) = f(x), \quad x \in D,\ t \ge s, \tag{3.4}$$

$$v(s) = u(s) - z(s), \quad x \in D,\ s \in \mathbb{R}, \tag{3.5}$$

$$\nabla v = \mathbf{0}, \quad v = 0,\ x \in \partial D,\ t \ge s, \tag{3.6}$$

where $g(x,u)$ satisfies (1.4)–(1.6) and f is given in V', $2 \le q \le p < \infty$.

We define a nonlinear operator Ψ on V

$$\Psi(v) = \Delta\big(\Phi_p(\Delta(v+z))\big) + g(x, v+z) - f(x), \tag{3.7}$$

for $v \in V$, $x \in D$. Then we have

$$\Psi(v) = \overline{\Psi}(u), \tag{3.8}$$

where we define $\overline{\Psi}(u) = \Delta(\Phi_p(\Delta u)) + g(x,u) - f(x)$ with $u = v + z$ as in (3.3). So we can deduce problem (1.1) to the problem

$$\frac{dv}{dt} + \Psi(v) = 0, \quad t \ge s, \tag{3.9}$$

with initial condition $v(s) = u(s) - z(s)$ for $s \in \mathbb{R}$. Moreover, by (3.9), it follows that the solutions w-wise satisfy the following:

$$v(t) = v(s) - \int_s^t \Psi(v(\tau))d\tau, \tag{3.10}$$

with $v(s) = u(s) - z(s)$ and $t \geq s$.

Since $p \geq q$, by our assumption (1.4)–(1.6) and $f \in V'$, it is easy to check that the operator $\Psi : v \mapsto \Psi(v)$ mapping $W^{2,p}(D)$ into $W^{-2,p'}(D)$ is well-defined, where $p' = p/(p-1)$. We now prove the existence and uniqueness of solution to (3.4).

Theorem 3.1. *Assume that g satisfies* (1.4)–(1.6) *and f is given in V', $2 \leq q \leq p < \infty$. Then for all $s \in \mathbb{R}$ and $v_0 \in H$ with $v_0 = v(s)$,* (3.4) *has a unique solution*

$$v(t, w; s, v_0) \in L^p_{\text{loc}}([s, \infty), V) \bigcap C([s, \infty), H) \tag{3.11}$$

for $t \geq s$ and P-a.s. $w \in \Omega$. Furthermore, the mapping $v_0 \mapsto v(t, w; s, v_0)$ from H into H is continuous for all $t \geq s$.

Proof. We will show that $\Psi(v)$ possesses hemicontinuity, monotonicity, coercivity, and boundedness properties. Then for every $v_0 \in H$ with $v_0 = v(s)$, the existence and uniqueness of solution $v(t) = v(t, w; s, v_0) \in L^p_{\text{loc}}([s, \infty), V)$ follow from [13, Theorem 4.2.4]. If the solution $v \in L^p([s, T], V)$, $T > 0$, then it is elementary to check that $\Psi(v)$ belongs to $L^{p'}([s, T], V')$ by our assumption $p \geq q$ and $f \in V'$. Thus, from (3.9), we get that $v_t \in L^{p'}([s, T], V')$. Now by the general fact (see [1, page 164, line 1–3]) it follows that v is almost everywhere equal to a function belonging to $C([s, T], H)$. The continuity of the mapping $v_0 \mapsto v(t, w; s, v_0)$ from H into H is easily proved by using the monotonicity of Ψ.

By [13, Theorem 4.2.4], it remains to show that $\Psi(v)$ possesses hemicontinuity, monotonicity, coercivity, and boundedness properties. For convenience of our discussion in the following, we decompose $\Psi(v) = \Psi_1(v) + \Psi_2(v)$, where $\Psi_1(v) = \Delta(\Phi_p(\Delta u))$ and $\Psi_2(v) = \Psi(v) - \Psi_1(v)$, where Ψ is as in (3.7).

We first prove the hemicontinuity, that is, for every $v_1, v_2, v_3 \in V$, the function $\lambda \to (\Psi(v_1 + \lambda v_2), v_3)$ is continuous from $\mathbb{R} \to \mathbb{R}$. But it suffices to prove the continuity at $\lambda = 0$. So we assume that $|\lambda| < 1$. For $v_1, v_2, v_3 \in V$, by integration by parts, we see that

$$(\Psi_1(v_1 + \lambda v_2), v_3) = \int_D |\Delta(v_1 + \lambda v_2 + z)|^{p-2} \Delta(v_1 + \lambda v_2 + z) \Delta v_3 dx. \tag{3.12}$$

By Hölder's inequality and Young's inequality, it yields that

$$\begin{aligned}
&\left| |\Delta(v_1 + \lambda v_2 + z)|^{p-2} \Delta(v_1 + \lambda v_2 + z) \Delta v_3 \right| \\
&\quad \leq |\Delta(v_1 + \lambda v_2 + z)|^{p-1} |\Delta v_3| \\
&\quad \leq 2^{p-2} \Big(|\Delta(v_1 + z)|^{p-1} + |\Delta v_2|^{p-1} \Big) |\Delta v_3| \\
&\quad \leq 2^{p-2} \big(|\Delta(v_1 + z)|^p + |\Delta v_2|^p + 2|\Delta v_3|^p \big),
\end{aligned} \tag{3.13}$$

which the right-hand side is in $L^1(D)$ for $v_1, v_2, v_3 \in V$. Hence the expression of the right-hand side of inequality (3.13) is the control function for the integrant in (3.12). Then the Lebesgue's dominated convergence theorem can be apply to (3.12) when we take the limit $\lambda \to 0$. This proves the hemicontinuity of $\Psi_1(v)$. As for the hemicontinuity of $\Psi_2(v)$, noting that by our assumption (1.5) and $f \in V'$ we have

$$\begin{aligned}(\Psi_2(v_1 + \lambda v_2), v_3) &= \int_D g(x, v_1 + \lambda v_2 + z) v_3 dx - \int_D f(x) v_3 dx \\ &\leq C_2 \int_D |v_1 + \lambda v_2 + z|^{q-1} |v_3| dx + \int_D \Lambda_2(x) |v_3| dx - \int_D f(x) v_3 dx.\end{aligned} \tag{3.14}$$

It suffices to find the control function for the first integrand above, but we can get this by noting that $q \leq p$ and using approach similar to (3.13).

Second, we prove the monotonicity of $\Psi(v)$. We first prove the monotonicity for Ψ_1. For $v_1, v_2 \in V$, since $v_1 = u_1 - z$, $v_2 = u_2 - z$, we have

$$\begin{aligned}&(\Psi_1(v_1) - \Psi_1(v_2), v_1 - v_2) \\ &\quad = \left(|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2, \Delta u_1 - \Delta u_2\right) \\ &\quad = \int_D \left(|\Delta u_1|^p + |\Delta u_2|^p - |\Delta u_1|^{p-2} \Delta u_1 \Delta u_2 - |\Delta u_2|^{p-2} \Delta u_2 \Delta u_1\right) dx \\ &\quad \geq \int_D \left(|\Delta u_1|^p + |\Delta u_2|^p - |\Delta u_1|^{p-1} |\Delta u_2| - |\Delta u_2|^{p-1} |\Delta u_1|\right) dx \\ &\quad = \int_D \left(|\Delta u_1|^{p-1} - |\Delta u_2|^{p-1}\right)(|\Delta u_1| - |\Delta u_2|) dx \geq 0.\end{aligned} \tag{3.15}$$

Since $p \geq 2$, the function s^{p-1} is increasing for $s \geq 0$, which shows that the last inequality in the above proof is correct. On the other hand, by our assumption (1.6), we have $(\Psi_2(v_1) - \Psi_2(v_2), v_1 - v_2) \geq C_3 \|v_1 - v_2\|_2^2$, and therefore it follows that for $v_1, v_2 \in V$

$$(\Psi(v_1) - \Psi(v_2), v_1 - v_2) \geq C_3 \|v_1 - v_2\|^2, \tag{3.16}$$

where C_3 is as in (1.6). Hence, we have showed the monotonicity of $\Psi(v)$.

As for the coercivity, for $v \in V$, by our assumptions (1.4) and (1.5), using Hölder' inequality, we have

$$\begin{aligned}(\Psi(v), v) &= \int_D \Delta\left(|\Delta u|^{p-2}\Delta u\right) v\,dx + \int_D g(x, v+z) v\,dx - \int_D f(x) v\,dx \\ &= \|\Delta u\|_p^p - \int_D \left(|\Delta u|^{p-2}\Delta u\right)\Delta z\,dx + \int_D g(x,u) u\,dx - \int_D g(x,u) z\,dx - \int_D f(x) v\,dx \\ &\geq \|\Delta u\|_p^p - \|\Delta u\|_p^{p-1}\|\Delta z\|_p + C_1\|u\|_q^q - \|\Lambda_1\|_1 - C_2\|u\|_q^{q-1}\|z\|_q \\ &\quad - \|\Lambda_2\|_{q'}\|z\|_q - \|f\|_{V'}\|v\|_V,\end{aligned} \tag{3.17}$$

where C_1 and C_2 are defined in (1.4) and (1.5). By employing the ε-Young's inequality, that is, $ab \leq \varepsilon(a^r/r) + \varepsilon^{-r'/r}(b^{r'}/r')$ for $r > 1$ and $1/r + 1/r' = 1$, we find that

$$\|\Delta u\|_p^{p-1}\|\Delta z\|_p \leq \frac{1}{4}\frac{p-1}{p}\|\Delta u\|_p^p + \frac{2^{2p-2}}{p}\|\Delta z\|_p^p \leq \frac{1}{4}\|\Delta u\|_p^p + 2^{2p-2}\|\Delta z\|_p^p. \tag{3.18}$$

Similarly, we have

$$C_2\|u\|_q^{q-1}\|z\|_q \leq \frac{C_1}{2}\|u\|_q^q + 2^{q-1}C_1^{1-q}C_2^q\|z\|_q^q, \tag{3.19}$$

$$\|f\|_{V'}\|v\|_V \leq \frac{1}{4}\|\Delta u\|_p^p + 2^{2/(p-1)}\|f\|_{V'}^{p/(p-1)} + \|f\|_{V'}\|\Delta z\|_p, \tag{3.20}$$

then, by (3.17)–(3.20), we obtain that

$$(\Psi(v), v) \geq \frac{1}{2}\|\Delta u\|_p^p + \frac{C_1}{2}\|u\|_q^q - p_1(t, w), \tag{3.21}$$

with

$$\begin{aligned}p_1(t, w) = 2^{2p-2}\|\Delta z\|_p^p &+ 2^{q-1}C_1^{1-q}C_2^q\|z\|_q^q + \|\Lambda_2\|_{q'}\|z\|_q \\ &+ 2^{2/(p-1)}\|f\|_{V'}^{p/(p-1)} + \|f\|_{V'}\|\Delta z\|_p \geq 0,\end{aligned} \tag{3.22}$$

where q' is the dual number of q. At the same time, (3.21) is one form of coercivity which will be used in Section 4, but in order to prove the existence and uniqueness of solution to (3.4), we will give another form.

Noting that by Hölder's inequality it follows with $u(t) = v(t) + z(t)$ that

$$\|u\|_q^q = \|v(t) + z(t)\|_q^q \geq 2^{1-q}\|v\|_q^q - \|z\|_q^q, \tag{3.23}$$

then by the inverse ε-Young's inequality, that is, $ab \geq \varepsilon(a^r/r) + \varepsilon^{-r'/r}(b^{r'}/r')$ when $r < 1$ and $1/r + 1/r' = 1$, we get from (3.23) that

$$\|u\|_q^q \geq 2^{1-q}\|v\|_q^q - \|z\|_q^q \geq 2^{1-q}\eta_0^q\|v\|_2^q - \|z\|_q^q \geq \frac{q}{2}\|v\|_2^2 - \|z\|_q^q - C', \tag{3.24}$$

where $C' = ((q-2)/2)2^{(2-2q)/(2-q)}\eta_0^{2q/(2-q)}$ and η_0 is the Sobolev embedding coefficient of $L^q(D) \hookrightarrow L^2(D)$. Hence, it follows from (3.21) that

$$(\Psi(v), v) \geq \frac{1}{2}\|\Delta u\|_p^p + \frac{q}{2}\|v\|_2^2 - p_2(t, w), \tag{3.25}$$

with $p_2(t,w) = p_1(t,w) + \|z\|_q^q + C' \geq 0$, where $p_1(t,w)$ is defined as in (3.22). Note that if $q = 2$, we omit this procedure and directly (3.21) passes to (3.25). Hence we have proved the coercivity for Ψ.

We finally prove the the boundedness for $\Psi(v)$ for fixed $v \in V$, that is, for fixed $v \in V$, $\Psi(v)$ is a linear bounded functional on $W_0^{2,p}(D)$. Indeed, for $v, h \in V$, by applying Hölder's inequality and repeatedly using Sobolev's embedding inequality, we have

$$\begin{aligned}
(\Psi(v), h) &\leq \int_D |\Delta u|^{p-1}|\Delta h|dx + \int_D \left|g(x,u)\right||h|dx + \int_D \left|f(x)\right||h|dx \\
&\leq \|\Delta u\|_p^{p-1}\|\Delta h\|_p + C_2\|u\|_q^{q-1}\|h\|_q + \|\Lambda_2\|_{q'}\|h\|_q + \left\|f\right\|_{V'}\|h\|_V \\
&\leq \left(\|\Delta u\|_p^{p-1} + c_1\|\Delta u\|_p^{q-1} + c_2\|\Lambda_2\|_{q'} + \left\|f\right\|_{V'}\right)\|\Delta h\|_p \\
&\leq \left(2\|\Delta u\|_p^{p-1} + c_3 + c_2\|\Lambda_2\|_{q'} + \left\|f\right\|_{V'}\right)\|\Delta h\|_p \\
&\leq \left(2^{p-1}\|\Delta v\|_p^{p-1} + p_3(t,w)\right)\|\Delta h\|_p
\end{aligned} \tag{3.26}$$

with the random variable $p_3(t,w) = 2^{p-1}\|\Delta z\|_p^{p-1} + c_2\|\Lambda_2\|_{q'} + c_2\|f\|_{V'} + c_3 \geq 0$ and the positive constants c_i $(i = 1, 2, 3)$ independent of v, h. Therefore, from (3.26) we finally find that

$$\|\Psi(v)\|_{V'} \leq 2^{p-1}\|\Delta v\|_p^{p-1} + p_3(t,w), \tag{3.27}$$

is a bounded linear operator on $W_0^{2,p}(D)$ for fixed $v \in V$. From the proof we know that the assumption $p \geq q \geq 2$ is necessary. This completes the proof of Theorem 3.1. □

We now define

$$S(t,s;w)u_0 = v(t,w;s,u_0 - z(s,w)) + z(t,w), \quad t \geq s \in \mathbb{R}, \tag{3.28}$$

with $u_0 = u(s)$. Then $S(t,s;w)u_0$ is the solution to (1.1) in certain meaning for every $u_0 \in H$ and $t \geq s \in \mathbb{R}$. By the uniqueness part of solution in Theorem 3.1, we immediately get that $S(t,s,w)$ is a stochastic flow, that is, for every $u_0 \in H$ and $t \geq r \geq s \in \mathbb{R}$

$$S(t,s;w)u_0 = S(t,r;w)S(r,s;w)u_0, \tag{3.29}$$

$$S(t,s;w)u_0 = S(t-s,0;\theta_s w)u_0. \tag{3.30}$$

Hence if we define

$$\varphi(t,w)u_0 = S(t,0;w)u_0 = v(t,w;0,u_0 - z(0,w)) + z(t,w) \tag{3.31}$$

with $u_0 = u(0)$, then by Theorem 3.1 φ is a continuous stochastic dynamical system associated with quasilinear partial differential equation (1.1), with the following fact

$$\varphi(t,\theta_{-t}w)u_0 = u(0,w;-t,u_0), \quad \forall t \geq 0, \tag{3.32}$$

that is to say, $\varphi(t,\theta_{-t}w)u_0$ can be interpreted as the position of the trajectory at time 0, which was in u_0 at time $-t$ (see [5]).

4. Existence of Compact Random Attractor for RDS

In this section, we will compute some estimates in space $H = L^2(D)$ and $V_0 = H_0^1(D)$. Note that $p_i(t,w)$ $(i = 1,2,3)$ appearing in the proofs are given in Section 3. In the following computation, $w \in \Omega$; the results will hold for P-a.s. $w \in \Omega$.

Lemma 4.1. *Suppose that g satisfies* (1.4)–(1.6) *and f is given in V'. Then there exist random radii $r_1(w), r_2(w) > 0$, such that for all $\rho > 0$ there exists $s = s(w,\rho) \leq -1$ such that for all $s \leq s(w,\rho)$ and all $u_0 \in H$ with $\|u_0\|_2 \leq \rho$, the following inequalities hold for P-a.s. $w \in \Omega$*

$$\begin{gathered}
\|v(t,w;s,u_0 - z(s))\|_2^2 \leq r_1^2(w), \quad \forall t \in [-1,0], \\
\int_{-1}^{0} \left(\|\Delta u(\tau,w;s,u_0)\|_p^p + \|u(\tau,w;s,u_0)\|_q^q \right) d\tau \leq r_2^2(w),
\end{gathered} \tag{4.1}$$

where $v(t,w;s,u_0 - z(s))$ is the solution to (3.4) *with $v(t,w;s,u_0 - z(s)) = u(t,w;s,u_0) - z(t,w)$ and $u_0 = u(s)$.*

Proof. For simplicity, we abbreviate $v(t) := v(t,w;s,u_0 - z(s))$ and $u(t) := u(t,w;s,u_0)$ for fixed $u_0, w \in \Omega$ and $t \geq s$ with $u_0 = u(s)$. Multiplying both sides of (3.9) by $v(t)$ and then integrating over D, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + (\Psi(v), v) = 0. \tag{4.2}$$

Then, by (3.25), we have

$$\frac{d}{dt}\|v\|_2^2 + \|\Delta u\|_p^p + q\|v\|_2^2 \le 2p_2(t,w). \tag{4.3}$$

Applying the Gronwall's lemma to (4.3) from s to t, $t \in [-1,0]$, it yields that

$$\begin{aligned}\|v(t)\|_2^2 &\le e^{-q(t-s)}\|v(s)\|_2^2 + 2\int_s^t p_2(\tau,w)e^{-q(t-\tau)}d\tau \\ &\le 2e^q\left(e^{qs}\|u_0\|_2^2 + e^{qs}\|z(s)\|_2^2 + \int_{-\infty}^0 p_2(\tau,w)e^{q\tau}d\tau\right),\end{aligned} \tag{4.4}$$

where $p_2(\tau,w)$ grows at most polynomially as $\tau \to -\infty$ (see [5]). Since $p_2(\tau,w)$ is multiplied by a function which decays exponentially, the integral in (4.4) is convergent.

Given every fixed $\rho > 0$, we can choose $s(w,\rho)$, depending only on w and ρ, such that $e^{qs}\|u_0\|_2^2 \le 1$. Similarly, $\|z(s)\|_2^2$ grows at most polynomially as $s \to -\infty$, and $\|z(s)\|_2^2$ is multiplied by a function which decays exponentially. Then we have

$$\sup_{s\le 0} e^{qs}\|z(s)\|_2^2 < +\infty. \tag{4.5}$$

Hence by (4.4) we can give the final estimate for $\|v(t)\|_2^2$

$$\|v(t)\|_2^2 \le r_1^2(w) := 2e^q\left(1 + \sup_{s\le 0} e^{qs}\|z(s)\|_2^2 + \int_{-\infty}^0 p_2(\tau,w)e^{q\tau}d\tau\right), \tag{4.6}$$

for $t \in [-1,0]$. Following (4.2), by using (3.21), we find that

$$\frac{d}{dt}\|v\|_2^2 + \|\Delta u\|_p^p + C_1\|u\|_q^q \le 2p_1(t,w), \tag{4.7}$$

where $p_1(t,w)$ is the same as in (3.22). Integrating (4.7) for t on $[-1,0]$, we get that

$$\int_{-1}^0 \|\Delta u(\tau)\|_p^p + C_1\|u(\tau)\|_q^q d\tau \le 2\int_{-1}^0 p_1(\tau,w)d\tau + \|v(-1)\|_2^2, \tag{4.8}$$

which gives an expression for $r_2^2(w)$. □

In the following, we give the estimate of $\|\nabla u(t)\|_2$. This is the most difficult part in our discussion. Because the nonlinearity of Ψ_1 and Ψ_2 in (3.4) or (3.9), it seems impossible to derive the V-norm estimate by the way as [1, page 169]. So we relax to bound the solution in a weaker Sobolev $V_0 = H_0^1(D)$ with equivalent norms denoted by $\|\nabla u\|_2$ for $u \in V_0$. Here, just as our statement in the introduction, we make the inner product over the resolvent $R(\lambda, A)$ which is defined in Section 2, then by using the Dirichlet forms of A we obtain technically the estimate of $\|\nabla u(t)\|_2$.

Lemma 4.2. *Suppose that g satisfies* (1.4)–(1.6) *and f is given in V'. Then there exists a random radius $r_3(w) > 0$, such that for all $\rho > 0$ there exists $s = s(w,\rho) \le -1$ such that for all $s \le s(w,\rho)$ and all $u_0 \in H$ with $\|u_0\|_2 \le \rho$, the following inequality holds for P-a.s. $w \in \Omega$*

$$\|\nabla u(t,w;s,u_0)\|_2^2 \le r_3^2(w), \quad \forall t \in [-1,0], \tag{4.9}$$

where $u(t,w;s,u_0)$ is the solution to (1.1) *with $u_0 = u(s)$. In particular,*

$$\|\nabla u(0,w;s,u_0)\|_2^2 \le r_3^2(w). \tag{4.10}$$

Proof. Taking the inner product of (3.9) with $-\lambda AR(\lambda,A)v$ where $\lambda > 0$, $v \in V$, we get

$$-\int_D v_t \lambda AR(\lambda,A)v\,dx = \int_D \Psi_1(u)\lambda AR(\lambda,A)v\,dx + \int_D \Psi_2(u)\lambda AR(\lambda,A)v\,dx. \tag{4.11}$$

By the semigroup theory (see [16]) we have

$$AR(\lambda,A)v = R(\lambda,A)Av = \lambda R(\lambda,A)v - v, \tag{4.12}$$

for $v \in D(A)$. We now estimate every terms on the right-hand side of (4.11). The first term on the right-hand side of (4.11) is rewritten as

$$\int_D \Psi_1(u)\lambda AR(\lambda,A)v\,dx = \int_D \Psi_1(u)\lambda AR(\lambda,A)u\,dx - \int_D \Psi_1(u)\lambda AR(\lambda,A)z\,dx. \tag{4.13}$$

Employing (4.12) and by integration by parts, it yields that

$$\begin{aligned}
\int_D \Psi_1(u)\lambda AR(\lambda,A)u\,dx &= \lambda\int_D \Psi_1(u)(\lambda R(\lambda,A)u - u)dx \\
&= -\lambda\int_D \Delta\left(|\Delta u|^{p-2}\Delta u\right)u\,dx + \lambda\int_D \Delta\left(|\Delta u|^{p-2}\Delta u\right)\lambda R(\lambda,A)u\,dx \\
&= -\lambda\|\Delta u\|_p^p + \lambda\int_D \left(|\Delta u|^{p-2}\Delta u\right)\lambda\Delta R(\lambda,A)u\,dx \\
&\le -\lambda\|\Delta u\|_p^p + \lambda\int_D |\Delta u|^{p-1}|\lambda R(\lambda,A)\Delta u|dx \\
&\le -\lambda\|\Delta u\|_p^p + \lambda\|\Delta u\|_p^{p-1}\|\lambda R(\lambda,A)\Delta u\|_p \\
&\le -\lambda\|\Delta u\|_p^p + \lambda\|\Delta u\|_p^p = 0,
\end{aligned} \tag{4.14}$$

where we use the contraction property of $\lambda R(\lambda, A)$ on $L^p(D)$, that is, $\|\lambda R(\lambda, A)\Delta u\|_p \leq \|\Delta u\|_p$ for $\Delta u \in L^p(D)$ and every $\lambda > 0$. We now bound the second term on the right-hand side of (4.13)

$$-\int_D \Psi_1(u)\lambda AR(\lambda, A)z\,dx \leq \|\Psi_1(u)\|_{V'}\|\lambda AR(\lambda, A)z\|_V$$
$$= \|\Psi_1(u)\|_{V'}\|\lambda R(\lambda, A)Az\|_V, \tag{4.15}$$

where we use our assumption ϕ_j $(1 \leq j \leq m) \in W_0^{4,p}(D)$. Since Ψ_1 maps V into V', then for fixed $u \in V$ and every $h \in V$, we have

$$(\Psi_1(u), h) = \int_D \Delta\left(|\Delta u|^{p-2}\Delta u\right)h\,dx = \int_D \left(|\Delta u|^{p-2}\Delta u\right)\Delta h\,dx$$
$$\leq \int_D |\Delta u|^{p-1}|\Delta h|dx \leq \|\Delta u\|_p^{p-1}\|\Delta h\|_p. \tag{4.16}$$

So for fixed $u \in V$, $\|\Psi_1(u)\|_{V'} \leq \|\Delta u\|_p^{p-1}$, and therefore by (4.15) we obtain that

$$-\int_D \Psi_1(u)\lambda AR(\lambda, A)z\,dx \leq \|\Delta u\|_p^{p-1}\|\lambda R(\lambda, A)Az\|_V$$
$$\leq C\|\Delta u\|_p^{p-1}\|Az\|_V \leq \|\Delta u\|_p^p + C^p\|Az\|_V^p, \tag{4.17}$$

where $\|\lambda R(\lambda, A)Az\|_V \leq C\|Az\|_V$ and C is a constant independent of λ, $v(t)$ and $u(t)$. Here we should note that $\lambda R(\lambda, A)$ is a bounded linear operator on V. Hence, by (4.14)–(4.17) the first term on the right-hand side of (4.11) is finally bounded by

$$\int_D \Psi_1(u)\lambda AR(\lambda, A)v\,dx \leq \|\Delta u\|_p^p + C^p\|Az\|_V^p. \tag{4.18}$$

By our assumption (1.5), the second term on the right-hand side of (4.11) is estimated as

$$\begin{aligned}
&\int_D \Psi_2(u)\lambda AR(\lambda, A)v\,dx \\
&\quad= \int_D (g(x,u) - f(x))\lambda R(\lambda, A)Av\,dx \\
&\quad\leq \int_D \left|g(x,u)\right||\lambda R(\lambda, A)Av|dx + \int_D \left|f(x)\right||\lambda AR(\lambda, A)v|dx \\
&\quad\leq \int_D \left(C_2|u|^{q-1} + \Lambda_2(x)\right)|\lambda R(\lambda, A)Av|dx + \int_D \left|f(x)\right||\lambda R(\lambda, A)Av|dx \\
&\quad\leq C_2\|u\|_q^{q-1}\|\lambda R(\lambda, A)Av\|_q + \|\Lambda_2\|_{q'}\|\lambda R(\lambda, A)Av\|_q + \left\|f\right\|_2\|\lambda R(\lambda, A)Av\|_2 \\
&\quad\leq C_2\|u\|_q^{q-1}\|Av\|_q + \|\Lambda_2\|_{q'}\|Av\|_q + \left\|f\right\|_2\|Av\|_2 \\
&\quad\leq \|u\|_q^q + \left(C_2^q + 1\right)\|Av\|_q^q + \|\Lambda_2\|_{q'}^{q'} + \left\|f\right\|_2^2 + \|Av\|_2^2,
\end{aligned} \tag{4.19}$$

where we employ Young's inequality $ab \le a^r + b^{r/(r-1)}$ for $r > 1$. But, by Sobolev's inequality and Young's inequality, it yields that

$$\|Av\|_q^q \le \eta_1^q \|Av\|_p^q \le \eta_1^p \|Av\|_p^p + 1 \le 2^{p-1}\eta_1^p \|Au\|_p^p + 2^{p-1}\eta_1^p \|Az\|_p^p + 1; \tag{4.20}$$

similarly

$$\|Av\|_2^2 \le 2^{p-1}\eta_2^p \|Au\|_p^p + 2^{p-1}\eta_2^p \|Az\|_p^p + 1, \tag{4.21}$$

where the positive constants η_1, η_2 are Sobolev's embedding constants independent of λ. Then by (4.19)–(4.21), there exist positive constants c_1, c_2 such that

$$\int_D \Psi_2(u)\lambda AR(\lambda, A)v\,dx \le \|u\|_q^q + c_1\|Au\|_p^p + c_2\|Az\|_p^p + \|\Lambda_2\|_{q'}^{q'} + \|f\|_2^2 + 2, \tag{4.22}$$

where $q' = q/(q-1)$. By (4.18) and (4.22), we find that (4.11) becomes

$$-\int_D v_t \lambda AR(\lambda, A)v\,dx \le c_3\|\Delta u\|_p^p + \|u\|_q^q + p_4(t,w), \tag{4.23}$$

where $p_4(t,w) = C^p\|Az\|_V^p + c_2\|Az\|_p^p + \|\Lambda_2\|_{q'}^{q'} + \|f\|_2^2 + 2 \ge 0$ and $c_3 = c_1 + 1$. On the other hand, by (4.12) and the Dirichlet forms (2.14), we have

$$-\int_D v_t \lambda AR(\lambda, A)v\,dx = \varepsilon^{(\lambda)}(v, v_t). \tag{4.24}$$

Hence by (4.24), (4.23) is rewritten as

$$\varepsilon^{(\lambda)}(v, v_t) \le c_3\|\Delta u\|_p^p + \|u\|_q^q + p_4(t,w). \tag{4.25}$$

Note that the right-hand side of (4.25) is independent of λ. So taking limit on both side of (4.25) for $\lambda \to \infty$, association with (2.15), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 \le c_3\|\Delta u\|_p^p + \|u\|_q^q + p_4(t,w). \tag{4.26}$$

Integrating (4.26) from s to t $(-1 \le s \le t \le 0)$, it yields that

$$\begin{aligned}\|\nabla v(t)\|_2^2 &\le 2c_3\int_s^t \|\Delta u(\tau)\|_p^p d\tau + 2\int_s^t \|u(\tau)\|_q^q d\tau + 2\int_s^t p_4(\tau,w)d\tau + \|\nabla v(s)\|_2^2 \\ &\le 2c_3\int_{-1}^0 \|\Delta u(\tau)\|_p^p d\tau + 2\int_{-1}^0 \|u(\tau)\|_q^q d\tau + 2\int_{-1}^0 p_4(\tau,w)d\tau + \|\nabla v(s)\|_2^2.\end{aligned} \tag{4.27}$$

Therefore, by Lemma 4.1, we find that

$$\|\nabla v(t)\|_2^2 \le 2(c_3+1)r_2^2(w) + 2\int_{-1}^{0} p_4(\tau,w)\,d\tau + \|\nabla v(s)\|_2^2. \tag{4.28}$$

Integrating (4.28) for s from -1 to 0, we have

$$\|\nabla v(t)\|_2^2 \le 2(c_3+1)r_2^2(w) + 2\int_{-1}^{0} p_4(\tau,w)\,d\tau + \int_{-1}^{0}\|\nabla v(s)\|_2^2 ds, \tag{4.29}$$

for all $t \in [-1,0]$. By Poincare's inequality, and Young's inequality, there exist positive constants c_4, c_5, c_6 such that

$$\begin{aligned}\|\nabla v(s)\|_2^2 &\le c_4\|\Delta v(s)\|_2^2 \le 2c_4\|\Delta u(s)\|_2^2 + 2c_4\|\Delta z(s)\|_2^2\\ &\le 2c_5\|\Delta u(s)\|_p^p + 2c_5\|\Delta z(s)\|_2^2 + c_6.\end{aligned} \tag{4.30}$$

Hence, by (4.30) and using Lemma 4.1 again, (4.29) follows

$$\|\nabla v(t)\|_2^2 \le 2(c_3+c_5+1)r_2^2(w) + 2\int_{-1}^{0} p_4(\tau,w)\,d\tau + 2c_5\int_{-1}^{0}\|\Delta z(\tau)\|_2^2 d\tau + c_6, \tag{4.31}$$

with $t \in [-1,0]$. See that $v(t) = u(t) - z(t)$. Then, we have

$$\begin{aligned}\|\nabla u(t)\|_2^2 &\le 2\|\nabla v(t)\|_2^2 + 2\|\nabla z(t)\|_2^2\\ &\le 2(c_3+c_5+1)r_2^2(w) + 2\int_{-1}^{0} p_4(\tau,w)\,d\tau\\ &\quad + 2c_5\int_{-1}^{0}\|\Delta z(\tau)\|_2^2 d\tau + 2\sup_{-1\le t\le 0}\|\nabla z(t)\|_2^2 + c_6,\end{aligned} \tag{4.32}$$

which gives an expression for $r_3^2(w)$. This completes the proof. □

By Theorem 2.2 and Lemma 4.2, we have obtained our main result in this section.

Theorem 4.3. *Assume that g satisfies* (1.4)–(1.6) *and f is given in V'. Then the RDS $\varphi(t,\omega)$ generated by the stochastic equation* (1.1) *possesses a random attractor $\mathcal{A}(w)$ defined as*

$$\mathcal{A}(w) = \overline{\bigcup_{B\in\mathcal{B}(H)}\bigcap_{s\ge 0}\overline{\bigcup_{t\ge s}\varphi(t,\theta_{-t}w)B}}, \tag{4.33}$$

where $\mathcal{B}(H)$ denotes all the bounded subsets of H and the closure is the H-norm.

Remark 4.4. As stated in Theorem 3.1, under the assumptions (1.4)–(1.6), the solutions of (1.1) are in $W_0^{2,p}(D)$. So it is possible in theory to obtain a compact random attractor in $W_0^{1,p}(D)$ or

$W_0^{2,p}(D)$. But it seems most difficulty to derive the estimate of solution in $W^{2,p}(D)$ due to the nonlinear principle part $\Delta(\Phi_p(\Delta u))$.

5. The Single Point Attractor

In this section, we consider the attracting by a single point. In order to derive our anticipating result, we assume that $C_3 > 0$ in (1.6). This leads to the following fact that for every fixed $t \in \mathbb{R}$ and $w \in \Omega$, the solution $u(t, w; s, u(s))$ to (1.1) is a Cauchy sequence in H for the initial time s with initial value $u(s)$ belonging to the bounded subset of H. Then we obtain a compact attractor consisting of a single point which is the limit of $u(0, w; s, u(s))$ as $s \to -\infty$.

Lemma 5.1. *Assume that g satisfies (1.4)–(1.6) and f is given in V', $C_3 > 0$. Then for $s_1 \le s_2 \le t$ and $u_{01}, u_{02} \in H$ with $u(s_1) = u_{01}$ and $u(s_2) = u_{02}$, there exists a positive constant $k < C_3$ such that*

$$\|u(t, w; s_1, u_{01}) - u(t, w; s_2, u_{02})\|_2^2$$
$$\le 2e^{-C_3 t}\left(4e^{(C_3-k)s_2}\left(\|u_{01}\|_2^2 + \|z(s_1)\|_2^2 + \int_{-\infty}^{0} p_2(\tau, w)e^{k\tau}d\tau\right) + 2e^{ks_2}\|z(s_2)\|_2^2 + e^{ks_2}\|u_{02}\|_2^2\right). \tag{5.1}$$

In particular, for each fixed $t \in \mathbb{R}$ and $w \in \Omega$ there exists a single point $\xi_t(w)$ in H such that

$$\lim_{s \to -\infty} S(t, s; w)u_0 = \xi_t(w), \tag{5.2}$$

where $u_0 = u(s)$ and $S(t, s; w)$ is the stochastic flow defined as in (3.28) which is a version of solution to (1.1). Furthermore, the limit in the above is independent of u_0 for all u_0 belonging to a bounded subset of H.

Proof. For $s_1 \le s_2 \le t$ and $u_{01}, u_{02} \in H$ with $u(s_1) = u_{01}$ and $u(s_2) = u_{02}$, we can deduce from (3.9) and (3.8) that

$$\frac{d}{dt}(u(t, w; s_1, u_{01}) - u(t, w; s_2, u_{02})) + \overline{\Psi}(u(t, w; s_1, u_{01})) - \overline{\Psi}(u(t, w; s_2, u_{02})) = 0, \tag{5.3}$$

where $u(t) = v(t) + z(t)$ is the solution to problem (1.1). On the other hand, by (3.8) and (3.16), we immediately deduce that

$$\left(\overline{\Psi}(u_1) - \overline{\Psi}(u_2), u_1 - u_2\right) \ge C_3\|u_1 - u_2\|_2^2. \tag{5.4}$$

Then, multiplying (5.3) by $u(t, w; s_1, u_{01}) - u(t, w; s_2, u_{02})$, integrating over D, and using (5.4), we find that

$$\frac{d}{dt}\|u(t, w; s_1, u_{01}) - u(t, w; s_2, u_{02})\|_2^2 + C_3\|u(t, w; s_1, u_{01}) - u(t, w; s_2, u_{02})\|_2^2 \le 0. \tag{5.5}$$

Now, applying Gronwall's lemma to (5.5) from s_2 to t, it yields that

$$\begin{aligned}\|u(t,w;s_1,u_{01})-u(t,w;s_2,u_{02})\|_2^2 &\le \|u(s_2,w;s_1,u_{01})-u_{02}\|_2^2 e^{-C_3(t-s_2)} \\ &\le 2\Big(\|u(s_2,w;s_1,u_{01})\|_2^2+\|u_{02}\|_2^2\Big)e^{-C_3(t-s_2)}.\end{aligned} \tag{5.6}$$

We then estimate $\|u(s_2,w;s_1,u_{01})\|_2^2$. To this end, we rewrite (3.21) as

$$(\Psi(v),v) \ge \frac{C_1}{2}\|u\|_q^q - p_1(t,w). \tag{5.7}$$

Since $q \ge 2$, by Hölder's inequality and inverse Young's inequality we can choose constant $0<k<C_3$ such that

$$\frac{C_1}{2}\|u\|_q^q \ge \frac{k}{2}\|v\|_2^2 - c_1\|z\|_q^q - c_2. \tag{5.8}$$

Then, by (5.7)-(5.8) it follows from (4.2) that

$$\frac{d}{dt}\|v\|_2^2 + k\|v\|_2^2 \le 2p_5(t,w), \tag{5.9}$$

where $p_5(t,w) = p_1(t,w) + c_1\|z\|_q^q + c_2$ and $v(t) = v(t,w;s_1,u_{01}-z(s_1))$. Using Gronwall's lemma to (5.9) from s_1 to s_2 with $s_1 \le s_2 \le 0$, we get that

$$\begin{aligned}\|v(s_2,w;s_1,u_{01}-z(s_1))\|_2^2 &\le \|u_{01}-z(s_1)\|_2^2 e^{-k(s_2-s_1)} + \int_{s_1}^{s_2} 2p_5(\tau,w)e^{-k(s_2-\tau)}d\tau \\ &\le 2e^{-ks_2}\left(\|u_{01}\|_2^2+\|z(s_1)\|_2^2+\int_{-\infty}^{0} p_5(\tau,w)e^{k\tau}d\tau\right).\end{aligned} \tag{5.10}$$

Similar to the argument of (4.4), we know that the integral in the above is convergent. Therefore, we have

$$\begin{aligned}\|u(s_2,w;s_1,u_{01})\|_2^2 &\le 2\|v(s_2,w;s_1,u_{01})\|_2^2 + 2\|z(s_2)\|_2^2 \\ &\le 4e^{-ks_2}\left(\|u_{01}\|_2^2+\|z(s_1)\|_2^2+\int_{-\infty}^{0} p_5(\tau,w)e^{k\tau}d\tau\right) + 2\|z(s_2)\|_2^2,\end{aligned} \tag{5.11}$$

from which and (5.6) it follows for every fixed $t \in \mathbb{R}$ that

$$\begin{aligned}
&\|u(t,w;s_1,u_{01}) - u(t,w;s_2,u_{02})\|_2^2 \\
&\quad \le 2e^{-C_3 t}\left[4e^{(C_3-k)s_2}\left(\|u_{01}\|_2^2 + \|z(s_1)\|_2^2 + \int_{-\infty}^{0} p_5(\tau,w)e^{k\tau}d\tau\right) + 2e^{ks_2}\|z(s_2)\|_2^2 + e^{ks_2}\|u_{02}\|_2^2\right] \\
&\quad \longrightarrow 0, \quad \text{as}, \ s_1, s_2 \longrightarrow -\infty,
\end{aligned} \tag{5.12}$$

where the convergence is uniform with respect to u_{01}, u_{02} belonging to every bounded subset of H. Then (5.12) implies that for fixed $t \in \mathbb{R}$, $u(t,w;s,u(s))$ is a Cauchy sequence in H for $s \in \mathbb{R}$. Thus, by the completeness of H, for every fixed $t \in \mathbb{R}$ and $w \in \Omega$, $u(t,w;s,u(s))$ has a limit in H denoted by $\xi_t(w)$, that is,

$$\lim_{s\to-\infty} u(t,w;s,u(s)) = \xi_t(w). \tag{5.13}$$

□

Theorem 5.2. *Assume that g satisfies* (1.4)–(1.6) *and f is given in V', $C_3 > 0$. Then the RDS $\varphi(t,w)$ generated by the solution to* (1.1) *possesses a single point attractor $\mathcal{A}(w)$, that is, there exists a single point $\xi_0(w)$ in H such that*

$$\mathcal{A}(w) = \{\xi_0(w)\}. \tag{5.14}$$

Proof. By Lemma 5.1 we define

$$\xi_0(w) = \lim_{s\to-\infty} S(0,s;w)u_0, \tag{5.15}$$

where $S(0,s;w) = u(0,w;s,u(s))$ by (3.28). Then we need prove that $\mathcal{A}(w) = \{\xi_0(w)\}$ is a compact attractor. It is obvious that $\{\xi_0(w)\}$ is a compact random set. Hence by Definition 2.1 it suffices to prove the invariance and attracting property for $\{\xi_0(w)\}$. Since by the continuity of $\varphi(t,w)$, and relations (3.29)–(3.32), we have

$$\begin{aligned}
\varphi(t,w)\xi_0(w) &= \varphi(t,w)\lim_{s\to-\infty} S(0,s;w)u_0 = \lim_{s\to-\infty} \varphi(t,w)S(0,s;w)u_0 \\
&= \lim_{s\to-\infty} S(t,0;w)S(0,s;w)u_0 = \lim_{s\to-\infty} S(t,s;w)u_0 \\
&= \lim_{s\to-\infty} S(t-s,0;\theta_s w)u_0 = \lim_{s\to-\infty} S(0,s-t;\theta_t w)u_0 = \xi(\theta_t w)
\end{aligned} \tag{5.16}$$

then it follows that $\varphi(t, w)\mathcal{A}(w) = \mathcal{A}(\theta_t w)$. On the other hand, by Lemma 5.1, the convergence is uniform with respect to u_0 belonging to a bounded subset. Then for every bounded subset $B \subset H$, by relations (3.32) and (3.28), it follows that

$$\begin{aligned}
\operatorname{dist}\left(\varphi(t, \theta_{-t} w)B, \mathcal{A}(w)\right) &= \sup_{u_0 \in B} \left\| \varphi(t, \theta_{-t} w)u_0 - \xi_0(w) \right\|_2 \\
&= \sup_{u_0 \in B} \| S(0, -t, w)u_0 - \xi_0(w) \|_2 \\
&= \sup_{u_0 \in B} \| u(0, w; -t, u_0) - \xi_0(w) \|_2 \longrightarrow 0
\end{aligned} \tag{5.17}$$

as $t \to +\infty$. That is to say $\mathcal{A}(w)$ is a attracting set which attracts every deterministic bounded set of H, and therefore we complete the proof. □

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