

## Research Article

# A de Casteljau Algorithm for $q$ -Bernstein-Stancu Polynomials

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This paper is concerned with a generalization of the  $q$ -Bernstein polynomials and Stancu operators, where the function is evaluated at intervals which are in geometric progression. It is shown that these polynomials can be generated by a de Casteljau algorithm, which is a generalization of that relating to the classical case and  $q$ -Bernstein case.

## 1. Introduction

Let  $q > 0$ . For any fixed real number  $q > 0$  and for  $n \in Z = \{0, \pm 1, \pm 2, \dots\}$ , the  $q$ -integers of the number  $[n]$  are defined by

$$[n] = \frac{(1 - q^n)}{(1 - q)}, \quad \text{for } q \neq 1, \quad [n] = n, \quad \text{for } q = 1. \quad (1.1)$$

The  $q$ -factorial  $[n]!$ , for  $n \in N_0 = \{0, 1, 2, \dots\}$ , is defined by

$$[n]! = [1][2] \cdots [n] \quad (n = 1, 2, \dots), \quad [0]! = 1. \quad (1.2)$$

For the integers  $n, k$ , ( $n \geq k \geq 0$ ), the  $q$ -binomial or the Gaussian coefficients are defined by (see [1, page 12])

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}. \quad (1.3)$$

For  $f \in C[0;1]$ ,  $q > 0$ ,  $\alpha \geq 0$  and each positive integer  $n$ , we introduce (see [2]) the following generalized  $q$ -Bernstein operators:

$$B_n^{q,\alpha}(f; x) = \sum_{k=0}^n p_{n,k}^{q,\alpha}(x) f\left(\frac{[k]}{[n]}\right), \quad (1.4)$$

where

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{\prod_{i=0}^{k-1} (x + \alpha[i]) \prod_{s=0}^{n-1-k} (1 - q^s x + \alpha[s])}{\prod_{i=0}^{n-1} (1 + \alpha[i])}. \quad (1.5)$$

Note, that an empty product in (1.5) denotes 1. In the case where  $\alpha = 0$ ,  $B_n^{q,\alpha}(f; x)$  reduces to the well-known  $q$ -Bernstein polynomials introduced by Phillips [3, 4] in 1997

$$B_{n,q}(f; x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{i=0}^{n-k-1} (1 - q^i x) f\left(\frac{[k]}{[n]}\right). \quad (1.6)$$

In the case where  $q = 1$ ,  $B_n^{q,\alpha}(f; x)$  reduces to Bernstein-Stancu polynomials, introduced by Stancu [5] in 1968

$$S_n(f; x) = \sum_{k=0}^n \binom{n}{k} \frac{\prod_{i=0}^{k-1} (x + \alpha i) \prod_{s=0}^{n-k-1} (1 - x + s\alpha)}{\prod_{i=0}^{n-1} (1 + i\alpha)} f\left(\frac{k}{n}\right). \quad (1.7)$$

When  $q = 1$  and  $\alpha = 0$ , we obtain the classical Bernstein polynomial defined by

$$B_n(f; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \quad (1.8)$$

Basic facts on Bernstein polynomials, their generalizations, and applications can be found for example in [6–8]. In recent years, the  $q$ -Bernstein polynomials have attracted much interest, and a great number of interesting results related to the  $B_{n,q}(f)$  polynomials have been obtained (see [3, 4, 9–12]). Some approximation properties of the Stancu operators are presented in [5, 13–15].

Let  $\Delta_q^0 f_j = f_j$ , for  $j = 0, 1, \dots, n$ , and recursively,

$$\Delta_q^{k+1} f_j = \Delta_q^k f_{j+1} - q^k \Delta_q^k f_j, \quad (1.9)$$

for  $k = 0, 1, \dots, n - j - 1$  and  $f_j = f([j]/[n])$ . It is easily established by induction that  $q$ -differences satisfy the relation

$$\Delta_q^k f_j = \sum_{i=0}^k (-1)^k q^{i(i-1)/2} \begin{bmatrix} k \\ i \end{bmatrix} f_{j+k-i}. \tag{1.10}$$

In [2], we prove that the operators  $B_n^{q,\alpha}(f; x)$  defined by (1.4) can be expressed in terms of  $q$ -differences

$$B_n^{q,\alpha} f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \Delta_q^k f_0 \prod_{i=0}^{k-1} \frac{x + \alpha[i]}{1 + \alpha[i]}, \tag{1.11}$$

which generalized the well-known result [3, 4] for the  $q$ -Bernstein polynomial. In this paper, we show that polynomials defined by (1.4) can be generated by a de Castljam algorithm, which is a generalization of that relating to the classical case [16] and  $q$ -Bernstein case [4, 11].

## 2. Auxiliary Results

We note that  $B_n^{q,\alpha}(f; x)$  defined by (1.4), is a monotone linear operator for any  $0 < q \leq 1$  and  $\alpha \geq 0$ . These operators reproduces linear functions [2], that is,

$$B_n^{q,\alpha}(ax + b; x) = ax + b, \quad a, b \in R. \tag{2.1}$$

They also satisfy the end point interpolation conditions  $B_n^{q,\alpha}(f; 0) = f(0)$  and  $B_n^{q,\alpha}(f; 1) = f(1)$ . These properties are significant in designing curves and surfaces.

Moreover, the following holds.

**Lemma 2.1.** *Let  $0 < q \leq 1$ ,  $\alpha \geq 0$ . Then,*

$$\prod_{u=0}^{m-1} (q^r - q^u x + \alpha([u] - [r])) = \sum_{s=0}^m (-1)^s q^{s(s-1)/2 + (m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]), \tag{2.2}$$

for all  $m \in N$ ,  $r \in N_0 = N \cup \{0\}$  and  $x \in [0; 1]$ .

*Proof.* We use induction on  $m$ . First, we see from equality  $[-r] = -q^{-r}[r]$ , ( $r \in N$ ), that (2.2) is evident for  $m = 1$ . Let us assume that (2.2) holds for a given  $m \in N$ . Then, using (2.2), we obtain

$$\begin{aligned}
& \prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r])) \\
&= (q^r - q^m x + \alpha([m] - [r])) \sum_{s=0}^m (-1)^s q^{s(s-1)/2+(m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \\
&\quad \cdot \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]) \\
&= \sum_{s=0}^m (-1)^s q^{s(s-1)/2+(m-s)r} (q^r + \alpha[m] - \alpha[r] + \alpha q^m [s]) \begin{bmatrix} m \\ s \end{bmatrix} \\
&\quad \cdot \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]) \\
&\quad + \sum_{s=1}^{m+1} (-1)^s q^{s(s-1)(s-2)/2+(m-s+1)r+m} (1 + \alpha[s-r-1]) \begin{bmatrix} m \\ s-1 \end{bmatrix} \\
&\quad \cdot \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]) \\
&= q^{mr} (q^r + \alpha[m] - \alpha[r]) \prod_{j=-r}^{m-r-1} (1 + \alpha[j]) \\
&\quad + (-1)^{m+1} q^{m(m-1)/2+m} (1 + \alpha[m-r]) \prod_{i=0}^m (x + \alpha[i]) \\
&\quad + \sum_{s=1}^m (-1)^s q^{s(s-1)/2+(m+1-s)r} U_s \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j]),
\end{aligned} \tag{2.3}$$

where

$$U_s = \begin{bmatrix} m \\ s \end{bmatrix} (q^r + \alpha[m] - \alpha[r] + \alpha q^m [s]) q^{-r} + q^{m-s+1} \begin{bmatrix} m \\ s-1 \end{bmatrix} (1 + \alpha[s-r-1]). \tag{2.4}$$

Using the obvious equalities

$$(q^r + \alpha[m] - \alpha[r])q^{-r} = 1 + \alpha[m - r],$$

$$\begin{bmatrix} m \\ s \end{bmatrix} [s] = \begin{bmatrix} m \\ s-1 \end{bmatrix} [m - s + 1],$$
(2.5)

we have

$$U_s = \begin{bmatrix} m \\ s \end{bmatrix} (1 + \alpha[m - r])$$

$$+ \begin{bmatrix} m \\ s-1 \end{bmatrix} q^{m-s+1} \left( 1 + \alpha \left( [m - s + 1]q^{s-r-1} + [s - r - 1] \right) \right).$$
(2.6)

It is easy to see that

$$[m - s + 1]q^{s-r-1} + [s - r - 1] = [m - r],$$

$$\begin{bmatrix} m \\ s \end{bmatrix} + \begin{bmatrix} m \\ s-1 \end{bmatrix} q^{m-s+1} = \begin{bmatrix} m+1 \\ s \end{bmatrix}.$$
(2.7)

Therefore,

$$U_s = (1 + \alpha[m - r]) \begin{bmatrix} m+1 \\ s \end{bmatrix}.$$
(2.8)

From last equality and (2.3), we obtain

$$\prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r]))$$

$$= q^{mr} (q^r + \alpha[m] - \alpha[r]) \prod_{j=-r}^{m-r-1} (1 + \alpha[j])$$

$$+ (-1)^{m+1} q^{m(m-1)/2+m} (1 + \alpha[m - r]) \prod_{i=0}^m (x + \alpha[i])$$
(2.9)

$$+ \sum_{s=1}^m (-1)^s q^{s(s-1)/2+(m+1-s)r} \begin{bmatrix} m+1 \\ s \end{bmatrix} (1 + \alpha[m - r]) \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r-1} (1 + \alpha[j])$$

$$= \sum_{s=0}^{m+1} (-1)^s q^{s(s-1)/2+(m+1-s)r} \begin{bmatrix} m+1 \\ s \end{bmatrix} \prod_{i=0}^{s-1} (x + \alpha[i]) \prod_{j=s-r}^{m-r} (1 + \alpha[j]).$$

This completes the proof of the lemma. □

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input:  $q; f([0]/[n]), f([1]/[n]), \dots, f([n]/[n])$ 
for  $r = 0$  to  $n$ 
     $f_r^{[0]} := f\left(\frac{[r]}{[n]}\right)$ 
next  $r$ 
for  $m = 1$  to  $n$ 
    for  $r = 0$  to  $n - m$ 
         $f_r^{[m]} := \frac{\{(q^r - q^{m-1}x + \alpha([m-1] - [r]))f_r^{[m-1]} + (x + \alpha[r])f_{r+1}^{[m-1]}\}}{1 + \alpha[m-1]}$ 
    next  $r$ 
next  $m$ 

```

**Algorithm 1:** De Casteljaou type algorithm.

### 3. Main Result

The generalized  $q$ -Bernstein polynomials, defined by (1.4), may be evaluated by Algorithm 1.

In the case, where  $\alpha = 0$ , this is the de Casteljaou algorithm for evaluating the  $q$ -Bernstein polynomial [3, 4]. Note that with  $q = 1$  and  $\alpha = 0$ , we recover the original classical de Casteljaou algorithm (see Hoschek and Lasser [16]). The algorithm is justified by the following theorem.

**Theorem 3.1.** *Each intermediate point  $f_r^{[m]}$  of the algorithm can be expressed as*

$$f_r^{[m]} = \left( \prod_{i=0}^{m-1} (1 + \alpha[i]) \right)^{-1} \cdot \sum_{t=0}^m f_{r+t} \binom{m}{t} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])), \quad (3.1)$$

and, in particular

$$f_0^{[n]} = B_n^{q,\alpha}(f; x). \quad (3.2)$$

*Proof.* We use induction on  $m$ . From the initial conditions in the algorithm,  $f_r^{[0]} = f([r]/[n]) = f_r$ ,  $0 \leq r \leq n$ , it is clear that (3.1) holds for  $m = 0$  and  $0 \leq r \leq n$ . Let us assume that (3.1) holds for some  $m$  such that  $0 \leq m < n$ , and for all  $r$  such that  $0 \leq r \leq n - m$ . Then, for  $0 \leq r \leq n - m - 1$ , it follows from the algorithm that

$$f_r^{[m+1]} := \left\{ (q^r - q^m x + \alpha([m] - [r]))f_r^{[m]} + (x + \alpha[r])f_{r+1}^{[m]} \right\} \frac{1}{1 + \alpha[m]}, \quad (3.3)$$

and using (3.1), we obtain

$$\begin{aligned}
f_r^{[m+1]} \left( \prod_{i=0}^m (1 + \alpha[i]) \right) &:= (q^r - q^m x + \alpha([m] - [r])) \\
&\cdot \sum_{t=0}^m f_{r+t} \begin{bmatrix} m \\ t \end{bmatrix} \prod_{s=0}^{t-1} (x + \alpha[r + s]) \cdot \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) \\
&+ (x + \alpha[r]) \cdot \sum_{t=0}^m f_{r+t+1} \begin{bmatrix} m \\ t \end{bmatrix} \prod_{s=0}^{t-1} (x + \alpha[r + s + 1]) \\
&\cdot \prod_{u=0}^{m-t-1} (q^{r+1} - q^u x + \alpha([u] - [r + 1])) \\
&= (q^r - q^m x + \alpha([m] - [r])) f_r \prod_{u=0}^{m-1} (q^r - q^u x + \alpha([u] - [r])) \\
&+ (q^r - q^m x + \alpha([m] - [r])) \cdot \sum_{t=1}^m f_{r+t} \begin{bmatrix} m \\ t \end{bmatrix} \\
&\cdot \prod_{s=0}^{t-1} (x + \alpha[r + s]) \cdot \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) \\
&+ (x + \alpha[r]) \cdot \sum_{t=1}^m f_{r+t} \begin{bmatrix} m \\ t-1 \end{bmatrix} \prod_{s=0}^{t-2} (x + \alpha[r + s + 1]) \\
&\cdot \prod_{u=0}^{m-t} (q^{r+1} - q^u x + \alpha([u] - [r + 1])) \\
&+ (x + \alpha[r]) f_{r+m+1} \prod_{s=0}^{m-1} (x + \alpha[r + s + 1]) \\
&= f_r \prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r])) \\
&+ \sum_{t=1}^m \left\{ (q^r - q^m x + \alpha([m] - [r])) \begin{bmatrix} m \\ t \end{bmatrix} \right. \\
&\quad \cdot \prod_{s=0}^{t-1} (x + \alpha[r + s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) \\
&\quad + (x + \alpha[r]) \begin{bmatrix} m \\ t-1 \end{bmatrix} \prod_{s=0}^{t-2} (x + \alpha[r + s + 1]) \\
&\quad \left. \prod_{u=0}^{m-t} (q^{r+1} - q^u x + \alpha([u] - [r + 1])) \right\} f_{r+t} \\
&+ f_{r+m+1} \prod_{s=0}^m (x + \alpha[r + s]).
\end{aligned} \tag{3.4}$$

We see that

$$\begin{aligned}
 & \prod_{u=0}^{m-t} (q^{r+1} - q^u x + \alpha([u] - [r+1])) \\
 &= (q^{r+1} - x - \alpha[r+1]) \prod_{u=0}^{m-t-1} (q^{r+1} - q^{u+1} x + \alpha([u+1] - [r+1])) \\
 &= (q^{r+1} - x - \alpha[r+1]) \prod_{u=0}^{m-t-1} (q^{r+1} - q^{u+1} x + \alpha q([u] - [r])) \\
 &= (q^{r+1} - x - \alpha[r+1]) q^{m-t} \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])),
 \end{aligned} \tag{3.5}$$

and hence,

$$\begin{aligned}
 & f_r^{[m+1]} \left( \prod_{i=0}^m (1 + \alpha[i]) \right) \\
 &:= f_r \prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r])) \\
 &+ \sum_{t=1}^m \left\{ \begin{bmatrix} m \\ t \end{bmatrix} (q^r - q^m x + \alpha([m] - [r])) + \begin{bmatrix} m \\ t-1 \end{bmatrix} (q^{r+1} - x - \alpha[r+1]) q^{m-t} \right\} \\
 &\cdot f_{r+t} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) + f_{r+m+1} \prod_{s=0}^m (x + \alpha[r+s]).
 \end{aligned} \tag{3.6}$$

It is easy to verify that

$$\begin{aligned}
 \begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1} \begin{bmatrix} m \\ t-1 \end{bmatrix} &= \begin{bmatrix} m+1 \\ t \end{bmatrix}, \\
 \begin{bmatrix} m \\ t-1 \end{bmatrix} + q^t \begin{bmatrix} m \\ t \end{bmatrix} &= \begin{bmatrix} m+1 \\ t \end{bmatrix}.
 \end{aligned} \tag{3.7}$$



Therefore,

$$\begin{aligned}
 & \begin{bmatrix} m \\ t \end{bmatrix} (q^r - q^m x + \alpha([m] - [r])) + \begin{bmatrix} m \\ t-1 \end{bmatrix} (q^{r+1} - x - \alpha[r+1]) q^{m-t} \\
 &= q^r \left( \begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1} \begin{bmatrix} m \\ t-1 \end{bmatrix} \right) - x q^{m-t} \left( \begin{bmatrix} m \\ t-1 \end{bmatrix} + q^t \begin{bmatrix} m \\ t \end{bmatrix} \right) \\
 &+ \frac{\alpha}{1-q} \left\{ q^r \left( \begin{bmatrix} m \\ t \end{bmatrix} + q^{m-t+1} \begin{bmatrix} m \\ t-1 \end{bmatrix} \right) - q^{m-t} \left( \begin{bmatrix} m \\ t-1 \end{bmatrix} + q^t \begin{bmatrix} m \\ t \end{bmatrix} \right) \right\} \\
 &= \begin{bmatrix} m+1 \\ t \end{bmatrix} \{ (q^r - x q^{m-t}) + \alpha([m-t] - [r]) \}.
 \end{aligned} \tag{3.8}$$

Consequently,

$$\begin{aligned}
 f_r^{[m+1]} \left( \prod_{i=0}^m (1 + \alpha[i]) \right) &:= f_r \prod_{u=0}^m (q^r - q^u x + \alpha([u] - [r])) \\
 &+ \sum_{t=1}^m \begin{bmatrix} m+1 \\ t \end{bmatrix} \{ (q^r - x q^{m-t}) + \alpha([m-t] - [r]) \} \\
 &\cdot f_{r+t} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t-1} (q^r - q^u x + \alpha([u] - [r])) \\
 &+ f_{r+m+1} \prod_{s=0}^m (x + \alpha[r+s]) \\
 &= \sum_{t=0}^{m+1} \begin{bmatrix} m+1 \\ t \end{bmatrix} \cdot f_{r+t} \prod_{s=0}^{t-1} (x + \alpha[r+s]) \prod_{u=0}^{m-t} (q^r - q^u x + \alpha([u] - [r])).
 \end{aligned} \tag{3.9}$$

Thus, one has the desired result. □

**Theorem 3.2.** For  $0 \leq m \leq n$  and  $0 \leq r \leq n - m$ , we have

$$f_r^{[m]} = \sum_{s=0}^m q^{(m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \Delta_q^s f_r \frac{\prod_{i=r}^{s+r-1} (x + \alpha[i])}{\prod_{j=0}^{s-1} (1 + \alpha[j])}, \tag{3.10}$$

for all  $x \in [0; 1]$ .

*Proof.* Using (2.2) and (3.1), we have

$$f_r^{[m]} \prod_{i=0}^{m-1} (1 + \alpha[i]) = \sum_{t=0}^m \begin{bmatrix} m \\ t \end{bmatrix} f_{r+t} S_t(m), \tag{3.11}$$

where

$$\begin{aligned}
 S_t(m) &= \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \begin{bmatrix} m-t \\ u \end{bmatrix} \\
 &\times \prod_{s=r}^{t+r-1} (x + \alpha[s]) \prod_{i=0}^{u-1} (x + \alpha[i]) \prod_{j=u-r}^{m-t-r-1} (1 + \alpha[j]) \quad (0 \leq t \leq m).
 \end{aligned} \tag{3.12}$$

First, we prove that

$$S_t(m) = \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \begin{bmatrix} m-t \\ u \end{bmatrix} \cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]) \tag{3.13}$$

for all  $m \in N_0 = \{0, 1, 2, \dots\}$ ,  $t \in N_0$ , and  $x \in [0; 1]$ . Note that an empty sum denotes 0.

We use the induction on  $m$ . First, we see that (3.13) holds for  $m = 0$  and all  $t \in N_0$ . Let us assume that (3.13) holds for a given  $m$ , and for all  $t \in N_0$ . Then, from (3.12) and (3.13), we obtain

$$\begin{aligned}
 S_t(m+1) &= (x + \alpha[t+r-1]) \sum_{u=0}^{m+1-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
 &\cdot \prod_{i=r}^{t+u-2+r} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]) \\
 &= \sum_{u=0}^{m+1-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
 &\cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]) \\
 &+ \alpha \sum_{u=1}^{m-t+1} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
 &\cdot ([t+r+1] - [t+u+r+1]) \prod_{i=r}^{t+u-2+r} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]).
 \end{aligned} \tag{3.14}$$

We see that

$$\begin{bmatrix} m-t+1 \\ u \end{bmatrix} ([t+r-1] - [t+u+r-1]) = -q^{t+r-1} [m-t-u+2] \begin{bmatrix} m-t+1 \\ u-1 \end{bmatrix}, \tag{3.15}$$

and hence,

$$\begin{aligned}
 S_t(m+1) &= \sum_{u=0}^{m-t+1} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
 &\quad \cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t-1}^{m-1} (1 + \alpha[j]) \\
 &\quad + \alpha \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} q^{u+t-1} \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
 &\quad \cdot [m-t-u+1] \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]) \\
 &= (-1)^{m-t+1} q^{(m-t+1)(m-t)/2} \prod_{i=r}^{m+r} (x + \alpha[i]) \\
 &\quad + \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t+1-u)r} \\
 &\quad \cdot \left( 1 + \alpha[u+t-1] + \alpha q^{u+t-1} [m-t-u+1] \right) \begin{bmatrix} m-t+1 \\ u \end{bmatrix} \\
 &\quad \cdot \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]).
 \end{aligned} \tag{3.16}$$

Next, in view of the equality

$$\left( 1 + \alpha[u+t-1] + \alpha q^{u+t-1} [m-t-u+1] \right) = 1 + \alpha[m], \tag{3.17}$$

we obtain (3.13). Consequently, in view of (3.11) and (3.13), we get

$$\begin{aligned}
 f_r^{[m]} \prod_{i=0}^{m-1} (1 + \alpha[i]) &= \sum_{t=0}^m \begin{bmatrix} m \\ t \end{bmatrix} f_{r+t} \sum_{u=0}^{m-t} (-1)^u q^{u(u-1)/2+(m-t-u)r} \\
 &\quad \cdot \begin{bmatrix} m-t \\ u \end{bmatrix} \prod_{i=r}^{t+u+r-1} (x + \alpha[i]) \prod_{j=u+t}^{m-1} (1 + \alpha[j]) \\
 &= \sum_{t=0}^m \sum_{u=t}^m \begin{bmatrix} m \\ t \end{bmatrix} f_{r+t} (-1)^{u-t} q^{(u-t)(u-t-1)/2+(m-u)r} \\
 &\quad \cdot \begin{bmatrix} m-t \\ u-t \end{bmatrix} \prod_{i=r}^{u+r-1} (x + \alpha[i]) \prod_{j=u}^{m-1} (1 + \alpha[j]).
 \end{aligned} \tag{3.18}$$

Next, in view of the equality

$$\begin{bmatrix} m \\ t \end{bmatrix} \begin{bmatrix} m-t \\ u-t \end{bmatrix} = \begin{bmatrix} m \\ u \end{bmatrix} \begin{bmatrix} u \\ t \end{bmatrix}, \quad (3.19)$$

we obtain

$$\begin{aligned} f_r^{[m]} \prod_{i=0}^{m-1} (1 + \alpha[i]) &= \sum_{u=0}^m \sum_{t=0}^u \begin{bmatrix} m \\ u \end{bmatrix} f_{r+t} (-1)^{u-t} q^{(u-t)(u-t-1)/2+(m-u)r} \\ &\quad \cdot \begin{bmatrix} u \\ t \end{bmatrix} \prod_{i=r}^{u+r-1} (x + \alpha[i]) \prod_{j=u}^{m-1} (1 + \alpha[j]) \\ &= \sum_{u=0}^m \begin{bmatrix} m \\ u \end{bmatrix} q^{(m-u)r} \prod_{i=r}^{u+r-1} (x + \alpha[i]) \prod_{j=u}^{m-1} (1 + \alpha[j]) \\ &\quad \cdot \sum_{t=0}^u \begin{bmatrix} u \\ t \end{bmatrix} (-1)^{u-t} q^{(u-t)(u-t-1)/2} f_{r+t}. \end{aligned} \quad (3.20)$$

The condition (1.10) completes the proof.  $\square$

Theorems 3.1 and 3.2 are generalizations of Theorems 2.1 and 2.3 in [11].  
Note that when  $m = n$  and  $r = 0$ , (3.10) does indeed reduce to (1.11)

## References

- [1] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, NY, USA, 2002.
- [2] G. Nowak, "Approximation properties for generalized  $q$ -Bernstein polynomials," *Journal of Mathematical Analysis and Applications*, vol. 350, no. 1, pp. 50–55, 2009.
- [3] G. M. Phillips, "Bernstein polynomials based on the  $q$ -integers," *Annals of Numerical Mathematics*, vol. 4, no. 1–4, pp. 511–518, 1997.
- [4] G. M. Phillips, *Interpolation and Approximation by Polynomials*, Springer, New York, NY, USA, 2003.
- [5] D. D. Stancu, "Approximation of functions by a new class of linear polynomial operators," *Académie de la République Populaire Roumaine. Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 13, pp. 1173–1194, 1968.
- [6] G. G. Lorentz, *Bernstein Polynomials*, vol. 8, University of Toronto Press, Toronto, Canada, 1953.
- [7] P. Pych-Taberska, "Some approximation properties of Bernštejn and Kantorovič polynomials," *Functiones et Approximatio Commentarii Mathematici*, vol. 6, pp. 57–67, 1978.
- [8] P. Pych-Taberska, "On the rate of pointwise convergence of Bernstein and Kantorovič polynomials," *Functiones et Approximatio Commentarii Mathematici*, vol. 16, pp. 63–76, 1988.
- [9] T. N. T. Goodman, H. Oruç, and G. M. Phillips, "Convexity and generalized Bernstein polynomials," *Proceedings of the Edinburgh Mathematical Society. Series 2*, vol. 42, no. 1, pp. 179–190, 1999.
- [10] A. Il'inskii and S. Ostrovska, "Convergence of generalized Bernstein polynomials," *Journal of Approximation Theory*, vol. 116, no. 1, pp. 100–112, 2002.
- [11] G. M. Phillips, "A de Casteljau algorithm for generalized Bernstein polynomials," *BIT. Numerical Mathematics*, vol. 37, no. 1, pp. 232–236, 1997.
- [12] G. M. Phillips, "A survey of results on the  $q$ -Bernstein polynomials," *IMA Journal of Numerical Analysis*, vol. 30, no. 1, pp. 277–288, 2010.
- [13] Z. Finta, "Direct and converse results for Stancu operator," *Periodica Mathematica Hungarica*, vol. 44, no. 1, pp. 1–6, 2002.

- [14] Z. Finta, "On approximation properties of Stancu's operators," *Mathematica*, vol. 47, no. 4, pp. 47–55, 2002.
- [15] H. H. Gonska and J. Meier, "Quantitative theorems on approximation by Bernstein-Stancu operators," *Calcolo*, vol. 21, no. 4, pp. 317–335, 1984.
- [16] J. Hoschek and D. Lasser, *Fundamentals of Computer Aided Geometric Design*, A. K. Peters, Wellesley, Mass, USA, 1993.