Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 586328, 28 pages
doi:10.1155/2011/586328

Research Article

A Final Result on the Oscillation of Solutions of the Linear Discrete Delayed Equation $\Delta x(n) = -p(n)x(n-k)$ with a Positive Coefficient

J. Baštinec,[1] L. Berezansky,[2] J. Diblík,[1,3] and Z. Šmarda[1]

[1] *Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, 61600 Brno, Czech Republic*
[2] *Department of Mathematics, Ben-Gurion University of the Negev, 84105 Beer-Sheva, Israel*
[3] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, 602 00 Brno, Czech Republic*

Correspondence should be addressed to J. Diblík, diblik@feec.vutbr.cz

Received 15 January 2011; Accepted 6 June 2011

Academic Editor: Yuri V. Rogovchenko

A linear $(k+1)$th-order discrete delayed equation $\Delta x(n) = -p(n)x(n-k)$ where $p(n)$ a positive sequence is considered for $n \to \infty$. This equation is known to have a positive solution if the sequence $p(n)$ satisfies an inequality. Our aim is to show that, in the case of the opposite inequality for $p(n)$, all solutions of the equation considered are oscillating for $n \to \infty$.

1. Introduction

The existence of positive solutions of difference equations is often encountered when analysing mathematical models describing various processes. This is a motivation for an intensive study of the conditions for the existence of positive solutions of discrete or continuous equations. Such analysis is related to investigating the case of all solutions being oscillating (for investigation in both directions, we refer, e.g., to [1–30] and to the references therein). The existence of monotonous and nontrivial solutions of nonlinear difference equations (the first one implies the existence of solutions of the same sign) also has attracted some attention recently (see, e.g., several, mostly asymptotic methods in [31–42] and the related references therein). In this paper, sharp conditions are derived for all the solutions being oscillating for a class of linear $(k+1)$-order delayed discrete equations.

We consider the delayed $(k+1)$-order linear discrete equation

$$\Delta x(n) = -p(n)x(n-k), \tag{1.1}$$

where $n \in \mathbb{Z}_a^\infty := \{a, a+1, \dots\}$, $a \in \mathbb{N} := \{1, 2, \dots\}$ is fixed, $\Delta x(n) = x(n+1) - x(n)$, $p : \mathbb{Z}_a^\infty \to \mathbb{R}$, $k \in \mathbb{N}$. In what follows, we will also use the sets $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{Z}_a^b := \{a, a+1, \dots, b\}$ for $a, b \in \mathbb{N}$, $a < b$. A solution $x = x(n) : \mathbb{Z}_a^\infty \to \mathbb{R}$ of (1.1) is positive (negative) on $\mathbb{Z}_a^\infty$ if $x(n) > 0$ ($x(n) < 0$) for every $n \in \mathbb{Z}_a^\infty$. A solution $x = x(n) : \mathbb{Z}_a^\infty \to \mathbb{R}$ of (1.1) is oscillating on $\mathbb{Z}_a^\infty$ if it is not positive or negative on $\mathbb{Z}_{a_1}^\infty$ for an arbitrary $a_1 \in \mathbb{Z}_a^\infty$.

Definition 1.1. Let us define the expression $\ln_q t$, $q \geq 1$, by $\ln_q t = \ln(\ln_{q-1} t)$, $\ln_0 t \equiv t$, where $t > \exp_{q-2} 1$ and $\exp_s t = \exp(\exp_{s-1} t)$, $s \geq 1$, $\exp_0 t \equiv t$, and $\exp_{-1} t \equiv 0$ (instead of $\ln_0 t$, $\ln_1 t$, we will only write t and $\ln t$).

In [4] difference equation (1.1) is considered and the following result on the existence of a positive solution is proved.

Theorem 1.2 (see [4]). *Let $q \in \mathbb{N}_0$ be a fixed integer, let $a \in \mathbb{N}$ be sufficiently large and*

$$0 < p(n) \leq \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n \ln n)^2} + \cdots + \frac{k}{8(n \ln n \cdots \ln_q n)^2}\right] \tag{1.2}$$

for every $n \in \mathbb{Z}_a^\infty$. Then there exists a positive integer $a_1 \geq a$ and a solution $x = x(n)$, $n \in \mathbb{Z}_{a_1}^\infty$ of equation (1.1) *such that*

$$0 < x(n) \leq \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln n \ln_2 n \cdots \ln_q n} \tag{1.3}$$

holds for every $n \in \mathbb{Z}_{a_1}^\infty$.

Our goal is to answer the open question whether all solutions of (1.1) are oscillating if inequality (1.2) is replaced with the opposite inequality

$$p(n) \geq \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n \ln n)^2} + \cdots + \frac{k\theta}{8(n \ln n \cdots \ln_q n)^2}\right] \tag{1.4}$$

assuming $\theta > 1$, and n is sufficiently large. Below we prove that if (1.4) holds and $\theta > 1$, then all solutions of (1.1) are oscillatory. This means that the result given by Theorem 1.2 is a final in a sense. This is discussed in Section 4. Moreover, in Section 3, we show that all solutions of (1.1) will be oscillating if (1.4) holds only on an infinite sequence of subintervals of $\mathbb{Z}_a^\infty$.

Because of its simple form, equation (1.1) (as well as its continuous analogue) attracts permanent attention of investigators. Therefore, in Section 4 we also discuss some of the known results.

The proof of our main result will use the next consequence of one of Domshlak's results [13, Theorem 4, page 66].

Lemma 1.3. *Let s and r be fixed natural numbers such that $r - s > k$. Let $\{\varphi(n)\}_1^\infty$ be a bounded sequence of real numbers and ν_0 be a positive number such that there exists a number $\nu \in (0, \nu_0)$*

satisfying

$$0 \le \sum_{n=s+1}^{i} \varphi(n) \le \frac{\pi}{\nu}, \quad i \in \mathbb{Z}_{s+1}^{r}, \quad \frac{\pi}{\nu} \le \sum_{n=s+1}^{i} \varphi(n) \le \frac{2\pi}{\nu}, \quad i \in \mathbb{Z}_{r+1}^{r+k}, \tag{1.5}$$

$$\varphi(i) \ge 0, \quad i \in \mathbb{Z}_{r+1-k}^{r}, \quad \sum_{n=i+1}^{i+k} \varphi(n) > 0, \quad i \in \mathbb{Z}_{a}^{\infty}, \qquad \sum_{n=i}^{i+k} \varphi(n) > 0, \quad i \in \mathbb{Z}_{a}^{\infty}. \tag{1.6}$$

Then, if $p(n) \ge 0$ *for* $n \in \mathbb{Z}_{s+1}^{s+k}$ *and*

$$p(n) \ge \mathcal{R} := \left(\prod_{\ell=n-k}^{n} \frac{\sin\left(\nu \sum_{i=\ell+1}^{\ell+k} \varphi(i)\right)}{\sin\left(\nu \sum_{i=\ell}^{\ell+k} \varphi(i)\right)} \right) \cdot \frac{\sin(\nu\varphi(n-k))}{\sin\left(\nu \sum_{i=n+1-k}^{n} \varphi(i)\right)} \tag{1.7}$$

for $n \in \mathbb{Z}_{s+k+1}^{r}$*, any solution of* (1.1) *has at least one change of sign on* $\mathbb{Z}_{s-k}^{r}$.

Throughout the paper, symbols "o" and "O" (for $n \to \infty$) will denote the well-known Landau order symbols.

2. Main Result

In this section, we give sufficient conditions for all solutions of (1.1) to be oscillatory as $n \to \infty$. It will be necessary to develop asymptotic decompositions of some auxiliary expressions. As the computations needed are rather cumbersome, some auxiliary computations are collected in the appendix to be utilized in the proof of the main result (Theorem 2.1) below.

Theorem 2.1. *Let* $a \in \mathbb{N}$ *be sufficiently large,* $q \in \mathbb{N}_0$ *and* $\theta > 1$*. Assuming that the function* $p : \mathbb{Z}_{a}^{\infty} \to (0, \infty)$ *satisfies inequality* (1.4) *for every* $n \in \mathbb{Z}_{a}^{\infty}$*, all solutions of* (1.1) *are oscillating as* $n \to \infty$.

Proof. As emphasized above, in the proof, we will use Lemma 1.3. We define

$$\varphi(n) := \frac{1}{n \ln n \ln_2 n \ln_3 n \cdots \ln_q n}, \tag{2.1}$$

where n is sufficiently large, and $q \ge 0$ is a fixed integer. Although the idea of the proof is simple, it is very technical and we will refer to notations and auxiliary computations contained in the appendix. We will develop an asymptotic decomposition of the right-hand side $\mathcal{R}$ of inequality (1.7) with the function $\varphi(n)$ defined by (2.1). We show that this will lead to the desired inequality (1.4). We set

$$\mathcal{R}_1 := \frac{\prod_{i=1}^{k} V(n+i)}{\prod_{i=0}^{k} V^{+}(n+i)} \cdot \varphi(n-k), \tag{2.2}$$

where V and V^+ are defined by (A.4) and (A.5). Moreover, $\mathcal{R}$ can be expressed as

$$\begin{aligned}\mathcal{R} &= \frac{\sin\left(\nu\sum_{i=n+1-k}^{n}\varphi(i)\right)\prod_{\ell=n-k+1}^{n}\sin\left(\nu\sum_{i=\ell+1}^{\ell+k}\varphi(i)\right)}{\prod_{\ell=n-k}^{n}\sin\left(\nu\sum_{i=\ell}^{\ell+k}\varphi(i)\right)}\cdot\frac{\sin(\nu\varphi(n-k))}{\sin\left(\nu\sum_{i=n+1-k}^{n}\varphi(i)\right)}\\ &= \frac{\prod_{\ell=n-k+1}^{n}\sin\left(\nu\sum_{i=\ell+1}^{\ell+k}\varphi(i)\right)}{\prod_{\ell=n-k}^{n}\sin\left(\nu\sum_{i=\ell}^{\ell+k}\varphi(i)\right)}\cdot\sin(\nu\varphi(n-k))\\ &= \frac{\prod_{p=1}^{k}\sin(\nu V(n+p))}{\prod_{p=0}^{k}\sin(\nu V^+(n+p))}\cdot\sin(\nu\varphi(n-k)).\end{aligned}\tag{2.3}$$

Recalling the asymptotic decomposition of $\sin x$ when $x \to 0$: $\sin x = x + O(x^3)$, we get (since $\lim_{n\to\infty}\varphi(n-k) = 0$, $\lim_{n\to\infty}V(n+p) = 0$, $p = 1,\dots,k$, and $\lim_{n\to\infty}V^+(n+p) = 0$, $p = 0,\dots,k$)

$$\begin{gathered}\sin\nu\varphi(n-k) = \nu\varphi(n-k) + O\left(\nu^3\varphi^3(n-k)\right),\\ \sin\nu V(n+p) = \nu V(n+p) + O\left(\nu^3 V^3(n+p)\right),\quad p = 1,\dots,k,\\ \sin\nu V^+(n+p) = \nu V^+(n+p) + O\left(\nu^3 (V^+)^3(n+p)\right),\quad p = 0,\dots,k\end{gathered}\tag{2.4}$$

as $n \to \infty$. Then, it is easy to see that, by (A.13), we have $\varphi(n-\ell) = O(\varphi(n))$, $n \to \infty$ for every $\ell \in \mathbb{R}$ and

$$\mathcal{R} = \mathcal{R}_1\cdot\left(1 + O\left(\nu^2\varphi^2(n)\right)\right),\quad n \longrightarrow \infty.\tag{2.5}$$

Moreover, for $\mathcal{R}_1$, we will get an asymptotic decomposition as $n \to \infty$. Using formulas (A.13), (A.57), and (A.60), we get

$$\begin{aligned}\mathcal{R}_1 = {} & \frac{k^k}{(k+1)^{k+1}}\cdot\frac{1-k\alpha(n)-(k/24)\left(k^2-12k+11\right)\alpha^2(n)+(k/6)\left(k^2+5\right)\sum_{i=0}^{q}\omega_i(n)+O\left(1/n^3\right)}{1-(k/24)\left(k^2+3k+2\right)\alpha^2(n)+(k/6)\left(k^2+3k+2\right)\sum_{i=0}^{q}\omega_i(n)+O\left(1/n^3\right)}\\ & \times\left(1 + k\alpha(n) + k^2\sum_{i=0}^{q}\omega_i(n) + O\left(\frac{1}{n^3}\right)\right).\end{aligned}\tag{2.6}$$

Since $\lim_{n\to\infty}\alpha(n) = 0$, $\lim_{n\to\infty}\omega_i(n) = 0$, $i = 1,\dots,q$, we can decompose the denominator of the second fraction as the sum of the terms of a geometric sequence. Keeping only terms with the order of accuracy necessary for further analysis (i.e. with order $O(1/n^3)$), we get

$$\left(1-\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)+\frac{k}{6}\left(k^2+3k+2\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)^{-1}$$
$$=1+\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)-\frac{k}{6}\left(k^2+3k+2\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right). \tag{2.7}$$

We perform an auxiliary computation in $\mathcal{R}_1$,

$$\left(1-k\alpha(n)-\frac{k}{24}\left(k^2-12k+11\right)\alpha^2(n)+\frac{k}{6}\left(k^2+5\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)$$
$$\times\left(1+\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)-\frac{k}{6}\left(k^2+3k+2\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)$$
$$\times\left(1+k\alpha(n)+k^2\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)$$
$$=\left(1-k\alpha(n)-\frac{k}{24}\left(k^2-12k+11\right)\alpha^2(n)+\frac{k}{6}\left(k^2+5\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)$$
$$\times\left(1+k\alpha(n)+\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)-\frac{k}{6}\left(k^2-3k+2\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)$$
$$=1-\frac{3}{8}k(k+1)\alpha^2(n)+\frac{1}{2}k(k+1)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)=(\text{we use formula (A.15)})$$
$$=1+\frac{1}{8}k(k+1)\Omega(n)+O\left(\frac{1}{n^3}\right)$$
$$=1+\frac{1}{8}k(k+1)\left(\frac{1}{n^2}+\frac{1}{(n\ln n)^2}+\frac{1}{(n\ln n\ln_2 n)^2}+\cdots+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}\right)$$
$$+O\left(\frac{1}{n^3}\right). \tag{2.8}$$

Thus, we have

$$\mathcal{R}_1=\frac{k^k}{(k+1)^{k+1}}\times\left[1+\frac{1}{8}k(k+1)\left(\frac{1}{n^2}+\frac{1}{(n\ln n)^2}+\frac{1}{(n\ln n\ln_2 n)^2}+\cdots\right.\right.$$
$$\left.\left.+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}\right)\right]+O\left(\frac{1}{n^3}\right) \tag{2.9}$$
$$=\left(\frac{k}{k+1}\right)^k\times\left[\frac{1}{k+1}+\frac{k}{8n^2}+\frac{k}{8(n\ln n)^2}+\cdots+\frac{k}{8(n\ln n\cdots\ln_q n)^2}\right]+O\left(\frac{1}{n^3}\right).$$

Finalizing our decompositions, we see that

$$\begin{aligned}
\mathcal{R} &= \mathcal{R}_1 \cdot \left(1 + O\left(\nu^2\varphi^2(n)\right)\right) \\
&= \left(\left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n\cdots\ln_q n)^2}\right] + O\left(\frac{1}{n^3}\right)\right) \\
&\quad \times \left(1 + O\left(\nu^2\varphi^2(n)\right)\right) \\
&= \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n\cdots\ln_q n)^2}\right] \\
&\quad + O\left(\frac{\nu^2}{(n\ln n\cdots\ln_q n)^2}\right).
\end{aligned} \tag{2.10}$$

It is easy to see that inequality (1.7) becomes

$$\begin{aligned}
p(n) \geq \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n\cdots\ln_q n)^2}\right] \\
+ O\left(\frac{\nu^2}{(n\ln n\cdots\ln_q n)^2}\right)
\end{aligned} \tag{2.11}$$

and will be valid if (see (1.4))

$$\begin{aligned}
&\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k\theta}{8(n\ln n\cdots\ln_q n)^2} \\
&\quad \geq \frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n\cdots\ln_q n)^2} + O\left(\frac{\nu^2}{(n\ln n\cdots\ln_q n)^2}\right)
\end{aligned} \tag{2.12}$$

or

$$\theta \geq 1 + O\left(\nu^2\right) \tag{2.13}$$

for $n \to \infty$. If $n \geq n_0$, where n_0 is sufficiently large, (2.13) holds for $\nu \in (0, \nu_0)$ with ν_0 sufficiently small because $\theta > 1$. Consequently, (2.11) is satisfied and the assumption (1.7) of Lemma 1.3 holds for $n \in \mathbb{Z}_{n_0}^{\infty}$. Let $s \geq n_0$ in Lemma 1.3 be fixed, $r > s + k + 1$ be so large (and ν_0 so small if necessary) that inequalities (1.5) hold. Such choice is always possible since the series $\sum_{n=s+1}^{\infty} \varphi(n)$ is divergent. Then Lemma 1.3 holds and any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{s-k}^{r}$. Obviously, inequalities (1.5) can be satisfied for another pair of (s, r), say (s_1, r_1) with $s_1 > r$ and $r_1 > s_1 + k$ sufficiently large and, by Lemma 1.3, any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{s_1-k}^{r_1}$. Continuing this process, we will get a sequence of pairs (s_j, r_j) with $\lim_{j\to\infty} s_j = \infty$ such that any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{s_j-k}^{r_j}$. This concludes the proof. □

3. A Generalization

The coefficient p in Theorem 2.1 is supposed to be positive on $\mathbb{Z}_a^\infty$. For all sufficiently large n, the expression $\mathcal{R}$, as can easily be seen from (2.10), is positive. Then, as follows from Lemma 1.3, any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{s-k}^r$ if p is nonnegative on $\mathbb{Z}_{s+1}^{s+k}$ and satisfies inequality (1.4) on $\mathbb{Z}_{s+k+1}^r$.

Owing to this remark, Theorem 2.1 can be generalized because (and the following argumentation was used at the end of the proof of Theorem 2.1) all solutions of (1.1) will be oscillating as $n \to \infty$ if a sequence of numbers $\{s_i, r_i\}$, $r_i > s_i + k + 1$, $s_1 \geq a$, $i = 1, 2, \dots$ exists such that $s_{i+1} > r_i$ (i.e., the sets $\mathbb{Z}_{s_i}^{r_i}$, $\mathbb{Z}_{s_{i+1}}^{r_{i+1}}$ are disjoint and $\lim_{i\to\infty} s_i = \infty$), and, for every pair (s_i, r_i), all assumptions of Lemma 1.3 are satisfied (because of the specification of function φ by (2.1), inequalities (1.6) are obviously satisfied). This means that, on the set

$$\mathcal{M} := \mathbb{Z}_a^\infty \setminus \bigcup_{i=1}^{\infty} \mathbb{Z}_{s_i}^{r_i}, \tag{3.1}$$

function p can assume even negative values, and, moreover, there is no restriction on the behavior of $p(n)$ for $n \in \mathcal{M}$. This leads to the following generalization of Theorem 2.1 with a proof similar to that of Theorem 2.1 and, therefore, omitted.

Theorem 3.1. *Let $a \in \mathbb{N}$ be sufficiently large, $q \in \mathbb{N}_0$, ν_0 be a positive number, $\theta > 1$ and $p : \mathbb{Z}_a^\infty \to \mathbb{R}$. Let there exists a sequence on integers $\{s_j, r_j\}$, $r_j > s_j + k + 1$, $j = 1, 2, \dots$, $s_1 \geq a$, s_1 sufficiently large and $s_{j+1} > r_j$ such that, for function φ (defined by (2.1)) and for each pair (s_j, r_j), $j = 1, 2, \dots$, there exists a number $\nu_j \in (0, \nu_0)$ such that*

$$0 \leq \sum_{n=s_j+1}^{i} \varphi(n) \leq \frac{\pi}{\nu_j}, \quad i \in \mathbb{Z}_{s_j+1}^{r_j}, \quad \frac{\pi}{\nu_j} \leq \sum_{n=s_j+1}^{i} \varphi(n) \leq \frac{2\pi}{\nu_j}, \quad i \in \mathbb{Z}_{r_j+1}^{r_j+k}, \tag{3.2}$$

$p(n) \geq 0$ for $n \in \mathbb{Z}_{s_j+1}^{s_j+k}$, and (1.4) holds for $n \in \mathbb{Z}_{s_j+k+1}^{r_j}$, then all solutions of (1.1) are oscillating as $n \to \infty$.

4. Comparisons, Concluding Remarks, and Open Problems

Equation (1.1) with $k = 1$ was considered in [5], where a particular case of Theorem 2.1 is proved. In [4], a hypothesis is formulated together with the proof of Theorem 1.2 (Conjecture 1) about the oscillation of all solutions of (1.1) almost coinciding with the formulation of Theorem 2.1. For its simple form, (1.1) is often used for testing new results and is very frequently investigated.

Theorems 1.2 and 2.1 obviously generalize several classical results. We mention at least some of the simplest ones (see, e.g., [16, Theorem 7.7] or [19, Theorem 7.5.1]),

Theorem 4.1. *Let $p(n) \equiv p =$ const. Then every solution of (1.1) oscillates if and only if*

$$p > \frac{k^k}{(k+1)^{k+1}}. \tag{4.1}$$

Or the following result holds as well (see, e.g., [16, Theorem 7.6]) [18, 19]).

Theorem 4.2. *Let* $p(n) \geq 0$ *and*

$$\sup_n p(n) < \frac{k^k}{(k+1)^{k+1}}. \tag{4.2}$$

Then (1.1) *has a nonoscillatory solution.*

In [9] a problem on oscillation of all solutions of equation

$$\Delta u(n) + p(n)u(\tau(n)) = 0, \quad n \in \mathbb{N} \tag{4.3}$$

is considered where $p : \mathbb{N} \to \mathbb{R}_+$, $\tau : \mathbb{N} \to \mathbb{N}$, and $\lim_{n\to\infty}\tau(n) = +\infty$. Since in (4.3) delay τ is variable, we can formulate

Open Problem 1. It is an interesting open question whether Theorems 1.2 and 2.1 can be extended to linear difference equations with a variable delay argument of the form, for example,

$$\Delta u(n) = -p(n)u(h(n)), \quad n \in \mathbb{Z}_a^{\infty}, \tag{4.4}$$

where $0 \leq n - h(n) \leq k$. For some of the related results for the differential equation

$$\dot{x}(t) = -p(t)x(h(t)), \tag{4.5}$$

see the results in [3, 12] that are described below.

Open Problem 2. It is well known [19, 22] that the following condition is also sufficient for the oscillation of all solutions of (4.5) with $h(n) = n - k$:

$$\liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}. \tag{4.6}$$

The right-hand side of (4.6) is a critical value for this criterion since this number cannot be replaced with a smaller one.

In [30] equation (1.1) is considered as well. The authors prove that all solutions oscillate if $p(n) \geq 0$, $\varepsilon > 0$ and

$$\limsup_{n\to\infty} p(n) > \frac{k^k}{(k+1)^{k+1}} - \frac{\varepsilon}{k} + 4k\varepsilon^{1/4}, \tag{4.7}$$

where

$$\varepsilon = \left(\frac{k}{k+1}\right)^{k+1} - \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i. \tag{4.8}$$

An open problem is to obtain conditions similar to Theorem 2.1 for this kind of oscillation criteria. Some results on this problem for delay differential equations were also obtained in paper [3].

In [26] the authors establish an equivalence between the oscillation of (1.1) and the equation

$$\Delta^2 y(n-1) + \frac{2(k+1)^k}{k^{k+1}}\left(p(n) - \frac{k^k}{(k+1)^{k+1}}\right)y(n) = 0 \tag{4.9}$$

under the critical state

$$\liminf_{n\to\infty} p(n) = \frac{k^k}{(k+1)^{k+1}}, \tag{4.10}$$

$$p(n) \geq \frac{k^k}{(k+1)^{k+1}}. \tag{4.11}$$

Then they obtain some sharp oscillation and nonoscillation criteria for (1.1). One of the results obtained there is the following.

Theorem 4.3. *Assume that, for sufficiently large n, inequality* (4.11) *holds. Then the following statements are valid.*

(i) *If*

$$\liminf_{n\to\infty}\left[\left(p(n) - \frac{k^k}{(k+1)^{k+1}}\right)n^2\right] > \frac{k^{k+1}}{8(k+1)^k}, \tag{4.12}$$

then every solution of (1.1) *is oscillatory.*

(ii) *If, on the other hand,*

$$\left(p(n) - \frac{k^k}{(k+1)^{k+1}}\right)n^2 \leq \frac{k^{k+1}}{8(k+1)^k}, \tag{4.13}$$

then (1.1) *has a nonoscillatory solution.*

Regarding our results, it is easy to see that statement (i) is a particular case of Theorem 2.1 while statement (ii) is a particular case of Theorem 1.2.

In [27], the authors investigate (1.1) for $n \geq n_0$ and prove that (1.1) is oscillatory if

$$\sum_{i=n_0}^{\infty} p(i)\left\{\frac{k+1}{k}\cdot\sqrt[k+1]{\sum_{j=i+1}^{i+k} p(j)} - 1\right\} = \infty. \tag{4.14}$$

Comparing (4.14) with Theorem 2.1, we can see that (4.14) gives not sharp sufficient condition. Set, for example, $k = 1$, $\theta > 1$ and

$$p(n) = \frac{1}{2}\left[\frac{1}{2} + \frac{\theta}{8n^2}\right]. \tag{4.15}$$

Then,

$$\frac{k+1}{k} \cdot \sqrt[k+1]{\sum_{j=i+1}^{i+k} p(j)} - 1 = 2 \cdot \sqrt{\frac{1}{4}\left(1 + \frac{\theta}{4(i+1)^2}\right)} - 1 = \frac{\theta/4(i+1)^2}{1 + \sqrt{1 + \theta/4(i+1)^2}} \tag{4.16}$$

and the series in the left-hand side of (4.14) converges since

$$\begin{aligned} &\sum_{i=n_0}^{\infty} p(i)\left\{\frac{k+1}{k} \sqrt[k+1]{\sum_{j=i+1}^{i+k} p(j)} - 1\right\} \\ &\quad = \sum_{i=n_0}^{\infty} \frac{1}{2}\left[\frac{1}{2} + \frac{\theta}{8i^2}\right] \frac{\theta/4(i+1)^2}{1 + \sqrt{1 + \theta/4(i+1)^2}} \le \theta \sum_{i=n_0}^{\infty} \frac{1}{i^2}\left[1 + \frac{\theta}{i^2}\right] < \infty. \end{aligned} \tag{4.17}$$

But, by Theorem 2.1 all solutions of (1.1) are oscillating as $n \to \infty$. Nevertheless (4.14) is not a consequence of Theorem 2.1.

Let us consider a continuous variant of (1.1): a delayed differential linear equation of the form

$$\dot{x}(t) = -a(t)x(t-\tau), \tag{4.18}$$

where $\tau > 0$ is a constant delay and $a : [t_0, \infty) \to (0, \infty)$ (or $a : [t_0, \infty) \to \mathbb{R}$), $t_0 \in \mathbb{R}$. This equation, too, for its simple form, is often used for testing new results and is very frequently investigated. It is, for example, well known that a scalar linear equation with delay

$$\dot{x}(t) + \frac{1}{e}x(t-1) = 0 \tag{4.19}$$

has a nonoscillatory solution as $t \to \infty$. This means that there exists an eventually positive solution. The coefficient $1/e$ is called critical with the following meaning: for any $\alpha > (1/e)$, all solutions of the equation

$$\dot{x}(t) + \alpha x(t-1) = 0 \tag{4.20}$$

are oscillatory while, for $\alpha \le (1/e)$, there exists an eventually positive solution. In [10], the third author considered (4.18), where $a : [t_0, \infty) \to (0, \infty)$ is a continuous function, and t_0 is sufficiently large. For the critical case, he obtained the following result (being a continuous variant of Theorems 1.2 and 2.1).

Theorem 4.4. *(a) Let an integer $k \geq 0$ exist such that $a(t) \leq a_k(t)$ if $t \to \infty$ where*

$$a_k(t) := \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2}. \tag{4.21}$$

Then there exists an eventually positive solution x of (4.18).

(b) Let an integer $k \geq 2$ and $\theta > 1$, $\theta \in \mathbb{R}$ exist such that

$$a(t) > a_{k-2}(t) + \frac{\theta\tau}{8e(t\ln t\ln_2 t\cdots\ln_{k-1} t)^2}, \tag{4.22}$$

if $t \to \infty$. Then all solutions of (4.18) *oscillate.*

Further results on the critical case for (4.18) can be found in [1, 11, 14, 17, 24].

In [12], Theorem 7 was generalized for equations with a variable delay

$$\dot{x}(t) + a(t)x(t - \tau(t)) = 0, \tag{4.23}$$

where $a : [t_0, \infty) \to (0, \infty)$ and $\tau : [t_0, \infty) \to (0, \infty)$ are continuous functions. The main results of this paper include the following.

Theorem 4.5 (see [12]). *Let $t - \tau(t) \geq t_0 - \tau(t_0)$ if $t \geq t_0$. Let an integer $k \geq 0$ exist such that $a(t) \leq a_{k\tau}(t)$ for $t \to \infty$, where*

$$a_{k\tau}(t) := \frac{1}{e\tau(t)} + \frac{\tau(t)}{8et^2} + \frac{\tau(t)}{8e(t\ln t)^2} + \cdots + \frac{\tau(t)}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2}. \tag{4.24}$$

If moreover

$$\begin{gathered}\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \leq 1 \quad \text{when } t \longrightarrow \infty, \\ \lim_{t\to\infty} \tau(t) \cdot \left(\frac{1}{t}\ln t\ln_2 t\cdots\ln_k t\right) = 0,\end{gathered} \tag{4.25}$$

then there exists an eventually positive solution x of (4.23) *for $t \to \infty$.*

Finally, the last results were generalized in [3]. We reproduce some of the results given there.

Theorem 4.6. *(A) Let $\tau > 0$, $0 \leq \tau(t) \leq \tau$ for $t \to \infty$, and let condition (a) of Theorem 4.4 holds. Then* (4.23) *has a nonoscillatory solution.*

(B) Let $\tau(t) \geq \tau > 0$ for $t \to \infty$, and let condition (b) of Theorem 4.4 holds. Then all solutions of (4.23) *oscillate.*

For every integer $k \geq 0$, $\delta > 0$ and $t \to \infty$, we define

$$A_k(t) := \frac{1}{e\delta\tau(t)} + \frac{\delta}{8e\tau(t)s^2} + \frac{\delta}{8e\tau(t)(s\ln s)^2} + \cdots + \frac{\delta}{8e\tau(t)(s\ln s\ln_2 s\cdots\ln_k s)^2}, \tag{4.26}$$

where

$$s = p(t) := \int_{t_0}^{t} \frac{1}{\tau(\xi)} d\xi. \tag{4.27}$$

Theorem 4.7. *Let for t_0 sufficiently large and $t \geq t_0$: $\tau(t) > 0$ a.e., $1/\tau(t)$ be a locally integrable function,*

$$\lim_{t\to\infty}(t-\tau(t)) = \infty, \quad \int_{t_0}^{\infty} \frac{1}{\tau(\xi)} d\xi = \infty, \tag{4.28}$$

and let there exists $t_1 > t_0$ such that $t - \tau(t) \geq t_0$, $t \geq t_1$.

(a) *If there exists a $\delta \in (0,\infty)$ such that*

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \leq \delta, \quad t \geq t_1, \tag{4.29}$$

and, for a fixed integer $k \geq 0$,

$$a(t) \leq A_k(t), \quad t \geq t_1, \tag{4.30}$$

then there exists an eventually positive solution of (4.23).

(b) *If there exists a $\delta \in (0,\infty)$ such that*

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \geq \delta, \quad t \geq t_1, \tag{4.31}$$

and, for a fixed integer $k \geq 2$ and $\theta > 1$, $\theta \in \mathbb{R}$,

$$a(t) > A_{k-2}(t) + \frac{\theta\delta}{8e\tau(t)(s\ln s \ln_2 s \cdots \ln_{k-1} s)^2}, \tag{4.32}$$

if $t \geq t_1$, then all solutions of (4.23) *oscillate.*

Appendix

A. Auxiliary Computations

This part includes auxiliary results with several technical lemmas proved. Part of them is related to the asymptotic decomposition of certain functions and the rest deals with computing the sums of some algebraic expressions. The computations are referred to in the proof of the main result (Theorem 2.1) in Section 2.

First we define auxiliary functions (recalling also the definition of function φ given by (2.1)):

$$\begin{aligned}
\varphi(n) &:= \frac{1}{n \ln n \ln_2 n \ln_3 n \cdots \ln_q n},\\
\alpha(n) &:= \frac{1}{n} + \frac{1}{n\ln n} + \frac{1}{n \ln n \ln_2 n} + \cdots + \frac{1}{n \ln n \ln_2 n \cdots \ln_q n},\\
\omega_0(n) &:= \frac{1}{n^2} + \frac{3}{2n^2 \ln n} + \frac{3}{2n^2 \ln n \ln_2 n} + \cdots + \frac{3}{2n^2 \ln n \ln_2 n \cdots \ln_q n},\\
\omega_1(n) &:= \frac{1}{(n\ln n)^2} + \frac{3}{2(n \ln n)^2 \ln_2 n} + \cdots + \frac{3}{2(n\ln n)^2 \ln_2 n \cdots \ln_q n},\\
&\vdots\\
\omega_{q-1}(n) &:= \frac{1}{(n \ln n \cdots \ln_{q-1} n)^2} + \frac{3}{2(n\ln n \cdots \ln_{q-1} n)^2 \ln_q n},\\
\omega_q(n) &:= \frac{1}{(n \ln n \cdots \ln_q n)^2},\\
\Omega(n) &:= \frac{1}{n^2} + \frac{1}{(n\ln n)^2} + \frac{1}{(n \ln n \ln_2 n)^2} + \cdots + \frac{1}{(n \ln n \ln_2 n \cdots \ln_q n)^2},
\end{aligned} \tag{A.1}$$

where n is sufficiently large and $q \in \mathbb{N}_0$. Moreover, we set (for admissible values of arguments)

$$\Sigma(p) := \sum_{\ell=1}^{k} (k - p - \ell), \tag{A.2}$$

$$\Sigma^+(p) := \Sigma(p) + (k - p), \tag{A.3}$$

$$V(n+p) := \sum_{\ell=1}^{k} \varphi(n + p - k + \ell), \tag{A.4}$$

$$V^+(n+p) := V(n+p) + \varphi(n+p-k), \tag{A.5}$$

$$S(p) := \sum_{\ell=1}^{k} (k - p - \ell)^2, \tag{A.6}$$

$$S^+(p) := S(p) + (k-p)^2. \tag{A.7}$$

A.1. Asymptotic Decomposition of Iterative Logarithms

In the proof of the main result, we use auxiliary results giving asymptotic decompositions of iterative logarithms. The following lemma is proved in [11].

Lemma A.1. *For fixed* $r, \sigma \in \mathbb{R} \setminus \{0\}$ *and a fixed integer* $s \geq 1$, *the asymptotic representation*

$$\begin{aligned}
\frac{\ln_s^{\sigma}(n-r)}{\ln_s^{\sigma} n} = 1 &- \frac{r\sigma}{n \ln n \cdots \ln_s n} - \frac{r^2\sigma}{2n^2 \ln n \cdots \ln_s n} \\
&- \frac{r^2\sigma}{2(n\ln n)^2 \ln_2 n \cdots \ln_s n} - \cdots - \frac{r^2\sigma}{2(n \ln n \cdots \ln_{s-1} n)^2 \ln_s n} \\
&+ \frac{r^2\sigma(\sigma-1)}{2(n \ln n \cdots \ln_s n)^2} - \frac{r^3\sigma(1+o(1))}{3n^3 \ln n \cdots \ln_s n}
\end{aligned} \tag{A.8}$$

holds for $n \to \infty$.

A.2. Formulas for $\Sigma(p)$ and for $\Sigma^+(p)$

Lemma A.2. *The following formulas hold:*

$$\Sigma(p) = \frac{k}{2} \cdot (k - 2p - 1), \tag{A.9}$$

$$\Sigma^+(p) = \frac{k+1}{2} \cdot (k - 2p). \tag{A.10}$$

Proof. It is easy to see that

$$\begin{aligned}
\Sigma(p) &= \sum_{\ell=-p}^{k-p-1} \ell = (k-p-1) + (k-p-2) + \cdots + (-p) \\
&= (k-(p+1)) + (k-(p+2)) + \cdots + (k-(p+k)) = \frac{k}{2} \cdot (k-2p-1), \\
\Sigma^+(p) &= \Sigma(p) + (k-p) = \frac{k+1}{2} \cdot (k-2p).
\end{aligned} \tag{A.11}$$

□

A.3. Formula for the Sum of the Terms of an Arithmetical Sequence

Denote by $u_1, u_2, \ldots, u_r$ the terms of an arithmetical sequence of kth order (kth differences are constant), $d'_1, d'_2, d'_3, \ldots$, the first differences ($d'_1 = u_2 - u_1$, $d'_2 = u_3 - u_2, \ldots$), $d''_1, d''_2, d''_3, \ldots$, the second differences ($d''_1 = d'_2 - d'_1, \ldots$), and so forth. Then the following result holds (see, e.g., [43]).

Lemma A.3. *For the sum of* r *terms of an arithmetical sequence of* k*th order, the following formula holds*

$$\sum_{i=1}^{r} u_i = \frac{r!}{(r-1)! \cdot 1!} \cdot u_1 + \frac{r!}{(r-2)! \cdot 2!} \cdot d'_1 + \frac{r!}{(r-3)! \cdot 3!} \cdot d''_1 + \cdots . \tag{A.12}$$

A.4. Asymptotic Decomposition of $\varphi(n-l)$

Lemma A.4. *For fixed* $\ell \in \mathbb{R}$ *and* $q \in \mathbb{N}_0$*, the asymptotic representation*

$$\varphi(n-\ell) = \varphi(n)\left(1 + \ell\alpha(n) + \ell^2 \sum_{i=0}^{q} \omega_q(n)\right) + O\left(\frac{\varphi(n)}{n^3}\right) \tag{A.13}$$

holds for $n \to \infty$.

Proof. The function $\varphi(n)$ is defined by (2.1). We develop the asymptotic decomposition of $\varphi(n-\ell)$ when n is sufficiently large and $\ell \in \mathbb{R}$. Applying Lemma A.1 (for $\sigma = -1$, $r = \ell$ and $s = 1, 2, \ldots, q$), we get

$$\begin{aligned}
\varphi(n-\ell) &= \frac{1}{(n-\ell)\ln(n-\ell)\ln_2(n-\ell)\ln_3(n-\ell)\cdots\ln_q(n-\ell)} \\
&= \frac{1}{n(1-\ell/n)\ln(n-\ell)\ln_2(n-\ell)\ln_3(n-\ell)\cdots\ln_q(n-\ell)} \\
&= \varphi(n)\cdot\frac{1}{1-\ell/n}\cdot\frac{\ln n}{\ln(n-\ell)}\cdot\frac{\ln_2 n}{\ln_2(n-\ell)}\cdot\frac{\ln_3 n}{\ln_3(n-\ell)}\cdots\frac{\ln_q n}{\ln_q(n-\ell)} \\
&= \varphi(n)\left(1 + \frac{\ell}{n} + \frac{\ell^2}{n^2} + O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\left(1 + \frac{\ell}{n\ln n} + \frac{\ell^2}{2n^2\ln n} + \frac{\ell^2}{(n\ln n)^2} + O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\left(1 + \frac{\ell}{n\ln n\ln_2 n} + \frac{\ell^2}{2n^2\ln n\ln_2 n} + \frac{\ell^2}{2(n\ln n)^2\ln_2 n} + \frac{\ell^2}{(n\ln n\ln_2 n)^2} + O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\left(1 + \frac{\ell}{n\ln n\ln_2 n\ln_3 n} + \frac{\ell^2}{2n^2\ln n\ln_2 n\ln_3 n} + \frac{\ell^2}{2(n\ln n)^2\ln_2 n\ln_3 n}\right. \\
&\qquad\left. + \frac{\ell^2}{2(n\ln n\ln_2 n)^2\ln_3 n} + \frac{\ell^2}{(n\ln n\ln_2 n\ln_3 n)^2} + O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\cdots\times\left(1 + \frac{\ell}{n\ln n\ln_2 n\ln_3 n\cdots\ln_q n} + \frac{\ell^2}{2n^2\ln n\cdots\ln_q n} + \frac{\ell^2}{2(n\ln n)^2\ln_2\cdots n\ln_q n}\right. \\
&\qquad\left. + \cdots + \frac{\ell^2}{2(n\ln n\cdots\ln_{q-1}n)^2\ln_q n} + \frac{\ell^2}{(n\ln n\cdots\ln_q n)^2} + O\left(\frac{1}{n^3}\right)\right).
\end{aligned} \tag{A.14}$$

Finally, gathering the same functional terms and omitting the terms having a higher order of accuracy than is necessary, we obtain the asymptotic decomposition (A.13). □

A.5. Formula for $\alpha^2(n)$

Lemma A.5. *For fixed* $q \in \mathbb{N}_0$*, the formula*

$$\alpha^2(n) = \frac{4}{3}\sum_{i=0}^{q}\omega_i(n) - \frac{1}{3}\Omega(n) \tag{A.15}$$

holds for all sufficiently large n.

Proof. It is easy to see that

$$\begin{aligned}
\alpha^2(n) &= \frac{1}{n^2} + \frac{2}{n^2\ln n} + \frac{2}{n^2\ln n\ln_2 n} + \cdots + \frac{2}{n^2\ln n\ln_2 n\cdots\ln_q n} \\
&\quad + \frac{1}{(n\ln n)^2} + \frac{2}{(n\ln n)^2\ln_2 n} + \cdots + \frac{2}{(n\ln n)^2\ln_2 n\cdots\ln_q n} \\
&\quad + \frac{1}{(n\ln n\ln_2 n)^2} + \frac{2}{(n\ln n\ln_2 n)^2\ln_3 n} + \cdots + \frac{2}{(n\ln n\ln_2 n)^2\cdots\ln_q n} \\
&\quad + \cdots + \frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2} \\
&= \frac{4}{3}\left(\frac{1}{n^2} + \frac{3}{2n^2\ln n} + \frac{3}{2n^2\ln n\ln_2 n} + \cdots + \frac{3}{2n^2\ln n\ln_2 n\cdots\ln_q n}\right. \\
&\qquad + \frac{1}{(n\ln n)^2} + \frac{3}{2(n\ln n)^2\ln_2 n} + \cdots + \frac{3}{2(n\ln n)^2\ln_2 n\cdots\ln_q n} \\
&\qquad + \frac{1}{(n\ln n\ln_2 n)^2} + \frac{2}{(n\ln n\ln_2 n)^2\ln_3 n} + \cdots + \frac{2}{(n\ln n\ln_2 n)^2\cdots\ln_q n} \\
&\qquad \left. + \cdots + \frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}\right) \\
&\quad - \frac{1}{3}\left(\frac{1}{n^2} + \frac{1}{(n\ \ln n)^2} + \frac{1}{(n\ln n\ln_2 n)^2} + \cdots + \frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}\right) \\
&= \frac{4}{3}\sum_{i=0}^{q}\omega_i(n) - \frac{1}{3}\Omega(n).
\end{aligned} \tag{A.16}$$

□

A.6. Asymptotic Decomposition of $V(n+p)$

Lemma A.6. *For fixed* $p \in \mathbb{N}$ *and* $q \in \mathbb{N}_0$*, the asymptotic representation*

$$V(n+p) = \varphi(n)\left[k + \Sigma(p)\alpha(n) + S(p)\sum_{i=0}^{q}\omega_i(n)\right] + O\left(\frac{\varphi(n)}{n^3}\right) \tag{A.17}$$

holds for $n \to \infty$.

Proof. It is easy to deduce from formula (A.13) with $\ell = k-p-1, k-p-2, \ldots, -p$ that

$$V(n+p) := \varphi(n+p-k+1) + \varphi(n+p-k+2) + \cdots + \varphi(n+p) = \sum_{\ell=-p}^{k-p-1} \varphi(n-\ell) = \varphi(n)$$

$$\times \sum_{\ell=-p}^{k-p-1} \left(1 + \frac{\ell}{n} + \frac{\ell}{n\ln n} + \frac{\ell}{n\ln n \ln_2 n} + \cdots + \frac{\ell}{n \ln n \ln_2 n \cdots \ln_q n} \right.$$
$$+ \frac{\ell^2}{n^2} + \frac{3\ell^2}{2n^2 \ln n} + \cdots + \frac{3\ell^2}{2n^2 \ln n \ln_2 n \cdots \ln_q n} + \frac{\ell^2}{(n\ln n)^2}$$
$$+ \frac{3\ell^2}{2(n\ln n)^2 \ln_2 n} + \frac{3\ell^2}{2(n\ln n)^2 \ln_3 n} + \cdots + \frac{3\ell^2}{2(n\ln n)^2 \ln_3 n \cdots \ln_q n}$$
$$+ \frac{\ell^2}{(n\ln n\ln_2 n)^2} + \frac{3\ell^2}{2(n\ln n\ln_2 n)^2 \ln_3 n} + \cdots + \frac{3\ell^2}{2(n\ln n\ln_2 n)^2 \ln_3 n \cdots \ln_q n}$$
$$+ \frac{\ell^2}{(n\ln n\ln_2 n \ln_3 n)^2} + \cdots + \frac{3\ell^2}{2(n\ln n\ln_2 n\ln_3 n)^2 \ln_4 n \cdots \ln_q n}$$
$$+ \cdots + \frac{\ell^2}{(n\ln n\ln_2 n \cdots \ln_{q-1} n)^2} + \frac{3\ell^2}{2(n\ln n\ln_2 n \cdots \ln_{q-1} n)^2 \ln_q n}$$
$$\left. + \frac{\ell^2}{(n\ln n\ln_2 n \cdots \ln_q n)^2} + O\left(\frac{1}{n^3}\right)\right). \tag{A.18}$$

Then

$$\begin{aligned}
V(n+p) &:= \varphi(n) \sum_{\ell=-p}^{k-p-1} \left[1 + \ell\alpha(n) + \ell^2 \sum_{i=0}^{q} \omega_i(n) + O\left(\frac{1}{n^3}\right)\right] \\
&= \varphi(n) \left[\sum_{\ell=-p}^{k-p-1} 1 + \alpha(n) \cdot \sum_{\ell=-p}^{k-p-1} \ell + \sum_{\ell=-p}^{k-p-1} \ell^2 \cdot \sum_{i=0}^{q} \omega_i(n) + O\left(\frac{1}{n^3}\right)\right] \\
&= \varphi(n) \left[k + \Sigma(p)\alpha(n) + S(p) \cdot \sum_{i=0}^{q} \omega_i(n) + O\left(\frac{1}{n^3}\right)\right] \\
&= \varphi(n) \left[k + \Sigma(p)\alpha(n) + S(p) \sum_{i=0}^{q} \omega_i(n)\right] + O\left(\frac{\varphi(n)}{n^3}\right).
\end{aligned} \tag{A.19}$$

□

A.7. Asymptotic Decomposition of $V^+(n+p)$

Lemma A.7. *For fixed* $p \in \mathbb{N}_0$ *and* $q \in \mathbb{N}_0$, *the asymptotic representation*

$$V^+(n+p) = \varphi(n)\left[k+1+\Sigma^+(p)\alpha(n)+S^+(p)\sum_{i=0}^{q}\omega_i(n)\right]+O\left(\frac{\varphi(n)}{n^3}\right) \tag{A.20}$$

holds for $n \to \infty$.

Proof. By (A.5), (A.13), (A.17), (A.10), and (A.7), we get

$$\begin{aligned}
V^+(n+p) &:= V(n+p)+\varphi(n+p-k)\\
&= \varphi(n)\left[k+\Sigma(p)\alpha(n)+S(p)\sum_{i=0}^{q}\omega_i(n)\right]+O\left(\frac{\varphi(n)}{n^3}\right)+\varphi(n+p-k)\\
&= \varphi(n)\left[k+\Sigma(p)\alpha(n)+S(p)\sum_{i=0}^{q}\omega_i(n)\right]+O\left(\frac{\varphi(n)}{n^3}\right)\\
&\quad+\varphi(n)\Big(1+(k-p)\alpha(n)+(k-p)^2\omega_0(n)+(k-p)^2\omega_1(n)\\
&\qquad+\cdots+(k-p)^2\omega_{q-1}(n)+(k-p)^2\omega_q(n)+O\left(\frac{1}{n^3}\right)\Big)\\
&= \varphi(n)\left[k+1+(\Sigma(p)+(k-p))\alpha(n)+\left(S(p)+(k-p)^2\right)\sum_{i=0}^{q}\omega_i(n)\right]+O\left(\frac{\varphi(n)}{n^3}\right)\\
&= \varphi(n)\left[k+1+\Sigma^+(p)\alpha(n)+S^+(p)\sum_{i=0}^{q}\omega_i(n)\right]+O\left(\frac{\varphi(n)}{n^3}\right).
\end{aligned} \tag{A.21}$$

□

A.8. Formula for $\sum_{p=1}^{k}\Sigma(p)$

Lemma A.8. *For the above sum, the following formula holds:*

$$\sum_{p=1}^{k}\Sigma(p) = -k^2. \tag{A.22}$$

Proof. Using formula (A.9), we get

$$\begin{aligned}
\sum_{p=1}^{k}\Sigma(p) &= \Sigma(1)+\Sigma(2)+\Sigma(3)+\cdots+\Sigma(k)\\
&= \frac{k}{2}\cdot[(k-3)+(k-5)+(k-7)+\cdots+(k-(2k+1))]\\
&= \frac{k}{2}\cdot(-2k) = -k^2.
\end{aligned} \tag{A.23}$$

□

A.9. Formula for $\sum_{p=1}^{k} \Sigma^2(p)$

Lemma A.9. *For the above sum, the following formula holds:*

$$\sum_{p=1}^{k} \Sigma^2(p) = \frac{k^3}{12}\left(k^2 + 11\right). \tag{A.24}$$

Proof. Using formula (A.9), we get

$$\begin{aligned}\sum_{p=1}^{k} \Sigma^2(p) &= \frac{k^2}{4}\sum_{p=1}^{k}(k-2p-1)^2 \\ &= \frac{k^2}{4} \cdot \left[(k-3)^2 + (k-5)^2 + (k-7)^2 + \cdots + (k-(2k+1))^2\right].\end{aligned} \tag{A.25}$$

We compute the sum in the square brackets. We use formula (A.12). In our case,

$$\begin{gathered} r = k, \quad u_1 = (k-3)^2, \quad u_2 = (k-5)^2, \quad u_3 = (k-7)^2, \ldots, \quad u_k = (k-2k-1)^2 = (k+1)^2, \\ d_1' = u_2 - u_1 = (k-5)^2 - (k-3)^2 = -4k + 16, \\ d_2' = u_3 - u_2 = (k-7)^2 - (k-5)^2 = -4k + 24, \end{gathered} \tag{A.26}$$

the second differences are constant, and

$$d_1'' = d_2' - d_1' = (-4k + 24) - (-4k + 16) = 8. \tag{A.27}$$

Then the sum in the square brackets equals

$$\frac{k!}{(k-1)! \cdot 1!} \cdot (k-3)^2 + \frac{k!(-4)}{(k-2)! \cdot 2!} \cdot (k-4) + \frac{k!}{(k-3)! \cdot 3!} \cdot 8 = \frac{k}{3}\left(k^2 + 11\right), \tag{A.28}$$

and formula (A.24) is proved. □

A.10. Formula for $2\prod_{i,j=0, i>j}^{k} \Sigma(i)\Sigma(j)$

Lemma A.10. *For the above product, the following formula holds:*

$$2\prod_{\substack{i,j=0 \\ i>j}}^{k} \Sigma(i)\Sigma(j) = k^4 - \frac{k^3}{12}\left(k^2 + 11\right). \tag{A.29}$$

Proof. We have

$$2\prod_{\substack{i,j=0\\ i>j}}^{k}\Sigma(i)\ \Sigma(j)=\left(\sum_{p=1}^{k}\Sigma(p)\right)^{2}-\sum_{p=1}^{k}(\Sigma(p))^{2}. \tag{A.30}$$

Then, using formulas (A.22), and (A.24), we get

$$\left(\sum_{p=1}^{k}\Sigma(p)\right)^{2}-\sum_{p=1}^{k}(\Sigma(p))^{2}=\left(-k^{2}\right)^{2}-\frac{k^{3}}{12}\left(k^{2}+11\right)=k^{4}-\frac{k^{3}}{12}\left(k^{2}+11\right). \tag{A.31}$$

□

A.11. Formula for $\sum_{p=0}^{k}\Sigma^{+}(p)$

Lemma A.11. *For the above sum, the following formula holds:*

$$\sum_{p=0}^{k}\Sigma^{+}(p)=0. \tag{A.32}$$

Proof. Using formulas (A.9), (A.10), and (A.22), we get

$$\sum_{p=0}^{k}\Sigma^{+}(p)=\Sigma(0)+\sum_{p=1}^{k}\Sigma(p)+\sum_{p=0}^{k}(k-p)=\frac{k}{2}(k-1)-k^{2}+\frac{k}{2}(k+1)=0. \tag{A.33}$$

□

A.12. Formula for $\sum_{p=0}^{k}(\Sigma^{+}(p))^{2}$

Lemma A.12. *For the above sum, the following formula holds:*

$$\sum_{p=0}^{k}\left(\Sigma^{+}(p)\right)^{2}=\frac{(k+1)^{2}k}{12}\cdot\left(k^{2}+3k+2\right). \tag{A.34}$$

Proof. Using formula (A.10), we get

$$\sum_{p=0}^{k}\left(\Sigma^{+}(p)\right)^{2}=\frac{(k+1)^{2}}{4}\left[(k-0)^{2}+(k-2)^{2}+(k-4)^{2}+\cdots+(k-2k)^{2}\right]. \tag{A.35}$$

We compute the sum in the square brackets. We use formula (A.12). In our case,

$$\begin{gathered}r=k+1,\quad u_{1}=k^{2},\quad u_{2}=(k-2)^{2},\quad u_{3}=(k-4)^{2},\ldots,\quad u_{k+1}=(k-2k)^{2}=k^{2},\\ d_{1}'=u_{2}-u_{1}=(k-2)^{2}-k^{2}=-4k+4,\\ d_{2}'=u_{3}-u_{2}=(k-4)^{2}-(k-2)^{2}=-4k+12,\end{gathered} \tag{A.36}$$

the second differences are constant, and

$$d_1'' = d_2' - d_1' = (-4k + 12) - (-4k + 4) = 8. \tag{A.37}$$

Then, the sum in the square brackets equals

$$\frac{(k+1)!}{k! \cdot 1!} \cdot k^2 + \frac{4(k+1)!}{(k-1)! \cdot 2!} \cdot (-k+1) + \frac{(k+1)!}{(k-2)! \cdot 3!} \cdot 8 = \frac{k}{3}\left(k^2 + 3k + 2\right), \tag{A.38}$$

and formula (A.34) is proved. □

A.13. Formula for $2\prod_{i,j=0,\ i>j}^{k} \Sigma^+(i)\Sigma^+(j)$

Lemma A.13. *For the above product, the following formula holds:*

$$2\prod_{\substack{i,j=0 \\ i>j}}^{k} \Sigma^+(i)\Sigma^+(j) \ = -\frac{(k+1)^2 k}{12}\left(k^2 + 3k + 2\right). \tag{A.39}$$

Proof. We have

$$2\prod_{\substack{i,j=0 \\ i>j}}^{k} \Sigma^+(i)\Sigma^+(j) = \left(\sum_{p=0}^{k} \Sigma^+(p)\right)^2 - \sum_{p=0}^{k} \left(\Sigma^+(p)\right)^2. \tag{A.40}$$

Then, using formulas (A.32), and (A.34), we get

$$\left(\sum_{p=1}^{k} \Sigma^+(p)\right)^2 - \sum_{p=1}^{k} \left(\Sigma^+(p)\right)^2 = -\sum_{p=1}^{k} \left(\Sigma^+(p)\right)^2 = -\frac{(k+1)^2 k}{12} \cdot \left(k^2 + 3k + 2\right). \tag{A.41}$$

□

A.14. Formula for $S(p)$

Lemma A.14. *For a fixed integer p, the formula*

$$S(p) = \frac{k}{6}\left[2k^2 - 3(1 + 2p)k + \left(6p^2 + 6p + 1\right)\right] \tag{A.42}$$

holds.

Proof. We use formula (A.12). In our case

$$
\begin{aligned}
&r = k, \quad u_1 = (k-p-1)^2, \dots, \quad u_k = (k-p-k)^2 = p^2, \\
&d_1' = u_2 - u_1 = (k-p-2)^2 - (k-p-1)^2 = (2k-2p-3)(-1), \\
&d_2' = u_3 - u_2 = (k-p-3)^2 - (k-p-2)^2 = (2k-2p-5)(-1),
\end{aligned}
\tag{A.43}
$$

the second differences are constant, and

$$
d_1'' = d_2' - d_1' = (2k-2p-5)(-1) - (2k-2p-3)(-1) = 2. \tag{A.44}
$$

Then the formula

$$
S(p) = \frac{k!}{(k-1)! \cdot 1!} \cdot (k-p-1)^2 + \frac{k!(-1)}{(k-2)! \cdot 2!} \cdot (2k-2p-3) + \frac{k!}{(k-3)! \cdot 3!} \cdot 2 \tag{A.45}
$$

directly follows from (A.12). After some simplification, we get

$$
\begin{aligned}
S(p) &= k \cdot (k-p-1)^2 - \frac{k(k-1)}{2} \cdot (2k-2p-3) + \frac{k(k-1)(k-2)}{3} \\
&= \frac{k}{6} \cdot \left[6\left(k^2 - 2k(p+1) + (p+1)^2\right) - 3\left(2k^2 - k(2p+5) + (2p+3)\right) + 2\left(k^2 - 3k + 2\right)\right] \\
&= \frac{k}{6}\left[2k^2 - 3(1+2p)k + \left(6p^2 + 6p + 1\right)\right].
\end{aligned}
\tag{A.46}
$$

Formula (A.42) is proved. □

A.15. Formula for $\sum_{p=1}^{k} S(p)$

Lemma A.15. *For a fixed integer p, the formula*

$$
\sum_{p=1}^{k} S(p) = \frac{k}{6}\left(k^3 + 5k\right) \tag{A.47}
$$

holds.

Proof. Since, by (A.42),

$$
\frac{6}{k} S(p) = 2k^2 - 3(1+2p)k + \left(6p^2 + 6p + 1\right), \tag{A.48}
$$

we get

$$\frac{6}{k}\sum_{p=1}^{k}S(p) = 2\sum_{p=1}^{k}k^2 - 3k\sum_{p=1}^{k}(1+2p) + 6\sum_{p=1}^{k}p^2 + 6\sum_{p=1}^{k}p + \sum_{p=1}^{k}1$$
$$= 2k^3 - 3k\left(k^2+2k\right) + k\left(2k^2+3k+1\right) + 3\left(k^2+k\right) + k \tag{A.49}$$
$$= k^3 + 5k.$$

This yields (A.47). □

A.16. Formula for $S^+(p)$

Lemma A.16. *The above expression equals*

$$S^+(p) = \frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right]. \tag{A.50}$$

Proof. We have the following:

$$\begin{aligned}
S^+(p) &= (k-p)^2 + S(p)\\
&= \left[(k-p)^2 + \frac{k}{6}\left[2k^2 - 3(1+2p)k + \left(6p^2+6p+1\right)\right]\right.\\
&\qquad \left. -\frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right]\right] + \frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right]\\
&= \frac{1}{6}\left[6k^2 - 12kp + 6p^2 + k[-4k+6p+1] - 2k^2 + (6p-1)k - 6p^2\right]\\
&\qquad + \frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right]\\
&= \frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right].
\end{aligned} \tag{A.51}$$

This yields (A.50). □

A.17. Formula for $\sum_{p=0}^{k} S^+(p)$

Lemma A.17. *The above expression equals*

$$\sum_{p=0}^{k}S^+(p) = \frac{(k+1)k}{6}\left(k^2+3k+2\right). \tag{A.52}$$

Proof. Since, by (A.50),

$$\frac{6}{k+1}S^+(p) = 2k^2 + (-6p+1)k + 6p^2, \tag{A.53}$$

we get

$$\begin{aligned}\frac{6}{k+1}\sum_{p=0}^{k}S^{+}(p) &= 2\sum_{p=0}^{k}k^2 + k\sum_{p=0}^{k}(-6p+1) + 6\sum_{p=0}^{k}p^2 \\ &= 2k^2(k+1) + k(-3k(k+1) + (k+1)) + k\left(2k^2+3k+1\right) \\ &= k^3 + 3k^2 + 2k.\end{aligned} \tag{A.54}$$

This yields (A.52). □

A.18. Formula for $(1/k)\sum_{p=1}^{k}S(p) - (1/(k+1))\sum_{p=0}^{k}S^{+}(p)$

Lemma A.18. *The above expression equals*

$$\frac{1}{k}\sum_{p=1}^{k}S(p) - \frac{1}{k+1}\sum_{p=0}^{k}S^{+}(p) = \frac{1}{2}\cdot\left(-k^2+k\right). \tag{A.55}$$

Proof. By (A.47) and (A.50), we obtain

$$\begin{aligned}\frac{1}{k}\sum_{p=1}^{k}S(p) - \frac{1}{k+1}\sum_{p=0}^{k}S^{+}(p) &= \frac{1}{6}\cdot\left(k^3+5k\right) - \frac{1}{6}\cdot\left(k^3+3k^2+2k\right) \\ &= \frac{1}{6}\cdot\left(-3k^2+3k\right) = \frac{1}{2}\cdot\left(-k^2+k\right).\end{aligned} \tag{A.56}$$

This yields (A.55). □

A.19. Asymptotic Decomposition of $\prod_{p=1}^{k}V(n+p)$

Lemma A.19. *For a fixed* $q \in \mathbb{N}_0$, *the asymptotic representation*

$$\begin{aligned}\prod_{p=1}^{k}V(n+p) = k^k\varphi^k(n)&\left[1 - k\alpha(n) - \frac{k}{24}\left(k^2-12k+11\right)\alpha^2(n) + \frac{k}{6}\left(k^2+5\right)\sum_{i=0}^{q}\omega_i(n)\right] \\ &+ O\left(\frac{\varphi^k(n)}{n^3}\right)\end{aligned} \tag{A.57}$$

holds for $n \to \infty$.

Proof. Using formula (A.17), we get

$$\prod_{p=1}^{k} V(n+p) = \prod_{p=1}^{k}\left[\varphi(n)\left[k+\Sigma(p)\alpha(n)+S(p)\sum_{i=0}^{q}\omega_i(n)\right]+O\left(\frac{\varphi(n)}{n^3}\right)\right]$$
$$= \varphi^k(n)\left[k^k + k^{k-1}\alpha(n)\sum_{i=1}^{k}\Sigma(i) + k^{k-2}\alpha^2(n)\prod_{\substack{i,j=0\\ i>j}}^{k}\Sigma(i)\ \Sigma(j) + k^{k-1}\sum_{i=1}^{k}S(i)\sum_{j=0}^{q}\omega_j(n)\right]$$
$$+O\left(\frac{\varphi^k(n)}{n^3}\right) = (*). \tag{A.58}$$

Now, by (A.22), (A.29), and (A.47)

$$(*) = \varphi^k(n)\left[k^k + k^{k-1}(-k)^2\alpha(n) + \frac{1}{2}k^{k-2}\left(k^4 - \frac{k^3}{12}\left(k^2+11\right)\right)\alpha^2(n)\right.$$
$$\left. +\frac{1}{6}k^{k-1}k\left(k^3+5k\right)\sum_{j=0}^{q}\omega_j(n)\right] + O\left(\frac{\varphi^k(n)}{n^3}\right)$$
$$= k^k\varphi^k(n)\left[1 - k\alpha(n) - \frac{k}{24}\left(k^2-12k+11\right)\alpha^2(n)\right.$$
$$\left. +\frac{k}{6}\left(k^2+5\right)\sum_{j=0}^{q}\omega_j(n)\right] + O\left(\frac{\varphi^k(n)}{n^3}\right). \tag{A.59}$$

□

A.20. Asymptotic Decomposition of $\prod_{p=0}^{k} V^+(n+p)$

Lemma A.20. *For a fixed* $q \in \mathbb{N}_0$*, the asymptotic representation*

$$\prod_{p=0}^{k} V^+(n+p) = (k+1)^{k+1}\varphi^{k+1}(n)\left[1 - \frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n) + \frac{k}{6}\left(k^2+3k+2\right)\sum_{i=0}^{q}\omega_i(n)\right]$$
$$+O\left(\frac{\varphi^{k+1}(n)}{n^3}\right) \tag{A.60}$$

holds for $n \to \infty$.

Proof. Using formula (A.20), we get

$$\begin{aligned}\prod_{p=0}^{k} V^{+}(n+p) &= \prod_{p=0}^{k}\left[\varphi(n)\left[k+1+\Sigma^{+}(p)\alpha(n)+S^{+}(p)\sum_{i=0}^{q}\omega_i(n)\right]+O\left(\frac{\varphi(n)}{n^3}\right)\right] \\ &= \varphi^{k+1}(n)\Bigg[(k+1)^{k+1}+(k+1)^{k}\alpha(n)\sum_{i=0}^{k}\Sigma^{+}(i) \\ &\quad + (k+1)^{k-1}\alpha^2(n)\prod_{\substack{i,j=0\\ i>j}}^{k}\Sigma^{+}(i)\Sigma^{+}(j) \\ &\quad +(k+1)^{k}\sum_{i=0}^{k}S^{+}(i)\sum_{j=0}^{q}\omega_j(n)\Bigg]+O\left(\frac{\varphi^{k+1}(n)}{n^3}\right) = (*).\end{aligned} \tag{A.61}$$

Now, by (A.32), (A.39), and (A.52), we derive

$$\begin{aligned}(*) &= \varphi^{k+1}(n)\Bigg[(k+1)^{k+1}-(k+1)^{k-1}\frac{(k+1)^2k}{24}\left(k^2+3k+2\right)\alpha^2(n) \\ &\quad +(k+1)^{k}\frac{(k+1)k}{6}\left(k^2+3k+2\right)\sum_{j=0}^{q}\omega_j(n)\Bigg]+O\left(\frac{\varphi^{k+1}(n)}{n^3}\right) \\ &= (k+1)^{k+1}\varphi^{k+1}(n)\left[1-\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)+\frac{k}{6}\left(k^2+3k+2\right)\sum_{j=0}^{q}\omega_j(n)\right] \\ &\quad +O\left(\frac{\varphi^{k+1}(n)}{n^3}\right).\end{aligned} \tag{A.62}$$

□

Acknowledgment

This research was supported by Grants P201/10/1032 and P201/11/0768 of the Czech Grant Agency (Prague), by the Council of Czech Government MSM 00216 30503 and by the project FEKT-S-11-2(921) Brno University of Technology.

References

[1] R. P. Agarwal, M. Bohner, and W.-T. Li, *Nonoscillation and Oscillation: Theory for Functional Differential Equations*, vol. 267 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2004.

[2] R. P. Agarwal and A. Zafer, "Oscillation criteria for second-order forced dynamic equations with mixed nonlinearities," *Advances in Difference Equations*, vol. 2009, Article ID 938706, 2009 pages, 2009.

[3] J. Baštinec, L. Berezansky, J. Diblík, and Z. Šmarda, "On the critical case in oscillation for differential equations with a single delay and with several delays," *Abstract and Applied Analysis*, vol. 2010, Article ID 417869, 20 pages, 2010.

[4] J. Baštinec, J. Diblík, and Z. Šmarda, "Existence of positive solutions of discrete linear equations with a single delay," *Journal of Difference Equations and Applications*, vol. 16, no. 9, pp. 1047–1056, 2010.

[5] J. Baštinec, J. Diblík, and Z. Šmarda, "Oscillation of solutions of a linear second-order discretedelayed equation," *Advances in Difference Equations*, vol. 2010, Article ID 693867, 12 pages, 2010.

[6] L. Berezansky and E. Braverman, "On existence of positive solutions for linear difference equations with several delays," *Advances in Dynamical Systems and Applications*, vol. 1, no. 1, pp. 29–47, 2006.

[7] L. Berezansky and E. Braverman, "Oscillation of a logistic difference equation with several delays," *Advances in Difference Equations*, vol. 2006, Article ID 82143, 12 pages, 2006.

[8] M. Bohner, B. Karpuz, and Ö. Öcalan, "Iterated oscillation criteria for delay dynamic equations of first order," *Advances in Difference Equations*, vol. 2008, Article ID 458687, 12 pages, 2008.

[9] G. E. Chatzarakis, R. Koplatadze, and I. P. Stavroulakis, "Oscillation criteria of first order linear difference equations with delay argument," *Nonlinear Analysis*, vol. 68, no. 4, pp. 994–1005, 2008.

[10] J. Diblík, "Positive and oscillating solutions of differential equations with delay in critical case," *Journal of Computational and Applied Mathematics*, vol. 88, no. 1, pp. 185–202, 1998.

[11] J. Diblík and N. Koksch, "Positive solutions of the equation $\dot{x}(t) = -c(t)x(t-\tau)$ in the critical case," *Journal of Mathematical Analysis and Applications*, vol. 250, no. 2, pp. 635–659, 2000.

[12] J. Diblík, Z. Svoboda, and Z. Šmarda, "Explicit criteria for the existence of positive solutions for a scalar differential equation with variable delay in the critical case," *Computers & Mathematics with Applications*, vol. 56, no. 2, pp. 556–564, 2008.

[13] Y. Domshlak, "Oscillation properties of discrete difference inequalities and equations: the new approach," in *Functional-Differential Equations*, vol. 1 of *Functional Differential Equations Israel Seminar*, pp. 60–82, Coll. Judea Samaria, Ariel, Israel, 1993.

[14] Y. Domshlak and I. P. Stavroulakis, "Oscillations of first-order delay differential equations in a critical state," *Applicable Analysis*, vol. 61, no. 3-4, pp. 359–371, 1996.

[15] B. Dorociaková and R. Olach, "Existence of positive solutions of delay differential equations," *Tatra Mountains Mathematical Publications*, vol. 43, pp. 63–70, 2009.

[16] S. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 3rd edition, 2005.

[17] Á. Elbert and I. P. Stavroulakis, "Oscillation and nonoscillation criteria for delay differential equations," *Proceedings of the American Mathematical Society*, vol. 123, no. 5, pp. 1503–1510, 1995.

[18] L. H. Erbe and B. G. Zhang, "Oscillation of discrete analogues of delay equations," *Differential and Integral Equations*, vol. 2, no. 3, pp. 300–309, 1989.

[19] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, NY, USA, 1991.

[20] L. Hanuštiaková and R. Olach, "Nonoscillatory bounded solutions of neutral differential systems," *Nonlinear Analysis*, vol. 68, no. 7, pp. 1816–1824, 2008.

[21] L. K. Kikina and I. P. Stavroulakis, "A survey on the oscillation of solutions of first order delay difference equations," *Cubo*, vol. 7, no. 2, pp. 223–236, 2005.

[22] G. Ladas, Ch. G. Philos, and Y. G. Sficas, "Sharp conditions for the oscillation of delay difference equations," *Journal of Applied Mathematics and Simulation*, vol. 2, no. 2, pp. 101–111, 1989.

[23] R. Medina and M. Pituk, "Nonoscillatory solutions of a second-order difference equation of Poincaré type," *Applied Mathematics Letters*, vol. 22, no. 5, pp. 679–683, 2009.

[24] V. E. Slyusarchuk, "Necessary and sufficient conditions for the oscillation of solutions of nonlinear differential equations with impulse action in a Banach space," *Ukrainian Mathematical Journal*, vol. 51, no. 1, pp. 98–109, 1999.

[25] I. P. Stavroulakis, "Oscillation criteria for first order delay difference equations," *Mediterranean Journal of Mathematics*, vol. 1, no. 2, pp. 231–240, 2004.

[26] X. H. Tang and J. S. Yu, "Oscillations of delay difference equations in a critical state," *Applied Mathematics Letters*, vol. 13, no. 2, pp. 9–15, 2000.

[27] X. H. Tang and J. S. Yu, "Oscillation of delay difference equations," *Computers & Mathematics with Applications*, vol. 37, pp. 11–20, 1999.

[28] B. G. Zhang and S. S. Cheng, "Oscillation criteria and comparison theorems for delay difference equations," *Fasciculi Mathematici*, no. 25, pp. 13–32, 1995.

[29] Y. Zhou and B. G. Zhang, “Oscillations of delay difference equations in a critical state,” *Computers & Mathematics with Applications*, vol. 39, no. 7-8, pp. 71–80, 2000.
[30] R. Y. Zhang, Z. C. Wang, and X. Zou, “Oscillation of delay difference equations: near a critical case,” *Dynamics of Continuous, Discrete & Impulsive Systems. Series A. Mathematical Analysis*, vol. 9, no. 1, pp. 89–103, 2002.
[31] L. Berg, “On the asymptotics of nonlinear difference equations,” *Zeitschrift für Analysis und Ihre Anwendungen*, vol. 21, no. 4, pp. 1061–1074, 2002.
[32] L. Berg, “Inclusion theorems for non-linear difference equations with applications,” *Journal of Difference Equations and Applications*, vol. 10, no. 4, pp. 399–408, 2004.
[33] L. Berg, “On the asymptotics of the difference equation $x_{n+1} = x_n x_{n-2} - 1$,” *Journal of Difference Equations and Applications*, vol. 14, no. 1, pp. 105–108, 2008.
[34] B. Iričanin and S. Stević, “Eventually constant solutions of a rational difference equation,” *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 854–856, 2009.
[35] C. M. Kent, W. Kosmala, and S. Stevič, “On the difference equation,” *Abstract and Applied Analysis*, vol. 2011, Article ID 815285, 15 pages, 2011.
[36] S. Stević, “On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$,” *Taiwanese Journal of Mathematics*, vol. 6, no. 3, pp. 405–414, 2002.
[37] S. Stević, “Global stability and asymptotics of some classes of rational difference equations,” *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 60–68, 2006.
[38] S. Stević, “On monotone solutions of some classes of difference equations,” *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 53890, 9 pages, 2006.
[39] S. Stević, “On positive solutions of a$(k+1)$-th order difference equation,” *Applied Mathematics Letters*, vol. 19, no. 5, pp. 427–431, 2006.
[40] S. Stević, “Asymptotics of some classes of higher-order difference equations,” *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 56813, 20 pages, 2007.
[41] S. Stević, “Existence of nontrivial solutions of a rational difference equation,” *Applied Mathematics Letters*, vol. 20, no. 1, pp. 28–31, 2007.
[42] S. Stevich, “Nontrivial solutions of higher-order rational difference equations,” *Rossiĭskaya Akademiya Nauk*, vol. 84, no. 5, pp. 772–780, 2008.
[43] H. B. Dwight, *Tables of Integrals and Other Mathematical Data*, The Macmillan Company, New York, NY, USA, 4th edition, 1961.