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Research Article

Bifurcations of Traveling Wave Solutions for the Coupled Higgs Field Equation

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By using the bifurcation theory of dynamical systems, we study the coupled Higgs field equation and the existence of new solitary wave solutions, and uncountably infinite many periodic wave solutions are obtained. Under different parametric conditions, various sufficient conditions to guarantee the existence of the above solutions are given. All exact explicit parametric representations of the above waves are determined.

1. Introduction

Recently, by using an algebraic method, Hon and Fan [1] studied the following coupled Higgs field equation:

$$u_{tt} - u_{xx} - \alpha u + \beta |u|^2 u - 2uv = 0, \qquad v_{tt} + v_{xx} - \beta\left(|u|^2\right)_{xx} = 0. \tag{1.1}$$

The Higgs field equation [2] describes a system of conserved scalar nucleons interacting with neutral scalar mesons. Here, real constant v represents a complex scalar nucleon field and $u(x,t)$ a real scalar meson field. Equation (1.1) is the coupled nonlinear Klein-Gordon equation for $\alpha < 0,\ \beta < 0$ and the coupled Higgs field equation for $\alpha > 0,\ \beta > 0$. The existence of N-soliton solutions for (1.1) has been shown by the Hirota bilinear method [3].

It is very important to consider the bifurcation behavior for the traveling wave solutions of (1.1). In this paper, we consider (1.1) and its traveling wave solutions in the form of

$$u(x,t) = \phi(\xi) e^{i\eta(\xi)}, \quad v(x,t) = v(\xi), \quad \xi = x - ct. \tag{1.2}$$

Substitute (1.2) into (1.1) and for $c^2-1\neq 0$ reduce system (1.1) to the following system of ordinary differential equations:

$$\begin{gathered}\left(c^2-1\right)\phi''-\left(c^2-1\right)\phi\left(\eta'\right)^2-\alpha\phi+\beta\phi^3-2\phi v=0,\\ 2\phi'\eta'+\phi\eta''=0,\\ \left(c^2+1\right)v''-\beta\left(\phi^2\right)''=0,\end{gathered} \tag{1.3}$$

where "′" is the derivative with respect to ξ. Integrating second equation of (1.3) once and integrating third equation of (1.3) twice, respectively, we have

$$\eta'=\frac{g_2}{\phi^2},\qquad v=\frac{\beta\phi^2+g_1}{c^2+1}, \tag{1.4}$$

where $g_2\neq 0$, g_1 are integral constants. Substituting (1.4) into first equation of (1.3), we have

$$\left(c^2-1\right)\phi''-\left(c^2-1\right)\frac{g_2^2}{\phi^3}-\left(\alpha+\frac{2g_1}{c^2+1}\right)\phi+\frac{\beta(c^2-1)}{c^2+1}\phi^3=0. \tag{1.5}$$

Equation (1.5) is equivalent to the two-dimensional systems as follows:

$$\frac{d\phi}{d\xi}=y,\qquad \frac{dy}{d\xi}=a\left[\phi^3+b\phi+e\phi^{-3}\right] \tag{1.6}$$

with the first integral

$$y^2=a\left(\frac{1}{2}\phi^4+b\phi^2-e\phi^{-2}+h\right), \tag{1.7}$$

$$H(\phi,y)=\frac{y^2}{a}-\frac{1}{2}\phi^4-b\phi^2+e\phi^{-2}=h, \tag{1.8}$$

where $a=-\beta/(c^2+1)$, $b=-(\alpha(c^2+1)+2g_1)/\beta(c^2-1)$, $e=-g_2^2(c^2+1)/\beta\neq 0$, $ae>0$.

System (1.6) is a 3-parameter planar dynamical system depending on the parameter group $(a,\ b,\ e)$. For a fixed a, we will investigate the bifurcations of phase portraits of (1.6) in the phase plane (ϕ,y) as the parameters b,e are changed. Here we are considering a physical model where only bounded traveling waves are meaningful. So we only pay attention to the bounded solutions of (1.6).

Suppose that $\phi(\xi)$ is a continuous solution of (1.6) for $\xi\in(-\infty,\infty)$ and $\lim_{\xi\to\infty}\phi(\xi)=a_1$, $\lim_{\xi\to-\infty}\phi(\xi)=a_2$. Recall that (i) $\phi(x,t)$ is called a solitary wave solution if $a_1=a_2$; (ii) $\phi(x,t)$ is called a kink or antikink solution if $a_1\neq a_2$. Usually, a solitary wave solution of (1.6) corresponds to a homoclinic orbit of (1.6); a kink (or antikink) wave solution (1.6) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of (1.6). Similarly, a periodic orbit of (1.6) corresponds to a periodically traveling wave solution of (1.6). Thus, to investigate all possible bifurcations of solitary waves and periodic waves of (1.6), we need to find all periodic annuli and homoclinic orbits of (1.6), which depend on the system parameters. The bifurcation theory of dynamical systems (see [4–11]) plays an important role in our study.

The paper is organized as follows. In Section 2, we discuss bifurcations of phase portraits of (1.6), where explicit parametric conditions will be derived. In Section 3, all explicit parametric representations of bounded traveling wave solutions are given. Section 4 contains the concluding remarks.

2. Bifurcations of Phase Portraits of (1.6)

In this section, we study all possible periodic annuluses defined by the vector fields of (1.6) when the parameters b, e are varied.

Let $d\xi = \phi^3 d\zeta$. Then, except on the straight lines $\phi = 0$, the system (1.6) has the same topological phase portraits as the following system:

$$\frac{d\phi}{d\zeta} = \phi^3 y, \qquad \frac{dy}{d\zeta} = a\left[\phi^6 + b\phi^4 + e\right]. \tag{2.1}$$

Now, the straight lines $\phi = 0$ is an integral invariant straight line of (2.1).

Denote that

$$f(\phi) = \phi^6 + b\phi^4 + e, \qquad f'(\phi) = 2\phi^3\left(3\phi^2 + 2b\right). \tag{2.2}$$

When $\phi = \phi_\pm = \pm\sqrt{-2b/3}$, $f'(\phi_\pm) = 0$. We have

$$f(\phi_\pm) = \frac{4b^3}{27} + e, \tag{2.3}$$

which implies the relations in the (b, e)-parameter plane

$$L : e = -\frac{4b^3}{27}. \tag{2.4}$$

Thus, we have the following.

(i) If $f(\phi_\pm) < 0$, $f(0) > 0$, there exist 4 equilibrium points of (2.1): $\phi_1 < \phi_2 \leq 0 < \phi_3 < \phi_4$.

(ii) If $f(\phi_\pm) < 0$, $f(0) < 0$, there exist 2 equilibrium points of (2.1): $\phi_1 \leq 0 < \phi_2$.

(iii) If $f(\phi_\pm) > 0$, $f(0) > 0$, there exist no equilibrium points of (2.1).

Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point (ϕ_e, y_e). Then, we have

$$J(\phi_e, 0) = \det(M(\phi_e, 0)) = a\phi_e^3 f'(\phi_e) = -2a\phi_e^6\left(3\phi_e^2 + 2b\right). \tag{2.5}$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$ and Trace $(M(\phi_e, y_e)) = 0$, then it is a center point; if $J > 0$ and (Trace $(M(\phi_e, y_e))^2 - 4J(\phi_e, y_e) > 0$ then it is a node; if $J = 0$ and the index of the equilibrium point is 0, then it is a cusp; otherwise, it is a high-order equilibrium point.

For the function defined by (1.8), we denote that

$$h_i = H(\phi_i, 0) = -\frac{1}{2}\left(3b\phi_i^2 - e\phi_i^{-2}\right), \quad i = 1\text{–}4. \tag{2.6}$$

We next use the above statements to consider the bifurcations of the phase portraits of (2.1). In the (b, e)-parameter plane, the curves L and the straight line $e = 0$ partition it into 4 regions shown in Figure 1.

We use Figures 2 and 3 to show the bifurcations of the phase portraits of (2.1). Notice that for $a > 0$, $e < 0$, $(b, e) \in (III) \bigcup (IV)$ or for $a < 0$, $e > 0$, $(b, e) \in (I) \bigcup (II)$, and we have $ae < 0$, So that we would not give the phase portrait of (2.1) for these cases.

Case 1 $(a > 0)$. We use Figure 2 to show the bifurcations of the phase portraits of (2.1).

Case 2 $(a < 0)$. We use Figure 3 to show the bifurcations of the phase portraits of (2.1).

3. Exact Explicit Parametric Representations of Traveling Wave Solutions of (1.6)

In this section, we give all exact explicit parametric representations of solitary wave solutions and periodic wave solutions. Denote that $sn(x, k)$ is the Jacobian elliptic functions with the modulus k and $\prod(\varphi, \alpha^2, k)$ is Legendre's incomplete elliptic integral of the third kind (see [12]).

(1) Suppose that $a > 0$, $(b, e) \in (II)$. Notice that $H(\phi_1, 0) = -(1/2)\phi_1^4 - b\phi_1^2 + e\phi_1^{-2} = h_1$, corresponding to $H(\phi, y) = h_1$ defined by (1.8), and we see from (1.6) that the arch curve connects $A(\phi_1, 0)$ (see Figure 2(b)). The arch curve has the algebraic equation

$$\begin{aligned} y^2 &= a\left(\frac{1}{2}\phi^4 + b\phi^2 - e\phi^{-2} + h_1\right) \\ &= a\left(\phi^2 - \psi_3\right)^2\left[\frac{1}{2} + (b + \psi_3)\phi^{-2}\right], \end{aligned} \tag{3.1}$$

where $\psi_3 > \psi_2 > 0 > \psi_1$ satisfies the equation

$$\psi^3 + b\psi^2 + e = 0. \tag{3.2}$$

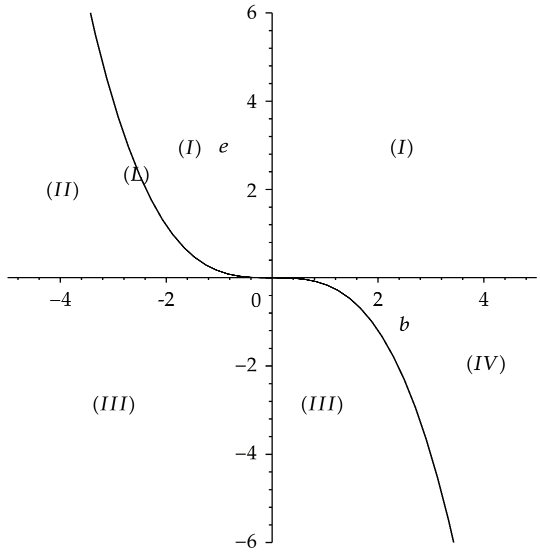
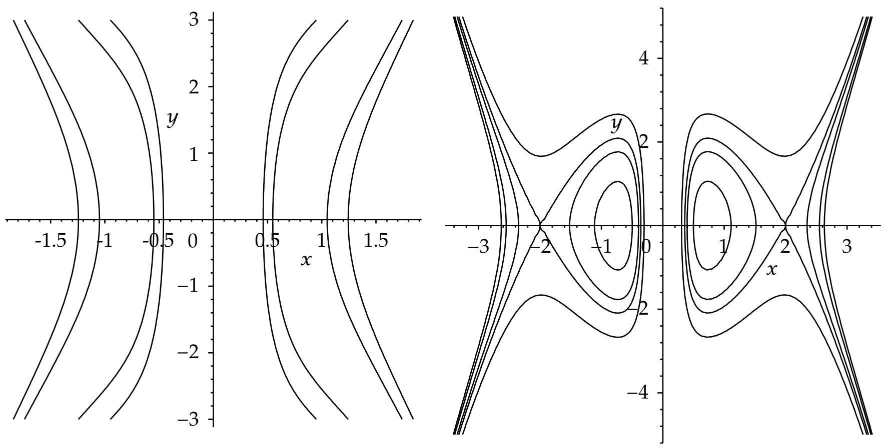


Figure 1: The bifurcation set of (1.6) in (b, e)-parameter plane.

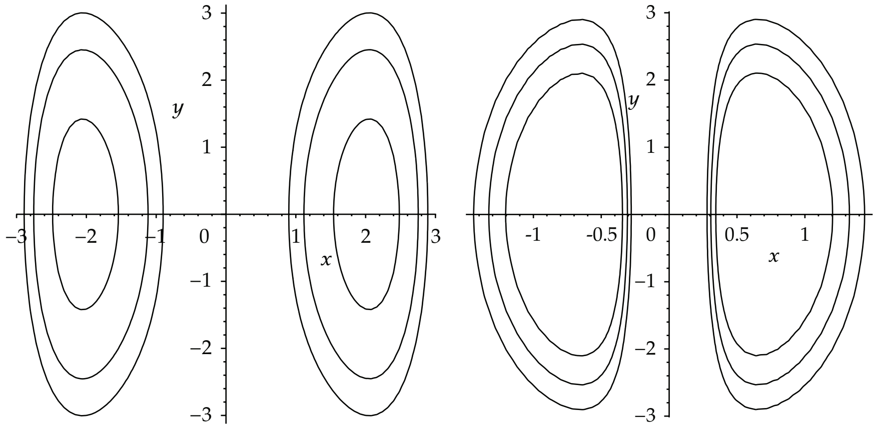


(a) $(b, e) \in (I)$ (b) $(b, e) \in (II)$

Figure 2: The phase portraits of (2.1) for $a > 0$.

By using the first equations of (1.6) and (3.1), we obtain the parametric representation of (1.6), a smooth solitary wave solution of valley type and a smooth solitary wave solution of peak type as follows:

$$\phi(\xi) = \pm\sqrt{-2(b+\varphi_3) + 2\left(b + \frac{3}{2}\varphi_3\right)\tanh^2\sqrt{a\left(b+\frac{3}{2}\varphi_3\right)}\xi}. \tag{3.3}$$



(a) $(b, e) \in (III)$ (b) $(b, e) \in (IV)$

Figure 3: The phase portraits of (2.1) for $a < 0$.

Thus, (1.1) has the following solitary wave solution of valley type and a solitary wave solution of peak type as follows:

$$u_1 = \pm\sqrt{-2(b+\varphi_3) + 2\left(b + \frac{3}{2}\varphi_3\right)\tanh^2\sqrt{a\left(b+\frac{3}{2}\varphi_3\right)}\xi e^{i\eta_1(\xi)}},$$

$$v_1 = \frac{-2\beta\left[(b+\varphi_3) - (\mathrm{b} + (3/2)\varphi_3)\ \tanh^2\sqrt{a(b+(3/2)\varphi_3)}\xi\right] + g_1}{\mathrm{c}^2+1}, \tag{3.4}$$

$$\eta_1 = \frac{g_2}{\varphi_3}\left[\xi + \sqrt{\frac{-1}{a(b+\varphi_3)}}\arctan\left(\sqrt{-\frac{b+(3/2)\varphi_3}{b+\varphi_3}}\tanh^2\sqrt{a(b+\varphi_3)}\xi\right)\right].$$

(2) Suppose that $a < 0$, $(b, e) \in (III)\bigcup(IV)$. Notice that $H(\phi_1, 0) = H(\phi_2 = -\phi_1, 0) = -(1/2)\phi_1^4 - b\phi_1^2 + e\phi_1^{-2} = h_1$, corresponding to $H(\phi, y) = h$, $h \in (-\infty, h_1)$ defined by (1.8), and system (1.6) has two families of periodic solutions enclosing the center $A_+(\phi_1, 0)$ and $A_-(-\phi_1, 0)$, respectively. These orbits determine uncountably infinite many periodic wave solutions of (1.1) (see Figures 3(a) and 3(b)). These orbits have the algebraic equation

$$y = \pm\sqrt{\frac{a}{\phi^2}\left(\frac{1}{2}\phi^6 + b\phi^4 - e + h\phi^2\right)}. \tag{3.5}$$

Integrating them along the periodic orbits, it follows that

$$\int\frac{\phi d\phi}{\sqrt{-\phi^6 - 2b\phi^4 - 2h\phi^2 + 2e}} = \pm\sqrt{\frac{-a}{2}}\xi. \tag{3.6}$$

Substituting $\phi^2 = \varphi$ into (3.6), we have

$$\int \frac{d\varphi}{\sqrt{(\varphi_M - \varphi)(\varphi - \varphi_l)(\varphi - \varphi_m)}} = \pm\frac{1}{2}\sqrt{\frac{-a}{2}}\xi, \tag{3.7}$$

where $\varphi_M > \varphi_l > 0 > \varphi_m$. From (3.7), we have

$$\phi = \pm\sqrt{\frac{\varphi_l(\varphi_M - \varphi_m) - \varphi_m(\varphi_M - \varphi_l)sn^2(\Omega_1\xi, k_1)}{\varphi_l(\varphi_M - \varphi_m)(1 - (\varphi_M - \varphi_l)sn^2(\Omega_1\xi, k_1))}}, \tag{3.8}$$

where $\Omega_1 = (1/4)\sqrt{-a(\varphi_M - \varphi_m)/2}$, $k_1^2 = (\varphi_M - \varphi_l)/(\varphi_M - \varphi_m)$.

Thus, (1.1) has the following uncountably infinite many periodic wave solutions as follows:

$$\begin{gathered} u_2 = \pm\sqrt{\frac{\varphi_l(\varphi_M - \varphi_m) - \varphi_m(\varphi_M - \varphi_l)sn^2(\Omega_1\xi, k_1)}{\varphi_l(\varphi_M - \varphi_m)(1 - (\varphi_M - \varphi_l)sn^2(\Omega_1\xi, k_1))}}e^{i\eta_2(\xi)}, \\ v_2 = \frac{1}{c^2+1}\left[\beta\frac{\varphi_l(\varphi_M - \varphi_m) - \varphi_m(\varphi_M - \varphi_l)sn^2(\Omega_1\xi, k_1)}{\varphi_l(\varphi_M - \varphi_m)(1 - (\varphi_M - \varphi_l)sn^2(\Omega_1\xi, k_1))} + g_1\right], \\ \eta_2 = \frac{g_2}{\alpha^2\Omega_1}\left[\left(\alpha^2 - \alpha_1^2\right)\prod\left(\varphi, \alpha^2, k_1\right) + \alpha_1^2\xi\right], \end{gathered} \tag{3.9}$$

where $\alpha_1^2 = \varphi_M - \varphi_l$, $\alpha^2 = \varphi_m(\varphi_M - \varphi_l)/\varphi_l(\varphi_M - \varphi_m)$, $\varphi = am\xi$.

4. Conclusion

In this paper, we have considered all traveling wave solutions for the coupled Higgs field equation (1.1) in its parameter space, by using the method of dynamical systems. We obtain all parametric representations for solitary wave solutions and uncountably infinite many periodic wave solutions of (1.1) in different parameter regions of the parameter space.

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