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Research Article

Part-Metric and Its Applications to Cyclic Discrete Dynamic Systems

Wanping Liu,[1] Xiaofan Yang,[1] Stevo Stević,[2] and Bratislav Iričanin[3]

[1] *College of Computer Science, Chongqing University, Chongqing 400044, China*
[2] *Mathematical Institute of the Serbian Academy of Sciences and Arts, Knez Mihailova 36/III, 11000 Beograd, Serbia*
[3] *Faculty of Electrical Engineering, Bulevar Kralja Aleksandra 73, 11120 Beograd, Serbia*

Correspondence should be addressed to Wanping Liu, wanping_liu@yahoo.cn and Stevo Stević, sstevic@ptt.rs

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We adapt the part metric and use it in studying positive solutions of a certain family of discrete dynamic systems. Some examples are presented, and we also compare some results in the literature.

1. Introduction

There has been an increasing interest in studying discrete dynamic systems recently (see, e.g., [1–37]). For some recent papers on the systems of difference equations which are not derived from differential equations, see, for example, [4, 12, 14, 15], and the related references therein. In particular, in [4], were considered some cyclic systems of difference equations for the first time. Motivated by [4], in [12], the global attractivity of four k-dimensional systems of higher-order difference equations with two or three delays was investigated. The results in [12] can be easily extended to the corresponding systems with arbitrary number of delays by using the main results in [28].

In [9], the authors used Thompson's *part-metric* [32] to investigate the behaviour of positive solutions to a difference equation from the William Lowell Putman Mathematical Competition [33] by applying a result on discrete dynamic systems in finite dimensional complete metric spaces. Further investigations devoted to applying various part-metric-related inequalities and some asymptotic methods in order to study (scalar) difference equations related to the equation in [33] can be found, for example, in [1, 3, 5, 18–22, 34–37] (see also the related references therein).

In this paper, we adapt the part-metric and apply it in studying of the behaviour of positive solutions to the following family of discrete dynamic systems

$$Y_n = \Phi\left(Y_{n-k_1}, Y_{n-k_2}, \dots, Y_{n-k_q}\right), \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $q \in \mathbb{N} \setminus \{1\}$, $1 \le k_1 < k_2 < \cdots < k_q$, $k_1, \dots, k_q \in \mathbb{N}$, $Y_n = (y_n^{(1)}, y_n^{(2)}, \dots, y_n^{(q)})^T$, $Y_{-k_q}, Y_{-k_q+1}, \dots, Y_{-1}$ are positive initial vectors and $\Phi : \mathbb{R}_+^{q\times q} \to \mathbb{R}_+^q$ is a continuous mapping which will be specified later.

In Section 2, we present some preliminary results which will be applied in the proofs of main results, given in Section 3. Some applications of the main result are given in Sections 4 and 5. In Section 6, we show that some recent results follow from a result in [9].

2. Auxiliary Results

Let $\mathbb{R}$ be the whole set of reals and let $\mathbb{R}_+ = (0, +\infty)$. Denote by $\mathbb{R}_+^n$ the set of all positive n-dimensional vectors and by $\mathbb{R}_+^{m\times n}$ the set of all $m \times n$ matrices with positive entries, that is, $\mathbb{R}_+^{m\times n} = \{(a_{ij})_{m\times n} \mid a_{ij} \in \mathbb{R}_+\}$, $m, n \in \mathbb{N}$.

The following theorem was proved in [9, Theorem 1].

Theorem 2.1. *Let (M, d) be a complete metric space, where d denotes a metric and M is an open subset of $\mathbb{R}^n$, and let $\mathcal{T} : M \to M$ be a continuous mapping with the unique equilibrium $x^* \in M$. Suppose that for the discrete dynamic system*

$$x_{n+1} = \mathcal{T}x_n, \quad n \in \mathbb{N}_0, \tag{2.1}$$

there is a $k \in \mathbb{N}$ such that for the kth iterate of $\mathcal{T}$, the next inequality holds

$$d\left(\mathcal{T}^k x, x^*\right) < d(x, x^*) \tag{2.2}$$

for all $x \ne x^*$. Then, x^* is globally asymptotically stable with respect to metric d.

The part-metric (see [9, 32]) is a metric defined on $\mathbb{R}_+^n$ by

$$p(X, Y) = \log_2 \max_{1\le i\le n} \left\{ \frac{x_i}{y_i}, \frac{y_i}{x_i} \right\}, \tag{2.3}$$

for arbitrary vectors $X = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}_+^n$ and $Y = (y_1, y_2, \dots, y_n)^T \in \mathbb{R}_+^n$.

Recall that the part-metric p has the following properties [9, 10]:

(1) p is a continuous metric on $\mathbb{R}_+^n$,

(2) $(\mathbb{R}_+^n, p)$ is a complete metric space,

(3) the distances induced by the part-metric and by the Euclidean norm $\|\cdot\|$ are equivalent on $\mathbb{R}_+^n$.

Based on these properties and Theorem 2.1, Kruse and Nesemann in [9] obtained the following result.

Lemma 2.2 (see [9, Corollary 2]). *Let* $\mathcal{T} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ *be a continuous mapping with a unique equilibrium* $x^* \in \mathbb{R}_+^n$*. Suppose that for the discrete dynamic system* (2.1) *there is some* $k \in \mathbb{N}$ *such that for the part-metric p inequality* $p(\mathcal{T}^k x, x^*) < p(x, x^*)$ *holds for all* $x \neq x^*$*. Then,* x^* *is globally asymptotically stable.*

Our idea is to adapt the part-metric to matrices. For any two matrices with positive entries $\mathbf{A} = (a_{ij})_{m\times n} \in \mathbb{R}_+^{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n} \in \mathbb{R}_+^{m\times n}$, we define the part-metric in the following natural way:

$$\mathcal{P}(\mathbf{A}, \mathbf{B}) = \log_2 \max_{1\le i\le m, 1\le j\le n} \left\{ \frac{a_{ij}}{b_{ij}}, \frac{b_{ij}}{a_{ij}} \right\}. \tag{2.4}$$

Note that an $m \times n$ matrix $(a_{ij})_{m\times n}$ is equivalent to a vector with mn elements, such as

$$(a_{ij})_{m\times n} \longleftrightarrow (a_{1,1}, a_{2,1}, \dots, a_{m,1}, a_{1,2}, a_{2,2}, \dots, a_{m,2}, \dots, a_{1,n}, a_{2,n}, \dots, a_{m,n})^T. \tag{2.5}$$

Thus, for the above matrices **A** and **B**, we have that

$$\mathcal{P}(\mathbf{A}, \mathbf{B}) = p\left(\left(A_1^T, A_2^T, \dots, A_n^T\right)^T, \left(B_1^T, B_2^T, \dots, B_n^T\right)^T\right) = \max_{1\le j\le n} \{p(A_j, B_j)\}, \tag{2.6}$$

where $A_j = (a_{1j}, a_{2j}, \dots, a_{mj})^T$, $B_j = (b_{1j}, b_{2j}, \dots, b_{mj})^T$, $j = 1, 2, \dots, n$.

From this and the above-mentioned properties for the part-metric we have the following:

(1) the part-metric $\mathcal{P}$ is a continuous metric on $\mathbb{R}_+^{m\times n}$,

(2) $(\mathbb{R}_+^{m\times n}, \mathcal{P})$ is a complete metric space,

(3) the distances induced by the part-metric $\mathcal{P}$ and the Euclidean norm are equivalent on $\mathbb{R}_+^{m\times n}$.

From this and by Lemma 2.2, we have that the next result holds.

Theorem 2.3. *Let* $\mathcal{T} : \mathbb{R}_+^{m\times n} \to \mathbb{R}_+^{m\times n}$ *be a continuous mapping with the unique equilibrium* $\mathbf{C} \in \mathbb{R}_+^{m\times n}$*. Suppose that for the discrete dynamic system*

$$\mathbf{X}_{n+1} = \mathcal{T}\mathbf{X}_n, \quad n \in \mathbb{N}_0, \tag{2.7}$$

there is a $k \in \mathbb{N}$ *such that for metric* $\mathcal{P}$*, the inequality* $\mathcal{P}(\mathcal{T}^k\mathbf{X}, \mathbf{C}) < \mathcal{P}(\mathbf{X}, \mathbf{C})$ *holds for each* $\mathbf{X} \neq \mathbf{C}$*. Then,* **C** *is globally asymptotically stable.*

Remark 2.4. Note that if we do not assume in Theorem 2.3 that **C** is the unique equilibrium of (2.7), then if $\widetilde{\mathbf{C}}$ is another equilibrium it must be

$$\mathcal{P}\left(\widetilde{\mathbf{C}}, \mathbf{C}\right) = \mathcal{P}\left(\mathcal{T}^k\widetilde{\mathbf{C}}, \mathbf{C}\right) < \mathcal{P}\left(\widetilde{\mathbf{C}}, \mathbf{C}\right), \tag{2.8}$$

which is impossible. Hence, there is only one equilibrium of (2.7).

3. Main Result

Let $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_q)$ be a square $q \times q$ matrix, where $Y_i = (y_i^{(1)}, y_i^{(2)}, \ldots, y_i^{(q)})^T$, $i = 1, 2, \ldots, q$, and Φ is defined by

$$\Phi(\mathbf{Y}) = \Phi(Y_1, Y_2, \ldots, Y_q) = \begin{pmatrix} \phi\left(y_1^{(1)}, y_2^{(2)}, \ldots, y_q^{(q)}\right) \\ \phi\left(y_1^{(2)}, y_2^{(3)}, \ldots, y_q^{(1)}\right) \\ \vdots \\ \phi\left(y_1^{(q)}, y_2^{(1)}, \ldots, y_q^{(q-1)}\right) \end{pmatrix}, \tag{3.1}$$

where $\phi : \mathbb{R}_+^q \to \mathbb{R}_+$ is a continuous mapping. Clearly, Φ is a continuous mapping and our system becomes

$$Y_n = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \\ \vdots \\ y_n^{(q)} \end{pmatrix} = \begin{pmatrix} \phi\left(y_{n-k_1}^{(1)}, y_{n-k_2}^{(2)}, \ldots, y_{n-k_q}^{(q)}\right) \\ \phi\left(y_{n-k_1}^{(2)}, y_{n-k_2}^{(3)}, \ldots, y_{n-k_q}^{(1)}\right) \\ \vdots \\ \phi\left(y_{n-k_1}^{(q)}, y_{n-k_2}^{(1)}, \ldots, y_{n-k_q}^{(q-1)}\right) \end{pmatrix}, \quad n \in \mathbb{N}_0, \tag{3.2}$$

where $q \in \mathbb{N} \setminus \{1\}$, $1 \le k_1 < k_2 < \cdots < k_q$, $k_1, k_2, \ldots, k_q \in \mathbb{N}$.

As an application of Theorem 2.3, we will establish a theorem regarding the global asymptotic stability of cyclic system of difference equations in (3.2), as follows.

Theorem 3.1. *Consider system* (3.2)*, where $\phi : \mathbb{R}_+^q \to \mathbb{R}_+$, $q \ge 2$ is a continuous mapping. Let $\overline{C} = (c, c, \ldots, c)^T$, $c > 0$, be an equilibrium of* (3.2)*. If for $(x_1, x_2, \ldots, x_q)^T \ne \overline{C}$,*

$$\min_{1 \le i \le q} \left\{ x_i, \frac{c^2}{x_i} \right\} < \phi(x_1, x_2, \ldots, x_q) < \max_{1 \le i \le q} \left\{ x_i, \frac{c^2}{x_i} \right\}, \tag{3.3}$$

then $\overline{C}$ is globally asymptotically stable.

Proof. Define a matrix mapping $\mathcal{T} : \mathbb{R}_+^{q \times k_q} \to \mathbb{R}_+^{q \times k_q}$ such that

$$\mathcal{T}\left(X_1, X_2, \ldots, X_{k_q}\right) = \left(\Phi\left(X_{k_1}, X_{k_2}, \ldots, X_{k_q}\right), X_1, X_2, \ldots, X_{k_q - 1}\right), \tag{3.4}$$

where $X_i = (x_{1i}, x_{2i}, \ldots, x_{qi})^T$, $i = 1, 2, \ldots, k_q$. Then, (3.2) can be converted into the first-order recursive $q \times k_q$ matrix equation

$$\mathbf{M}_n = \mathcal{T}(\mathbf{M}_{n-1}), \quad n \in \mathbb{N}, \tag{3.5}$$

with $\mathbf{M}_0$ initial matrix, with positive entries.

Clearly $\mathbf{C} = (\overline{C}, \overline{C}, \ldots, \overline{C})$ is an equilibrium of (3.5).

Let $(Y_n)_{n=-k_q}^{+\infty}$ be an arbitrary positive solution to (3.2), and denote

$$\mathbf{M}_n = \left(Y_{n-1}, Y_{n-2}, \ldots, Y_{n-k_q}\right), \quad n \in \mathbb{N}_0, \tag{3.6}$$

then we get a matrix sequence $(\mathbf{M}_n)_{n=0}^{\infty}$. Apparently, the matrix sequence $(\mathbf{M}_n)_{n=0}^{\infty}$ is a solution to (3.5).

When $\mathbf{M}_0 = \mathbf{C}$, it is clear that $\mathbf{M}_n = \mathbf{C}$ holds for $n \in \mathbb{N}_0$. Hence, in what follows, we assume that $\mathbf{M}_0 \neq \mathbf{C}$.

Let the relation "$\preceq_j$" be either "=" or "<" for each $j \in \mathbb{N}_0$. Since

$$\mathcal{T}(\mathbf{M}_n) = \mathbf{M}_{n+1} = \left(Y_n, Y_{n-1}, \ldots, Y_{n-k_q+1}\right), \quad n \in \mathbb{N}_0, \tag{3.7}$$

then

$$Y_n = \Phi\left(Y_{n-k_1}, Y_{n-k_2}, \ldots, Y_{n-k_q}\right). \tag{3.8}$$

By (3.2) and condition (3.3), we get that for each $j \in \{1, 2, \ldots, q\}$

$$\begin{aligned} \frac{y_n^{(j)}}{c} &= \frac{\phi\left(y_{n-k_1}^{(j)}, \ldots, y_{n-k_q}^{(\theta(j+q-1))}\right)}{c} \preceq_j \max_{1\le i\le q}\left\{\frac{y_{n-k_i}^{(\theta(j+i-1))}}{c}, \frac{c}{y_{n-k_i}^{(\theta(j+i-1))}}\right\}, \\ \frac{c}{y_n^{(j)}} &= \frac{c}{\phi\left(y_{n-k_1}^{(j)}, \ldots, y_{n-k_q}^{(\theta(j+q-1))}\right)} \preceq_j \max_{1\le i\le q}\left\{\frac{y_{n-k_i}^{(\theta(j+i-1))}}{c}, \frac{c}{y_{n-k_i}^{(\theta(j+i-1))}}\right\}, \end{aligned} \tag{3.9}$$

where $\theta(n) \equiv n \pmod{q}$ with $\theta(q) = q$.

Let $\mathbf{B}_n = (Y_{n-k_1}, Y_{n-k_2}, \ldots, Y_{n-k_q})$.

Case 1. If $\mathbf{B}_n \neq (\overline{C}, \overline{C}, \ldots, \overline{C})$, then there exists at least one $j \in \{1, 2, \ldots, q\}$ such that the relation "$\preceq_j$" in (3.9) is "<". Thus,

$$p\left(Y_n, \overline{C}\right) < \max\left\{p\left(Y_{n-k_1}, \overline{C}\right), p\left(Y_{n-k_2}, \overline{C}\right), \ldots, p\left(Y_{n-k_q}, \overline{C}\right)\right\}, \quad n \in \mathbb{N}_0. \tag{3.10}$$

Case 2. If $\mathbf{B}_n = (\overline{C}, \overline{C}, \ldots, \overline{C})$, then $Y_n = \overline{C}$ and "$\preceq_j$" is always "=" for each $j \in \{1, 2, \ldots, q\}$, which implies that

$$p\left(Y_n, \overline{C}\right) = \max\left\{p\left(Y_{n-k_1}, \overline{C}\right), p\left(Y_{n-k_2}, \overline{C}\right), \ldots, p\left(Y_{n-k_q}, \overline{C}\right)\right\} = 0, \quad n \in \mathbb{N}_0. \tag{3.11}$$

From relations (3.10) and (3.11), we obtain that

$$p\left(Y_n, \overline{C}\right) \preceq_n \max\left\{p\left(Y_{n-k_1}, \overline{C}\right), p\left(Y_{n-k_2}, \overline{C}\right), \ldots, p\left(Y_{n-k_q}, \overline{C}\right)\right\}, \quad n \in \mathbb{N}_0. \tag{3.12}$$

From the following set of inequalities

$$
\begin{aligned}
&p\left(Y_0,\overline{C}\right)\preceq_0 \max\left\{p\left(Y_{-k_1},\overline{C}\right),p\left(Y_{-k_2},\overline{C}\right),\ldots,p\left(Y_{-k_q},\overline{C}\right)\right\},\\
&p\left(Y_1,\overline{C}\right)\preceq_1 \max\left\{p\left(Y_{1-k_1},\overline{C}\right),p\left(Y_{1-k_2},\overline{C}\right),\ldots,p\left(Y_{1-k_q},\overline{C}\right)\right\},\\
&\quad\vdots\\
&p\left(Y_{k_q-1},\overline{C}\right)\preceq_{k_q-1} \max\left\{p\left(Y_{k_q-1-k_1},\overline{C}\right),p\left(Y_{k_q-1-k_2},\overline{C}\right),\ldots,p\left(Y_{k_q-1-k_q},\overline{C}\right)\right\},
\end{aligned}
\tag{3.13}
$$

and since $\mathbf{M}_0 \neq \mathbf{C}$, it follows that there exists at least one index $j \in \{0,1,\ldots,k_q-1\}$ such that the relation "$\preceq_j$" is "$<$", which implies

$$
\max_{0\le i\le k_q-1}\left\{p\left(Y_i,\overline{C}\right)\right\} < \max_{-k_q\le i\le -1}\left\{p\left(Y_i,\overline{C}\right)\right\}.
\tag{3.14}
$$

From the definition of the part-metric, we have that

$$
\mathcal{P}(\mathbf{M}_n,\mathbf{C}) = \log_2 \max_{1\le i\le k_q, 1\le j\le q}\left\{\frac{c}{y_{n-i}^{(j)}},\frac{y_{n-i}^{(j)}}{c}\right\} = \mathcal{P}\left(\left(Y_{n-1},\ldots,Y_{n-k_q}\right),\mathbf{C}\right).
\tag{3.15}
$$

Then, we derive

$$
\begin{aligned}
\mathcal{P}\left(T^{k_q}(\mathbf{M}_0),\mathbf{C}\right) &= \mathcal{P}\left(\mathbf{M}_{k_q},\mathbf{C}\right) = \mathcal{P}\left(\left(Y_{k_q-1},Y_{k_q-2},\ldots,Y_{k_q-k_q}\right),\mathbf{C}\right)\\
&= \max_{0\le i\le k_q-1}\left\{p\left(Y_i,\overline{C}\right)\right\} < \max_{-k_q\le i\le -1}\left\{p\left(Y_i,\overline{C}\right)\right\} = \mathcal{P}(\mathbf{M}_0,\mathbf{C}).
\end{aligned}
\tag{3.16}
$$

Because $\mathbf{M}_0$ is arbitrary and $\mathbf{M}_0 \neq \mathbf{C}$, then by Theorem 2.3 (see also Remark 2.4) we have that $\mathbf{C}$ is a globally asymptotically stable equilibrium of (3.5), which implies that the equilibrium $\overline{C}$ of system (3.2) is globally asymptotically stable, as desired. □

4. On Some Symmetric Discrete Dynamic Systems

For the sake of convenience, first we define two continuous mappings $f, g : \mathbb{R}_+^q \to \mathbb{R}_+$, $q \ge 2$, as follows:

$$
\begin{aligned}
f(x_1,x_2,\ldots,x_q) &= \frac{\prod_{j=1}^{q}\left(x_j^r+1\right)-\prod_{j=1}^{q}\left(x_j^r-1\right)}{\prod_{j=1}^{q}\left(x_j^r+1\right)+\prod_{j=1}^{q}\left(x_j^r-1\right)},\\
g(x_1,x_2,\ldots,x_q) &= \frac{\prod_{j=1}^{q}\left(x_j^r+1\right)+\prod_{j=1}^{q}\left(x_j^r-1\right)}{\prod_{j=1}^{q}\left(x_j^r+1\right)-\prod_{j=1}^{q}\left(x_j^r-1\right)},
\end{aligned}
\tag{4.1}
$$

where r is a real parameter belonging to the interval $(0,1]$.

Many researchers have studied the symmetric difference equation

$$x_n = \phi\left(x_{n-k_1}, x_{n-k_2}, \dots, x_{n-k_q}\right), \quad n \in \mathbb{N}_0, \tag{4.2}$$

where $q \in \mathbb{N} \setminus \{1\}$, $1 \le k_1 < k_2 < \cdots < k_q$, $k_1, k_2, \dots, k_q \in \mathbb{N}$, and $\phi \in \{f, g\}$.

In the following, we mainly investigate the behaviour of positive solutions to the following class of cyclic difference equation systems

$$\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \\ \vdots \\ y_n^{(q)} \end{pmatrix} = \begin{pmatrix} \phi\left(y_{n-k_1}^{(1)}, y_{n-k_2}^{(2)}, \dots, y_{n-k_q}^{(q)}\right) \\ \phi\left(y_{n-k_1}^{(2)}, y_{n-k_2}^{(3)}, \dots, y_{n-k_q}^{(1)}\right) \\ \vdots \\ \phi\left(y_{n-k_1}^{(q)}, y_{n-k_2}^{(1)}, \dots, y_{n-k_q}^{(q-1)}\right) \end{pmatrix}, \quad n \in \mathbb{N}_0, \tag{4.3}$$

where $q \in \mathbb{N} \setminus \{1\}$, $1 \le k_1 < k_2 < \cdots < k_q$, $k_1, k_2, \dots, k_q \in \mathbb{N}$, and $\phi \in \{f, g\}$.

In order to establish the main result concerning (4.3), we need some preliminary lemmas.

Lemma 4.1. *System* (4.3) *has unique positive equilibrium* $(\underbrace{1, 1, \dots, 1}_{q})^T$.

Proof. Let $(c_1, c_2, \dots, c_q)^T$ be an arbitrary positive equilibrium of system (4.3). Since the mappings f and g are both symmetric, then we derive that

$$c_i = \phi(c_1, c_2, \dots, c_q), \quad i = 1, 2, \dots, q, \tag{4.4}$$

from which it follows that $c_i = c > 0$, $i = 1, 2, \dots, q$, and then

$$c = \phi(c, c, \dots, c). \tag{4.5}$$

By Lemma 2.1 in [13], we obtain $c = 1$, as desired. □

Lemma 4.2. *Let a_1 and a_2 be positive real numbers with $(a_1, a_2) \neq (1, 1)$, and $\phi \in \{f, g\}$. Then,*

$$\min\left\{a_1, a_2, \frac{1}{a_1}, \frac{1}{a_2}\right\} < \phi(a_1, a_2) < \max\left\{a_1, a_2, \frac{1}{a_1}, \frac{1}{a_2}\right\}. \tag{4.6}$$

Proof. From the next identities

$$\begin{aligned} x - \frac{x+y}{1+xy} &= \frac{y(x^2-1)}{1+xy}, \qquad & y - \frac{x+y}{1+xy} &= \frac{x(y^2-1)}{1+xy}, \\ \frac{1}{x} - \frac{x+y}{1+xy} &= \frac{1-x^2}{x(1+xy)}, \qquad & \frac{1}{y} - \frac{x+y}{1+xy} &= \frac{1-y^2}{y(1+xy)}, \end{aligned} \tag{4.7}$$

it is easy to see that when $(a_1, a_2) \neq (1,1)$ the following inequalities hold:

$$\min\left\{a_1, a_2, \frac{1}{a_1}, \frac{1}{a_2}\right\} < \frac{a_1 + a_2}{1 + a_1 a_2} < \max\left\{a_1, a_2, \frac{1}{a_1}, \frac{1}{a_2}\right\}. \tag{4.8}$$

Because $r \in (0,1]$, then for the case $\phi = f$, we easily obtain that

$$\begin{aligned}\min\left\{a_1, a_2, \frac{1}{a_1}, \frac{1}{a_2}\right\} &\leq \min\left\{a_1^r, a_2^r, \frac{1}{a_1^r}, \frac{1}{a_2^r}\right\} < f(a_1, a_2)\\ &= \frac{a_1^r + a_2^r}{1 + a_1^r a_2^r} < \max\left\{a_1^r, a_2^r, \frac{1}{a_1^r}, \frac{1}{a_2^r}\right\} \leq \max\left\{a_1, a_2, \frac{1}{a_1}, \frac{1}{a_2}\right\}.\end{aligned} \tag{4.9}$$

The case $\phi = g$ follows immediately from the case $\phi = f$ due to the fact that $fg \equiv 1$. □

Lemma 4.3. *Let $q \geq 2$ be an integer and $\phi \in \{f, g\}$. Let $a_1, a_2, \ldots, a_q$ be positive real numbers with $(a_1, a_2, \ldots, a_q) \neq (1, 1, \ldots, 1)$. Then,*

$$\min_{1\leq i\leq q}\left\{a_i, \frac{1}{a_i}\right\} < \phi(a_1, a_2, \ldots, a_q) < \max_{1\leq i\leq q}\left\{a_i, \frac{1}{a_i}\right\}. \tag{4.10}$$

Proof. For the case $q = 2$, the assertion follows from Lemma 4.2. Next, we argue by the induction and assume that the assertion is true for $q = k$ $(k \geq 2)$. Then, it suffices to prove that the assertion holds when $q = k+1$. Now, let $a_1, a_2, \ldots, a_{k+1}$ be positive real numbers with $(a_1, a_2, \ldots, a_{k+1}) \neq (1, 1, \ldots, 1)$. Consider the following function h in a variable x

$$h(x; a_1, a_2, \ldots, a_k) = \frac{(x^r + 1)\prod_{j=1}^k \left(a_j^r + 1\right) - (x^r - 1)\prod_{j=1}^k \left(a_j^r - 1\right)}{(x^r + 1)\prod_{j=1}^k \left(a_j^r + 1\right) + (x^r - 1)\prod_{j=1}^k \left(a_j^r - 1\right)}, \tag{4.11}$$

where $a_1, a_2, \ldots, a_k$ are arbitrary (but fixed) positive numbers. Clearly,

$$h(a_{k+1}; a_1, a_2, \ldots, a_k) = f(a_1, a_2, \ldots, a_{k+1}). \tag{4.12}$$

The first derivative of the function h regarding the variable x is equal to

$$h'(x; a_1, a_2, \ldots, a_k) = \frac{-4rx^{r-1}\prod_{j=1}^k \left(a_j^{2r} - 1\right)}{\left((x^r + 1)\prod_{j=1}^k \left(a_j^r + 1\right) + (x^r - 1)\prod_{j=1}^k \left(a_j^r - 1\right)\right)^2}. \tag{4.13}$$

In the following, we distinguish three possibilities.

Case 1 ($\prod_{j=1}^{k}(a_j^{2r}-1)<0$). Then, $h'(x;a_1,a_2,\ldots,a_k)>0$ holds for all $x>0$, which implies that the function $h(x;a_1,a_2,\ldots,a_k)$ is strictly increasing in variable x. From this, we obtain

$$\begin{aligned}
h(a_{k+1};a_1,a_2,\ldots,a_k) &< \lim_{x\to+\infty} h(x;a_1,a_2,\ldots,a_k) = \frac{\prod_{j=1}^{k}\left(a_j^r+1\right)-\prod_{j=1}^{k}\left(a_j^r-1\right)}{\prod_{j=1}^{k}\left(a_j^r+1\right)+\prod_{j=1}^{k}\left(a_j^r-1\right)} \\
&= f(a_1,a_2,\ldots,a_k) \le \max_{1\le i\le k}\left\{a_i,\frac{1}{a_i}\right\} \le \max_{1\le i\le k+1}\left\{a_i,\frac{1}{a_i}\right\}, \\
h(a_{k+1};a_1,a_2,\ldots,a_k) &> \lim_{x\to 0^+} h(x;a_1,a_2,\ldots,a_k) = \frac{\prod_{j=1}^{k}\left(a_j^r+1\right)+\prod_{j=1}^{k}\left(a_j^r-1\right)}{\prod_{j=1}^{k}\left(a_j^r+1\right)-\prod_{j=1}^{k}\left(a_j^r-1\right)} \\
&= g(a_1,a_2,\ldots,a_k) \ge \min_{1\le i\le k}\left\{a_i,\frac{1}{a_i}\right\} \ge \min_{1\le i\le k+1}\left\{a_i,\frac{1}{a_i}\right\}.
\end{aligned} \tag{4.14}$$

Case 2 ($\prod_{j=1}^{k}(a_j^{2r}-1)>0$). Then, $h'(x;a_1,a_2,\ldots,a_k)<0$ holds for all $x>0$, implying that the function $h(x;a_1,a_2,\ldots,a_k)$ is strictly decreasing in variable x. From this, we obtain that

$$\begin{aligned}
h(a_{k+1};a_1,a_2,\ldots,a_k) &< \lim_{x\to 0^+} h(x;a_1,a_2,\ldots,a_k) = \frac{\prod_{j=1}^{k}\left(a_j^r+1\right)+\prod_{j=1}^{k}\left(a_j^r-1\right)}{\prod_{j=1}^{k}\left(a_j^r+1\right)-\prod_{j=1}^{k}\left(a_j^r-1\right)} \\
&= g(a_1,a_2,\ldots,a_k) \le \max_{1\le i\le k}\left\{a_i,\frac{1}{a_i}\right\} \le \max_{1\le i\le k+1}\left\{a_i,\frac{1}{a_i}\right\}, \\
h(a_{k+1};a_1,a_2,\ldots,a_k) &> \lim_{x\to+\infty} h(x;a_1,a_2,\ldots,a_k) = \frac{\prod_{j=1}^{k}\left(a_j^r+1\right)-\prod_{j=1}^{k}\left(a_j^r-1\right)}{\prod_{j=1}^{k}\left(a_j^r+1\right)+\prod_{j=1}^{k}\left(a_j^r-1\right)} \\
&= f(a_1,a_2,\ldots,a_k) \ge \min_{1\le i\le k}\left\{a_i,\frac{1}{a_i}\right\} \ge \min_{1\le i\le k+1}\left\{a_i,\frac{1}{a_i}\right\}.
\end{aligned} \tag{4.15}$$

Case 3 ($\prod_{j=1}^{k}(a_j^{2r}-1)=0$). This implies $h(x;a_1,a_2,\ldots,a_k)=1$ for all $x>0$. From this relation and the inspection that the condition $(a_1,a_2,\ldots,a_{k+1})\neq(1,1,\ldots,1)$ implies $\max_{1\le i\le k+1}\{a_i,1/a_i\}>1$ and $\min_{1\le i\le k+1}\{a_i,1/a_i\}<1$, we obtain that

$$\min_{1\le i\le k+1}\left\{a_i,\frac{1}{a_i}\right\} < h(a_{k+1};a_1,a_2,\ldots,a_k) < \max_{1\le i\le k+1}\left\{a_i,\frac{1}{a_i}\right\}. \tag{4.16}$$

Hence, by induction, the assertion immediately holds for $\phi=f$, and then the case $\phi=g$ follows directly from the case $\phi=f$ because $f\cdot g\equiv 1$ □

By Lemmas 4.1 and 4.3 and Theorem 3.1, we obtain the following theorem.

Theorem 4.4. *The unique equilibrium of system* (4.3) *is globally asymptotically stable.*

Remark 4.5. Note that the following two systems (particular cases $q = 2$ and $q = 3$ of the system (4.3), resp.)

$$\begin{aligned} &u_n = \phi(u_{n-k}, v_{n-l}), \\ &v_n = \phi(v_{n-k}, u_{n-l}), \end{aligned} \quad n \in \mathbb{N}_0,$$

$$\begin{aligned} &u_n = \phi(u_{n-k}, v_{n-l}, w_{n-m}), \\ &v_n = \phi(v_{n-k}, w_{n-l}, u_{n-m}), \quad n \in \mathbb{N}_0, \\ &w_n = \phi(w_{n-k}, u_{n-l}, v_{n-m}), \end{aligned} \tag{4.17}$$

where $1 \leq k < l < m$, $\phi \in \{f, g\}$, were studied in [12].

5. On a System of Difference Equations

Let $\mu : \mathbb{R}_+^q \to \mathbb{R}_+$, $q \geq 4$ be a continuous mapping defined by

$$\mu(t_1, t_2, \ldots, t_q) = \frac{\sum_{i=1}^{q-2} t_i + t_{q-1} t_q}{t_1 t_2 + \sum_{i=3}^{q} t_i}. \tag{5.1}$$

Then, the following difference equation

$$x_{n+1} = \mu(x_n, x_{n-1}, x_{n-2}, x_{n-3}), \quad n \in \mathbb{N}_0, \tag{5.2}$$

is an extension of the difference equation in [33], which was studied in [9].

First, we consider the next four-dimensional system of difference equations:

$$\begin{pmatrix} w_n \\ x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} \mu(w_{n-k_1}, x_{n-k_2}, y_{n-k_3}, z_{n-k_4}) \\ \mu(x_{n-k_1}, y_{n-k_2}, z_{n-k_3}, w_{n-k_4}) \\ \mu(y_{n-k_1}, z_{n-k_2}, w_{n-k_3}, x_{n-k_4}) \\ \mu(z_{n-k_1}, w_{n-k_2}, x_{n-k_3}, y_{n-k_4}) \end{pmatrix}, \quad n \in \mathbb{N}_0, \tag{5.3}$$

where $1 \leq k_1 < k_2 < k_3 < k_4$, $k_i \in \mathbb{N}$ for $i \in \{1, 2, 3, 4\}$.

Lemma 5.1. *System* (5.3) *has unique positive equilibrium* $(1, 1, 1, 1)^T$.

Proof. Let $(a, b, c, d)^T$ be an arbitrary positive equilibrium of the system (5.3). Then, we get

$$a = \frac{a + b + cd}{ab + c + d}, \qquad b = \frac{b + c + da}{bc + d + a}, \qquad c = \frac{c + d + ab}{cd + a + b}, \qquad d = \frac{d + a + bc}{da + b + c}, \tag{5.4}$$

which imply

$$ac = 1, \qquad bd = 1. \tag{5.5}$$

Applying (5.5), the system in (5.4) is reduced to

$$a\left(a^2-1\right)b^2+a(1-a)b+a^2-1=0,$$
$$b^3+a(a-1)b^2+(a-1)b-a^2=0. \tag{5.6}$$

If $a = 1$, then it follows from the second identity that $b = 1$. Now, assume $a \in \mathbb{R}_+ \setminus \{1\}$. By solving the first equation in (5.6) with respect to variable b, we get that the discriminant

$$\Delta = a^2(1-a)^2 - 4a\left(a^2-1\right)^2 = a(a-1)^2\left(-4a^2-7a-4\right) < 0, \tag{5.7}$$

which implies the first equation in (5.6) has no real roots. This contradicts $b > 0$. Hence, $a = b = 1$, which along with (5.5) implies $c = d = 1$, finishing the proof. □

The following lemma follows directly from Lemma 3.3 in [34] or the proof of Lemma 4 in [35].

Lemma 5.2. *Let a_1, a_2, a_3, and a_4 be positive real numbers with $(a_1, a_2, a_3, a_4) \neq (1,1,1,1)$. Then,*

$$\min_{1\le i\le 4}\left\{a_i, \frac{1}{a_i}\right\} < \mu(a_1,a_2,a_3,a_4) < \max_{1\le i\le 4}\left\{a_i, \frac{1}{a_i}\right\}. \tag{5.8}$$

By Lemmas 5.1 and 5.2, and Theorem 3.1, we easily derive the next theorem.

Theorem 5.3. *Unique positive equilibrium $(1,1,1,1)^T$ of system (5.3) is globally asymptotically stable.*

In the following, we consider the next q-dimensional ($q \ge 5$) generalization of system (5.3)

$$\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \\ \vdots \\ y_n^{(q)} \end{pmatrix} = \begin{pmatrix} \mu\left(y_{n-k_1}^{(1)}, y_{n-k_2}^{(2)}, \dots, y_{n-k_q}^{(q)}\right) \\ \mu\left(y_{n-k_1}^{(2)}, y_{n-k_2}^{(3)}, \dots, y_{n-k_q}^{(1)}\right) \\ \vdots \\ \mu\left(y_{n-k_1}^{(q)}, y_{n-k_2}^{(1)}, \dots, y_{n-k_q}^{(q-1)}\right) \end{pmatrix}, \quad n \in \mathbb{N}_0, \tag{5.9}$$

where $5 \le q \in \mathbb{N}$, $1 \le k_1 < k_2 < \cdots < k_q$, $k_1, k_2, \dots, k_q \in \mathbb{N}$.

It is easy to see that $(1,1,\dots,1)^T$ is a positive equilibrium of system (5.9), but it is not so easy to confirm its uniqueness as in the proof of Lemma 5.1. However, we have the following lemma which follows directly from Lemmas 3.4 and 3.5 in [34].

Lemma 5.4. *Let r be an integer with $r \geq 4$. Let $a_1, a_2, \dots, a_r$ be positive numbers with $(a_1, a_2, \dots, a_r) \neq (1, 1, \dots, 1)$. Then,*

$$\min_{1\le i\le r}\left\{a_i, \frac{1}{a_i}\right\} < \frac{\sum_{i=1}^{r-2} a_i + a_{r-1}a_r}{a_1a_2 + \sum_{i=3}^{r} a_i} < \max_{1\le i\le r}\left\{a_i, \frac{1}{a_i}\right\}. \tag{5.10}$$

Hence, we can apply Theorem 3.1 to establish the following theorem.

Theorem 5.5. *The positive equilibrium $(1, 1, \dots, 1)^T$ of system (5.9) is globally asymptotically stable.*

6. An Application of a Kruse-Nesseman Result

Numerous papers studied particular cases of (4.2) by using semi-cycle analysis of their solutions. It was shown by Berg and Stević in [1] that this analysis is unnecessarily complicated and useful only for lower-order difference equations. They also described some methods for determining rules of semi-cycles which can be used in many classes of difference equations. On the other hand, it has been noticed in several papers (see, e.g., [18]) that the stability results in many of these papers follow from the following result by Kruse and Nesemann in [9].

Theorem 6.1. *Consider the difference equation*

$$y_n = F(y_{n-1}, \dots, y_{n-m}), \quad n \in \mathbb{N}_0, \tag{6.1}$$

where $F : \mathbb{R}_+^m \to \mathbb{R}_+$ is a continuous function with a unique equilibrium $x^* \in \mathbb{R}_+$. Suppose that there is a $k_0 \in \mathbb{N}$ such that for each solution (y_n) of (6.1),

$$(y_n - y_{n-k_0})\left(y_n - \frac{(x^*)^2}{y_{n-k_0}}\right) \leq 0 \tag{6.2}$$

with equality if and only if $y_n = x^*$. Then, x^* is globally asymptotically stable.

Motivated by [18], in recent paper [2], Berg and Stević also applied Theorem 6.1 by proving the next result, which covers numerous particular cases appearing in the literature. We formulate the proposition here as a useful information to the reader. Before we formulate it we need some notation. Let $S_j = \{1, 2, \dots, j\}$, $j = 1, \dots, k$, let

$$T_r^k := \sum_{\substack{\{t_1, t_2, \dots, t_r\} \subseteq S_k \\ t_1 < t_2 < \cdots < t_r}} x_{t_1} x_{t_2} \cdots x_{t_r}, \tag{6.3}$$

for $r = 0, 1, \dots, k$, where we define $T_0^k = 1$ and $T_{-1}^k = T_k^{k-1} = 0$, and let

$$\sum{}' T_r^k = \sum_{\substack{r=1 \\ r \text{ odd}}}^{k} T_r^k, \qquad \sum{}'' T_r^k = \sum_{\substack{r=0 \\ r \text{ even}}}^{k} T_r^k, \tag{6.4}$$

or reversed, $\sum' T_r^k$ is the sum over the even, and $\sum'' T_r^k$ is the sum over the odd r.

Theorem 6.2. *Suppose χ is a nonnegative continuous function on $\mathbb{R}_+^k$, $k \in \mathbb{N}$, and $1 \leq i_1 < i_2 < \cdots < i_k$. If a sequence (y_i) satisfies the difference equation*

$$y_n = \frac{f(y_{n-i_1}, y_{n-i_2}, \ldots, y_{n-i_k})}{g(y_{n-i_1}, y_{n-i_2}, \ldots, y_{n-i_k})}, \quad n \in \mathbb{N}_0, \tag{6.5}$$

where

$$\begin{aligned} f(x_1, x_2, \ldots, x_k) &= \chi + \sum{}' T_r^k, \\ g(x_1, x_2, \ldots, x_k) &= \chi + \sum{}'' T_r^k, \end{aligned} \tag{6.6}$$

with $y_{-i_k}, \ldots, y_{-1} \in \mathbb{R}_+$, then it converges to the unique positive equilibrium 1.

Another proof of the previous result, in the case $\chi = 0$, can be also find in recent paper [28] by Stević.

Recently Sun and Xi in [31] gave an interesting proof of the following result. At first sight their result looked new and not so closely related to Theorem 6.1. However, we prove here that it is also a consequence of Theorem 6.1.

Theorem 6.3. *Let $f \in C(R_+^k, R_+)$ and $g \in C(R_+^l, R_+)$ with $k, l \in \mathbb{N}$, $0 \leq r_1 < \cdots < r_k$ and $0 \leq m_1 < \cdots < m_l$ and satisfy the following two conditions:*

(*H*1) $[f(u_1, u_2, \ldots, u_k)]^* = f(u_1^*, u_2^*, \ldots, u_k^*)$ *and* $[g(u_1, u_2, \ldots, u_l)]^* = g(u_1^*, u_2^*, \ldots, u_l^*)$.

(*H*2) $f(u_1^*, u_2^*, \ldots, u_k^*) \leq u_1^*$.

Then, $\overline{x} = 1$ is the unique positive equilibrium of the difference equation

$$x_n = \frac{f(x_{n-r_1-1}, \ldots, x_{n-r_k-1}) g(x_{n-m_1-1}, \ldots, x_{n-m_l-1}) + 1}{f(x_{n-r_1-1}, \ldots, x_{n-r_k-1}) + g(x_{n-m_1-1}, \ldots, x_{n-m_l-1})}, \quad n \in \mathbb{N}, \tag{6.7}$$

which is globally asymptotically stable (here $u^* = \max\{u, 1/u\}$).

Proof. Let

$$f_n = f(x_{n-r_1-1}, \ldots, x_{n-r_k-1}), \qquad g_n = g(x_{n-m_1-1}, \ldots, x_{n-m_l-1}). \tag{6.8}$$

We should determine the sign of the product of the next expressions

$$\begin{aligned} P_n &:= \frac{f_n g_n + 1}{f_n + g_n} - x_{n-r_1-1} \\ &= \frac{1}{f_n + g_n}\left(f_n g_n\left(1 - \frac{x_{n-r_1-1}}{f_n}\right) + 1 - x_{n-r_1-1} f_n\right), \end{aligned} \tag{6.9}$$

$$\begin{aligned} Q_n &:= \frac{f_n g_n + 1}{f_n + g_n} - \frac{1}{x_{n-r_1-1}} \\ &= \frac{1}{x_{n-r_1-1}(f_n + g_n)}\left(g_n(x_{n-r_1-1} f_n - 1) + f_n\left(\frac{x_{n-r_1-1}}{f_n} - 1\right)\right). \end{aligned} \tag{6.10}$$

From (6.9) and (6.10), we see if we show that $x_{n-r_1-1}f_n - 1$ and $(x_{n-r_1-1}/f_n) - 1$ have the same sign for $n \in \mathbb{N}$, then P_nQ_n will be nonpositive.

There are four cases to be considered.

Case 1 ($x_{n-r_1-1} \geq 1$, $f_n \geq 1$). Clearly, in this case, $x_{n-r_1-1}f_n - 1 \geq 0$. By $(H1)$ and $(H2)$, we have that

$$1 \leq f_n = (f_n)^* = f\left(x^*_{n-r_1-1}, \dots, x^*_{n-r_k-1}\right) \leq x^*_{n-r_1-1} = x_{n-r_1-1}. \tag{6.11}$$

Hence, $(x_{n-r_1-1}/f_n) - 1 \geq 0$ and consequently

$$(x_{n-r_1-1}f_n - 1)\left(\frac{x_{n-r_1-1}}{f_n} - 1\right) \geq 0. \tag{6.12}$$

Case 2 ($x_{n-r_1-1} \geq 1$, $f_n \leq 1$). Since $1/f_n \geq 1$ we obtain $(x_{n-r_1-1}/f_n) - 1 \geq 0$. On the other hand, by $(H1)$ and $(H2)$ we have

$$\frac{1}{f_n} = (f_n)^* = f\left(x^*_{n-r_1-1}, \dots, x^*_{n-r_k-1}\right) \leq x^*_{n-r_1-1} = x_{n-r_1-1}, \tag{6.13}$$

so that $x_{n-r_1-1}f_n - 1 \geq 0$. Hence, (6.12), follows in this case.

Case 3 (Case $x_{n-r_1-1} \leq 1$, $f_n \geq 1$). Then we have that $1/f_n \leq 1$ and consequently $(x_{n-r_1-1}/f_n) - 1 \leq 0$. On the other hand, we have

$$f_n = (f_n)^* = f\left(x^*_{n-r_1-1}, \dots, x^*_{n-r_k-1}\right) \leq x^*_{n-r_1-1} = \frac{1}{x_{n-r_1-1}}, \tag{6.14}$$

so that $x_{n-r_1-1}f_n - 1 \leq 0$. Hence, (6.12) follows in this case too.

Case 4 (Case $x_{n-r_1-1} \leq 1$, $f_n \leq 1$). Then $x_{n-r_1-1}f_n - 1 \leq 0$. On the other hand, we have

$$\frac{1}{f_n} = (f_n)^* = f\left(x^*_{n-r_1-1}, \dots, x^*_{n-r_k-1}\right) \leq x^*_{n-r_1-1} = \frac{1}{x_{n-r_1-1}}, \tag{6.15}$$

so that $(x_{n-r_1-1}/f_n) - 1 \leq 0$. Hence, (6.12) also holds in this case. Thus $P_nQ_n \leq 0$, for every $n \in \mathbb{N}$.

Assume that $P_nQ_n = 0$, then, $P_n = 0$ or $Q_n = 0$. Using (6.9) or (6.10) along with (6.12) in any of these two cases, we have that

$$f_n = \frac{1}{x_{n-r_1-1}} = x_{n-r_1-1}, \quad n \in \mathbb{N}. \tag{6.16}$$

Hence, $x_{n-r_1-1} = 1$, $n \in \mathbb{N}$.

Finally, let y^* be a solution of the equation

$$0 = \frac{f(\vec{y}_k^*)g(\vec{y}_l^*)+1}{f(\vec{y}_k^*)+g(\vec{y}_l^*)} - y^* = \frac{1}{f(\vec{y}_k^*)+g(\vec{y}_l^*)}\left(f(\vec{y}_k^*)g(\vec{y}_l^*)\left(1-\frac{y^*}{f(\vec{y}_k^*)}\right)+1-y^*f(\vec{y}_k^*)\right), \tag{6.17}$$

where $\vec{y}_j^* = (y^*,\dots,y^*)$ denotes the vector consisting of j copies of y^*. Then according to the considerations in Cases 1–4 it follows that $f(\vec{y}_k^*) = y^* = 1/y^*$, so that $y^* = 1$. Hence $y^* = 1$ is a unique positive equilibrium of (6.7).

From all above mentioned and by Theorem 6.1, we get the result. □

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References

[1] L. Berg and S. Stević, "Linear difference equations mod 2 with applications to nonlinear difference equations," *Journal of Difference Equations and Applications*, vol. 14, no. 7, pp. 693–704, 2008.
[2] L. Berg and S. Stević, "On the asymptotics of some difference equations," to appear in *Journal of Difference Equations and Applications*.
[3] Y. Z. Chen, "Global asymptotic stability of inhomogeneous iterates," *International Journal of Mathematics and Mathematical Sciences*, vol. 29, no. 3, pp. 133–142, 2002.
[4] B. D. Iričanin and S. Stević, "Some systems of nonlinear difference equations of higher order with periodic solutions," *Dynamics of Continuous, Discrete & Impulsive Systems. Series A*, vol. 13, no. 3-4, pp. 499–507, 2006.
[5] B. Iričanin and S. Stević, "Eventually constant solutions of a rational difference equation," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 854–856, 2009.
[6] B. Iričanin and S. Stević, "On a class of third-order nonlinear difference equations," *Applied Mathematics and Computation*, vol. 213, no. 2, pp. 479–483, 2009.
[7] C. M. Kent, W. Kosmala, M. A. Radin, and S. Stević, "Solutions of the difference equation $x_{n+1} = x_n x_{n-1} - 1$," *Abstract and Applied Analysis*, vol. 2010, Article ID 469683, 13 pages, 2010.
[8] C. M. Kent, W. Kosmala, and S. Stević, "Long-term behavior of solutions of the difference equation $x_{n+1} = x_{n-1}x_{n-2} - 1$," *Abstract and Applied Analysis*, vol. 2010, Article ID 152378, 17 pages, 2010.
[9] N. Kruse and T. Nesemann, "Global asymptotic stability in some discrete dynamical systems," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 1, pp. 151–158, 1999.
[10] U. Krause and R. D. Nussbaum, "A limit set trichotomy for self-mappings of normal cones in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 20, no. 7, pp. 855–870, 1993.
[11] M. Liao, X. Tang, and C. Xu, "On a conjecture for a higher-order rational difference equation," *Advances in Difference Equations*, vol. 2009, Article ID 394635, 9 pages, 2009.
[12] W. Liu, X. Yang, and B. D. Iričanin, "On some k-dimensional cyclic systems of difference equations," *Abstract and Applied Analysis*, vol. 2010, Article ID 528648, 11 pages, 2010.
[13] W. Liu, X. Yang, and L. Yang, "Global behavior of two families of nonlinear symmetric difference equations," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 367492, 15 pages, 2010.
[14] G. Stefanidou, G. Papaschinopoulos, and C. J. Schinas, "On a system of two exponential type difference equations," *Communications on Applied Nonlinear Analysis*, vol. 17, no. 2, pp. 1–13, 2010.
[15] S. Stević, "A global convergence results with applications to periodic solutions," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 1, pp. 45–53, 2002.

[16] S. Stević, "The recursive sequence $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$," *Applied Mathematics Letters*, vol. 15, no. 3, pp. 305–308, 2002.

[17] S. Stević, "On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$," *Taiwanese Journal of Mathematics*, vol. 6, no. 3, pp. 405–414, 2002.

[18] S. Stević, "Global stability and asymptotics of some classes of rational difference equations," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 60–68, 2006.

[19] S. Stević, "On positive solutions of a *(k+1)*-th order difference equation," *Applied Mathematics Letters*, vol. 19, no. 5, pp. 427–431, 2006.

[20] S. Stević, "Asymptotics of some classes of higher-order difference equations," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 56813, 20 pages, 2007.

[21] S. Stević, "Existence of nontrivial solutions of a rational difference equation," *Applied Mathematics Letters*, vol. 20, no. 1, pp. 28–31, 2007.

[22] S. Stević, "Nontrivial solutions of higher-order rational difference equations," *Mathematical Notes*, vol. 84, no. 5-6, pp. 718–724, 2008.

[23] S. Stević, "On the recursive sequence $x_{n+1} = \max\{c, x_n^p/x_{n-1}^p\}$," *Applied Mathematics Letters*, vol. 21, no. 8, pp. 791–796, 2008.

[24] S. Stević, "Boundedness character of a class of difference equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 2, pp. 839–848, 2009.

[25] S. Stević, "Global stability of a difference equation with maximum," *Applied Mathematics and Computation*, vol. 210, no. 2, pp. 525–529, 2009.

[26] S. Stević, "On a class of higher-order difference equations," *Chaos, Solitons and Fractals*, vol. 42, no. 1, pp. 138–145, 2009.

[27] S. Stević, "Global stability of a max-type difference equation," *Applied Mathematics and Computation*, vol. 216, no. 1, pp. 354–356, 2010.

[28] S. Stević, "Global stability of some symmetric difference equations," *Applied Mathematics and Computation*, vol. 216, no. 1, pp. 179–186, 2010.

[29] S. Stević, "On a generalized max-type difference equation from automatic control theory," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1841–1849, 2010.

[30] S. Stević, "On a nonlinear generalized max-type difference equation," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 1, pp. 317–328, 2011.

[31] T. Sun and H. Xi, "Global asymptotic stability of a higher order rational difference equation," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 462–466, 2007.

[32] A. C. Thompson, "On certain contraction mappings in a partially ordered vector space. ," *Proceedings of the American Mathematical Society*, vol. 14, pp. 438–443, 1963.

[33] "The William Lowell Putman Mathematical Competition," *The American Mathematical Monthly*, pp. 734–736, 1965.

[34] X. Yang, "Global asymptotic stability in a class of generalized Putnam equations," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 693–698, 2006.

[35] X. Yang, D. J. Evans, and G. M. Megson, "Global asymptotic stability in a class of Putnam-type equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 1, pp. 42–50, 2006.

[36] X. Yang, M. Yang, and H. Liu, "A part-metric-related inequality chain and application to the stability analysis of difference equation," *Journal of Inequalities and Applications*, vol. 2007, Article ID 19618, 9 pages, 2007.

[37] X. Yang, F. Sun, and Y. Y. Tang, "A new part-metric-related inequality chain and an application," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 193872, 7 pages, 2008.