Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 520648, 9 pages
doi:10.1155/2011/520648

Research Article

Optimal Lower Power Mean Bound for the Convex Combination of Harmonic and Logarithmic Means

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Received 1 February 2011; Accepted 15 May 2011

Academic Editor: Irena Lasiecka

We find the least value $\lambda \in (0,1)$ and the greatest value $p = p(\alpha)$ such that $\alpha H(a,b)+(1-\alpha)L(a,b) > M_p(a,b)$ for $\alpha \in [\lambda,1)$ and all $a,b > 0$ with $a \neq b$, where $H(a,b)$, $L(a,b)$, and $M_p(a,b)$ are the harmonic, logarithmic, and p-th power means of two positive numbers a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$, the p-th power mean $M_p(a,b)$ and logarithmic mean $L(a,b)$ of two positive numbers a and b are defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1.1}$$

$$L(a,b) = \begin{cases} \dfrac{b-a}{\log b - \log a}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

respectively.

It is well known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed a, $b > 0$ with $a \neq b$. In the recent past, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ and $L(a,b)$ can be found in the literature [1–17]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [18–20]. In [18], the authors study a variant of Jensen's functional equation involving L, which appears in a heat conduction problem. A representation of L as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [8]. In [21, 22], it is shown that L can be expressed in terms of Gauss's hypergeometric function ${}_2F_1$. And, in [21], the authors prove that the reciprocal of the logarithmic mean is strictly totally positive, that is, every $n \times n$ determinant with elements $1/L(a_i, b_i)$, where $0 < a_1 < a_2 < \cdots < a_n$ and $0 < b_1 < b_2 < \cdots < b_n$, is positive for all $n \geq 1$.

Let $A(a,b) = 1/2(a+b), I(a,b) = 1/e(b^b/a^a)^{1/(b-a)} (b \neq a), I(a,b) = a\ (b = a), G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the arithmetic, identric, geometric, and harmonic means of two positive numbers a and b, respectively, then it is well known that

$$\begin{aligned} \min\{a,b\} &< H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) \\ &< L(a,b) < I(a,b) < A(a,b) = M_1(a,b) < \max\{a,b\} \end{aligned} \tag{1.3}$$

for all $a, b > 0$ with $a \neq b$.

In [23], Alzer and Janous established the following best possible inequality:

$$M_{\log 2/\log 3}(a,b) < \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) < M_{2/3}(a,b) \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$.

In [8, 11, 24], the authors presented bounds for L in terms of G and A

$$G^{2/3}(a,b)A^{1/3}(a,b) < L(a,b) < \frac{2}{3}G(a,b) + \frac{1}{3}A(a,b) \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

The following companion of (1.3) provides inequalities for the geometric and arithmetic means of L and I. A proof can be found in [25]

$$G^{1/2}(a,b)A^{1/2}(a,b) < L^{1/2}(a,b)I^{1/2}(a,b) < \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b) < \frac{1}{2}G(a,b) + \frac{1}{2}A(a,b) \tag{1.6}$$

for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for L, I, $(LI)^{1/2}$, and $(L+I)/2$ in terms of the power means are proved in [4, 5, 7, 9, 16, 25, 26]:

$$\begin{gathered} M_0(a,b) < L(a,b) < M_{1/3}(a,b), \\ M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \\ M_0(a,b) < L^{1/2}(a,b)I^{1/2}(a,b) < M_{1/2}(a,b), \\ \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b) < M_{1/2}(a,b) \end{gathered} \tag{1.7}$$

for all $a, b > 0$ with $a \neq b$.

Alzer and Qiu [27] found the sharp bound of $1/2(L(a,b)+I(a,b))$ in terms of the power mean as follows:

$$M_c(a,b) < \frac{1}{2}(L(a,b)+I(a,b)) \tag{1.8}$$

for all $a,b>0$ with $a\neq b$, with the best possible parameter $c=\log 2/(1+\log 2)$.

The main purpose of this paper is to find the least value $\lambda\in(0,1)$ and the greatest value $p=p(\alpha)$ such that $\alpha H(a,b)+(1-\alpha)L(a,b)>M_p(a,b)$ for $\alpha\in[\lambda,1)$ and all $a,b>0$ with $a\neq b$.

2. Lemmas

In order to establish our main result we need three lemmas, which we present in this section.

Lemma 2.1. *Let $\alpha\in(1/4,1)$, $p=(1-4\alpha)/3\in(-1,0)$, and $f(t)=-4\alpha p(p+1)^2(p+2)t^{p-1}+2(1-\alpha)p^2(1-p^2)t^{p-2}+2(1-\alpha)p(1-p)^2(2-p)t^{p-3}+12(1-\alpha)(1-p)$. Then $f(t)>0$ for $t\in[1,+\infty)$.*

Proof. Simple computations lead to

$$f(1)=\frac{64}{81}(1-\alpha)^2\left(56\alpha^2+23\alpha+11\right)>0, \tag{2.1}$$

$$\lim_{t\to+\infty} f(t)=12(1-\alpha)(1-p)=8(1-\alpha)(1+2\alpha)>0, \tag{2.2}$$

$$f'(t)=-2p(1-p)t^{p-4}f_1(t), \tag{2.3}$$

where

$$\begin{gathered} f_1(t)=-2\alpha(p+1)^2(p+2)t^2+(1-\alpha)p(p+1)(2-p)t+(1-\alpha)(1-p)(2-p)(3-p),\\ f_1(1)=\frac{4}{27}(1-\alpha)\left(148\alpha^2-11\alpha+25\right)>0, \end{gathered} \tag{2.4}$$

$$\lim_{t\to+\infty} f_1(t)=-\infty, \tag{2.5}$$

$$\begin{aligned} f'_1(t) &= -4\alpha(p+1)^2(p+2)t+(1-\alpha)p(p+1)(2-p)\\ &= -\frac{4}{27}(1-\alpha)^2[16\alpha(7-4\alpha)t+(4\alpha-1)(4\alpha+5)]<0 \end{aligned} \tag{2.6}$$

for $t\in[1,+\infty)$.

Inequality (2.6) implies that $f_1(t)$ is strictly decreasing in $[1,+\infty)$, then from (2.4) and (2.5) we know that $\lambda_1>1$ exists such that $f_1(t)>0$ for $t\in[1,\lambda_1)$ and $f_1(t)<0$ for $t\in(\lambda_1,+\infty)$. Hence, equation (2.3) leads to the conclusion that $f(t)$ is strictly increasing in $[1,\lambda_1]$ and strictly decreasing in $[\lambda_1,+\infty)$. □

Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the piecewise monotonicity of $f(t)$.

Lemma 2.2. *Let* $\alpha \in (1/4, 1)$, $p = (1-4\alpha)/3 \in (-1, 0)$, *and* $g(t) = -(1-\alpha)(p+1)(p+2)^2(p+3)t^p + (p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha)t^{p-1} + (1-\alpha)p(p^3 - 8p^2 - p + 4)t^{p-2} + (1-\alpha)(1-p)(p^3 + 5p^2 - 14p + 4)t^{p-3} + 4(1-\alpha)(7-4p) - 4p(1-\alpha)t^{-1} + 4\alpha(1+p)t^{-2}$, *then* $g(t) > 0$ *for* $t \in [1, +\infty)$.

Proof. Let $g_1(t) = -(1-\alpha)(p+1)(p+2)^2(p+3)t^3 + (p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha)t^2 + (1-\alpha)p(p^3 - 8p^2 - p + 4)t + (1-\alpha)(1-p)(p^3 + 5p^2 - 14p + 4) + 4(1-\alpha)(7-4p)t^{3-p} - 4p(1-\alpha)t^{2-p} + 4\alpha(1+p)t^{1-p}$. Then simple computations lead to

$$g(t) = t^{p-3} g_1(t), \tag{2.7}$$

$$g_1(1) = \frac{16}{27}(1-\alpha)\left(80\alpha^2 + 110\alpha - 1\right) > 0, \tag{2.8}$$

$g_1'(t) = -3(1-\alpha)(p+1)(p+2)^2(p+3)t^2 + 2(p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha)t + (1-\alpha)p(p^3 - 8p^2 - p + 4) + 4(1-\alpha)(7-4p)(3-p)t^{2-p} - 4p(1-\alpha)(2-p)t^{1-p} + 4\alpha(1-p^2)t^{-p}$,

$$g_1'(1) = \frac{32}{27}(1-\alpha)\left(-16\alpha^3 + 38\alpha^2 + 176\alpha - 9\right) > 0, \tag{2.9}$$

$g_1''(t) = -6(1-\alpha)(p+1)(p+2)^2(p+3)t + 2(p+1)(p^3 - \alpha p^3 - 19\alpha p^2 + 3p^2 - 34\alpha p + 2p - 8\alpha) + 4(1-\alpha)(7-4p)(3-p)(2-p)t^{1-p} - 4p(1-\alpha)(2-p)(1-p)t^{-p} - 4\alpha p(1-p^2)t^{-p-1}$,

$$\begin{aligned} g_1''(1) &= \frac{8}{81}(1-\alpha)\left(-128\alpha^4 + 896\alpha^3 + 288\alpha^2 + 5294\alpha - 437\right) \\ &> 0, \end{aligned} \tag{2.10}$$

$g_1'''(t) = -6(1-\alpha)(p+1)(p+2)^2(p+3) + 4(1-\alpha)(7-4p)(3-p)(2-p)(1-p)t^{-p} + 4p^2(1-\alpha)(2-p)(1-p)t^{-p-1} + 4\alpha p(1+p)^2(1-p)t^{-p-2}$

$$\begin{aligned} g_1'''(1) &= \frac{8}{81}(1-\alpha)\left(576\alpha^4 + 3872\alpha^3 + 660\alpha^2 + 6612\alpha - 785\right) \\ &> 0, \end{aligned} \tag{2.11}$$

$$g_1^{(4)}(t) = -4p(1-p)t^{-p-3} g_2(t), \tag{2.12}$$

where

$$\begin{gathered} g_2(t) = (1-\alpha)(7-4p)(3-p)(2-p)t^2 + (1-\alpha)p(2-p)(p+1)t + \alpha(p+1)^2(p+2), \\ g_2(1) = \frac{4}{27}(1-\alpha)\left(96\alpha^3 + 232\alpha^2 + 388\alpha + 175\right) > 0, \end{gathered} \tag{2.13}$$

$$\begin{aligned} g_2'(t) &= 2(1-\alpha)(7-4p)(3-p)(2-p)t + (1-\alpha)p(2-p)(p+1) \\ &\geq g_2'(1) = \frac{4}{9}(1-\alpha)(5+4\alpha)\left(12\alpha^2 + 31\alpha + 23\right) > 0 \end{aligned} \tag{2.14}$$

for $t \in [1, +\infty)$. □

From (2.13) and (2.14), we clearly see that $g_2(t) > 0$ for $t \in [1, +\infty)$, then (2.12) leads to the conclusion that $g_1'''(t)$ is strictly in $[1, +\infty)$.

Therefore, Lemma 2.2 follows from (2.7)–(2.11) and the monotonicity of $g_1'''(t)$.

Lemma 2.3. *Let $\alpha \in (1/4, 1)$, $p = (1-4\alpha)/3 \in (-1, 0)$, and $h(t) = 2\alpha(1-t^{p+1})t\log^2 t + (1-\alpha)(1+t^{p-1})(1+t)^2 t\log t + (1-\alpha)(1+t)^2(1-t)(t^p+1)$, then $h(t) > 0$ for $t \in (1, +\infty)$.*

Proof. Let $h_1(t) = t^{-p}h''(t)$ and $h_2(t) = t^{p+2}h_1'(t)$, then simple computations lead to

$$h(1) = 0, \tag{2.15}$$

$h'(t) = 2\alpha[1-(p+2)t^{p+1}]\log^2 t + [(p+2-\alpha p-6\alpha)t^{p+1} + 2(1-\alpha)(p+1)t^p + (1-\alpha)pt^{p-1} + 3(1-\alpha)t^2 + 4(1-\alpha)t + 3\alpha + 1]\log t - (1-\alpha)[(p+3)t^{p+2} + (p+1)t^{p+1} - (p+3)t^p - (p+1)t^{p-1} + 2t^2 - 2]$,

$$h'(1) = 0, \tag{2.16}$$

$h_1(t) = -2\alpha(p+1)(p+2)\log^2 t + [(p^2 - \alpha p^2 + 3p - 11\alpha p - 14\alpha + 2) + 2(1-\alpha)p(p+1)t^{-1} - (1-\alpha)p(1-p)t^{-2} + 6(1-\alpha)t^{1-p} + 4(1-\alpha)t^{-p} + 4\alpha t^{-1-p}]\log t - (1-\alpha)(p+2)(p+3)t + (1-\alpha)(p^2+5p+2)t^{-1} + (1-\alpha)(p^2+p-1)t^{-2} - (1-\alpha)t^{1-p} + 4(1-\alpha)t^{-p} + (1+3\alpha)t^{-1-p} - (1-\alpha)p^2 - (1-\alpha)p - 5\alpha + 1$,

$$h_1(1) = 0, \tag{2.17}$$

$h_2(t) = -[4\alpha(p+1)(p+2)t^{p+1} + 2(1-\alpha)p(p+1)t^p - 2(1-\alpha)p(1-p)t^{p-1} - 6(1-\alpha)(1-p)t^2 + 4(1-\alpha)pt + 4\alpha(1+p)]\log t - (1-\alpha)(p+2)(p+3)t^{p+2} + (p^2 - \alpha p^2 + 3p - 11\alpha p - 14\alpha + 2)t^{p+1} + (1-\alpha)(p^2-3p-2)t^p - (1-\alpha)(p^2+3p-2)t^{p-1} + (1-\alpha)(p+5)t^2 + 4(1-\alpha)(1-p)t + \alpha - 3\alpha p - p - 1$,

$$h_2(1) = 0, \tag{2.18}$$

$h_2'(t) = -[4\alpha(p+1)^2(p+2)t^p + 2(1-\alpha)p^2(p+1)t^{p-1} + 2(1-\alpha)p(1-p)^2 t^{p-2} - 12(1-\alpha)(1-p)t + 4(1-\alpha)p]\log t - (1-\alpha)(p+2)^2(p+3)t^{p+1} + (p+1)(p^2 - \alpha p^2 - 15\alpha p + 3p - 22\alpha + 2)t^p + (1-\alpha)p(p^2 - 5p - 4)t^{p-1} + (1-\alpha)(1-p)(p^2+5p-2)t^{p-2} + 4(1-\alpha)(4-p)t - 4\alpha(p+1)t^{-1} + 4(1-\alpha)(1-2p)$,

$$\begin{aligned} h_2'(1) &= 0, \\ h_2''(t) &= f(t)\log t + g(t), \end{aligned} \tag{2.19}$$

where $f(t)$ and $g(t)$ are defined as in Lemmas 2.1 and 2.2, respectively.

From (2.19) and (2.10) together with Lemmas 2.1 and 2.2, we clearly see that $h_2(t)$ is strictly increasing in $[1, +\infty)$. □

Therefore, Lemma 2.3 follows from (2.15)–(2.18) and the monotonicity of $h_2(t)$.

3. Main Result

Theorem 3.1. *Inequality*

$$\alpha H(a,b) + (1-\alpha)L(a,b) > M_{(1-4\alpha)/3}(a,b) \tag{3.1}$$

holds for $\alpha \in [1/4, 1)$ and all $a, b > 0$ with $a \neq b$, and $M_{(1-4\alpha)/3}(a,b)$ is the best possible lower power mean bound for the sum $\alpha H(a,b) + (1-\alpha)L(a,b)$.

Proof. We divide the proof of inequality (3.1) into two cases.

Case 1 ($\alpha = 1/4$). Without loss of generality, we assume that $a > b$ and put $t = \sqrt{a/b} > 1$, then from (1.1) and (1.2), we have

$$\begin{aligned} &\alpha H(a,b) + (1-\alpha)L(a,b) - M_{(1-4\alpha)/3}(a,b) \\ &\quad = \frac{1}{4}[H(a,b) + 3L(a,b)] - \sqrt{ab} \\ &\quad = \frac{3t^4 - 4(2t^3 - t^2 + 2t)\log t - 3}{8(t^2+1)\log t} b. \end{aligned} \tag{3.2}$$

Let

$$F(t) = 3t^4 - 4\left(2t^3 - t^2 + 2t\right)\log t - 3, \tag{3.3}$$

then simple computations lead to

$$\begin{aligned} F(1) &= 0, \\ F'(t) &= 4\left(3t^3 - 2t^2 + t - 2\right) - 8\left(3t^2 - t + 1\right)\log t, \\ F'(1) &= 0, \\ F''(t) &= \frac{4}{t}F_1(t), \end{aligned} \tag{3.4}$$

where $F_1(t) = 9t^3 - 10t^2 + 3t - 2 - 2(6t-1)t\log t$,

$$\begin{aligned} F''(1) &= F_1(1) = 0, \\ F_1'(t) &= 27t^2 - 32t + 5 - 2(12t-1)\log t, \\ F_1'(1) &= 0, \\ F_1''(t) &= \frac{2}{t}F_2(t), \end{aligned} \tag{3.5}$$

where $F_2(t) = 27t^2 - 12t\log t - 28t + 1$,

$$F_1''(1) = F_2(1) = 0,$$
$$F_2'(t) = 54t - 12\log t - 40 > 0 \tag{3.6}$$

for $t > 1$.

Therefore, inequality (3.1) follows easily from (3.2)–(3.6).

Case 2 ($\alpha \in (1/4, 1)$). Without loss of generality, we assume that $a > b$. Let $p = (1-4\alpha)/3 \in (-1, 0)$ and $t = a/b > 1$, then from (1.1) and (1.2), one has

$$\begin{aligned} &\alpha H(a,b) + (1-\alpha)L(a,b) - M_{(1-4\alpha)/3}(a,b) \\ &\quad = \alpha H(a,b) + (1-\alpha)L(a,b) - M_p(a,b) \\ &\quad = b\left[\frac{2\alpha t}{t+1} + \frac{(1-\alpha)(t-1)}{\log t} - \left(\frac{t^p+1}{2}\right)^{1/p}\right]. \end{aligned} \tag{3.7}$$

Let

$$G(t) = \log\left[\frac{2\alpha t}{t+1} + \frac{(1-\alpha)(t-1)}{\log t}\right] - \frac{1}{p}\log\frac{t^p+1}{2}. \tag{3.8}$$

Then simple computations lead to

$$\lim_{t\to 1} G(t) = 0, \tag{3.9}$$

$$G'(t) = \frac{h(t)}{t(t+1)(t^p+1)\log t\left[2\alpha t\ \log t + (1-\alpha)(t^2-1)\right]}, \tag{3.10}$$

where $h(t)$ is defined as in Lemma 2.3.

From Lemma 2.3 and (3.10), we clearly see that $G(t)$ is strictly increasing in $(1, +\infty)$. $\square$

Therefore, inequality (3.1) follows from (3.7)–(3.9) and the monotonicity of $G(t)$.

Next, we prove that $M_{(1-4\alpha)/3}(a,b)$ is the best possible lower power mean bound for the sum $\alpha H(a,b) + (1-\alpha)L(a,b)$ if $\alpha \in [1/4, 1)$.

For any $\alpha \in [1/4, 1)$, $p > (1-4\alpha)/3$, and $x > 0$, one has

$$M_p(1+x, 1) - \alpha H(1+x, 1) - (1-\alpha)L(1+x, 1) = \frac{J(x)}{2^{1/p}\left(1+\dfrac{x}{2}\right)\log(1+x)}, \tag{3.11}$$

where $J(x) = (1+x/2)[1+(1+x)^p]^{1/p}\log(1+x) - 2^{1/p}[\alpha(1+x)\log(1+x) + (1-\alpha)x(1+x/2)]$. Letting $x \to 0$ and making use of Taylor expansion, we have

$$J(x) = \frac{2^{1/p}}{8}\left(p - \frac{1-4\alpha}{3}\right)x^3 + o\left(x^3\right). \tag{3.12}$$

Equations (3.11) and (3.12) imply that for any $\alpha \in [1/4, 1)$ and $p > (1-4\alpha)/3$ there exists $\delta > 0$, such that $\alpha H(1+x,1) + (1-\alpha)L(1+x,1) < M_p(1+x,1)$ for $x \in (0,\delta)$.

Remark 3.2. If $0 < \alpha < 1/4$, then from (1.1) and (1.2), we have

$$\lim_{x\to+\infty}\frac{M_{(1-4\alpha)/3}(1,x)}{\alpha H(1,x) + (1-\alpha)L(1,x)} = 2^{3/(4\alpha-1)} \times \lim_{x\to+\infty}\frac{\left(1+x^{(4\alpha-1)/3}\right)^{3/(1-4\alpha)}}{2\alpha/(x+1) + \left((1-1/x)(1-\alpha)/\log x\right)} = +\infty. \tag{3.13}$$

Equation (3.13) implies that for any $0 < \alpha < 1/4$, there exists $X > 1$, such that $M_{(1-4\alpha)/3}(1,x) > \alpha H(1,x) + (1-\alpha)L(1,x)$ for $x \in (X, +\infty)$. Therefore, $\lambda = 1/4$ is the least value of λ in $(0,1)$ such that inequality (3.1) holds for all $a, b > 0$ with $a \neq b$.

Acknowledgments

This research is supported by the N. S. Foundation of China under Grant 11071069, N. S. Foundation of Zhejiang province under Grants nos. Y7080106 and Y6100170, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant no. T200924.

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