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Research Article

Distribution of Maps with Transversal Homoclinic Orbits in a Continuous Map Space

Qiuju Xing and Yuming Shi

Department of Mathematics, Shandong University, Jinan, Shandong 250100, China

Correspondence should be addressed to Yuming Shi, ymshi@sdu.edu.cn

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This paper is concerned with distribution of maps with transversal homoclinic orbits in a continuous map space, which consists of continuous maps defined in a closed and bounded set of a Banach space. By the transversal homoclinic theorem, it is shown that the map space contains a dense set of maps that have transversal homoclinic orbits and are chaotic in the sense of both Li-Yorke and Devaney with positive topological entropy.

1. Introduction

Distribution of a set of maps with some dynamical properties in some continuous map space is a very interesting topic. In the 1960s, Smale [1] studied density of hyperbolicity. Some scholars believed that hyperbolic systems are dense in spaces of all dimensions, but it was shown that the conjecture is false in the late 1960s for diffeomorphisms on manifolds of dimension ≥2. The problem whether hyperbolic systems are dense in the one-dimension case was studied by many scholars. It was solved in the C^1 topology by Jakobson [2], a partial solution was given in the C^2 topology by Blokh and Misiurewicz [3], and C^2 density was finally proved by Shen [4]. In 2007, Kozlovki et al. got the result in C^k topology; that is, hyperbolic (i.e., Axiom A) maps are dense in the space of C^k maps defined in a compact interval or circle, $k = 1, 2, \ldots, \infty, \omega$ [5]. At the same time, some other scholars considered the distribution of hyperbolic diffeomorphisms in $\mathrm{Diff}(M)$, where M is a manifold. Just like the work of Smale, Palis [6, 7] gave the following conjecture: (1) any $f \in \mathrm{Diff}(M)$ can be approximated by a hyperbolic diffeomorphism or by a diffeomorphism exhibiting a homoclinic bifurcation (tangency or cycle), (2) any diffeomorphism can be C^r approximated by a Morse-Smale one or by one exhibiting transversal homoclinic orbit. Later, it was shown that the conjecture (1) holds for C^1 diffeomorphisms of surfaces [8]. And some good results have been obtained, such as any diffeomorphism can be C^1 approximated by a Morse-Smale one or by one displaying a transversal homoclinic orbit [9], any diffeomorphism can be C^1 approximated by one that exhibits either a homoclinic tangency or a heterodimensional cycle or by one that is essentially hyperbolic [10].

In 1963, Smale gave the well-known Smale-Birkhoff homoclinic theorem for diffeomorphisms [11], from which one can easily know that if a diffeomorphism F on a manifold M has a transversal homoclinic orbit, then there exists an integer $k > 0$ such that F^k is chaotic in the sense of both Li-Yorke and Devaney. Later, in 1986, Hale and Lin introduced a generalized definition of transversal homoclinic orbit for continuous maps and got the generalized Smale-Birkhoff homoclinic theorem, that is, a transversal homoclinic orbit implies chaos in the sense of both Li-Yorke and Devaney for continuous maps in Banach spaces [12]. In the meanwhile, some scholars studied the density of maps which are chaotic in the sense of Li-Yorke or Devaney. In particular, some results have been obtained in one-dimensional maps (cf. [13–15]).

Since 2004, Shi, Chen, and Yu extended the result about turbulent maps for one-dimensional maps introduced by Block and Coppel in 1992 [16] to maps in metric spaces. This map is termed by a new terminology: coupled-expanding map. Under certain conditions, the authors showed that a strictly coupled-expanding map is chaotic [17, 18]. Applying this coupled-expansion theory, they extended the criterion of chaos induced by snap-back repellers for finite-dimensional maps, introduced by Marotto in 1978 [19], to maps in metric spaces [17, 20, 21]. Recently, we studied the distribution of chaotic maps in continuous map spaces, in which maps are defined in general Banach spaces and finite-dimensional normed spaces, and obtained that the following several types of chaotic maps are dense in some continuous map spaces: (1) maps that are chaotic in the sense of both Li-Yorke and Devaney; (2) maps that are strictly coupled-expanding; (3) maps that have nondegenerated and regular snap-back repeller; (4) maps that have nondegenerate and regular homoclinic orbit to a repeller [22].

In the present paper, we will construct a set of continuous chaotic maps with generalized transversal homoclinic orbits, and show that the set is dense in the continuous map space. The method used in the present paper is motivated by the idea in [22].

This paper is organized as follows. In Section 2, we first introduce some notations and basic concepts including the Li-Yorke and Devaney chaos, hyperbolic fixed point, and transversal homoclinic orbit, and then give a useful lemma. In Section 3, we pay our attention to distribution of maps with generalized transversal homoclinic orbits in a continuous map space, in which every map is defined in a closed, bounded, and convex set or a closed bounded set in a general Banach space. Constructing a continuous map with a generalized transversal homoclinic orbit, we simultaneously show the density of maps which are chaotic in the sense of both Li-Yorke and Devaney in the map space.

2. Preliminaries

In this section, some notations and basic concepts are first introduced, including Li-Yorke and Devaney chaos, hyperbolic fixed point, and transversal homoclinic orbit. And then a useful lemma is given.

First, we give two definitions of chaos which will be used in the paper.

Definition 2.1 (see [23]). Let (X, d) be a metric space, $f : X \to X$ a map, and S a set of X with at least two points. Then, S is called a scrambled set of f if, for any two distinct points $x, y \in S$,

$$\liminf_{n\to\infty} d\left(f^n(x), f^n(y)\right) = 0, \qquad \limsup_{n\to\infty} d\left(f^n(x), f^n(y)\right) > 0. \tag{2.1}$$

The map f is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set S of f.

Definition 2.2 (see [24]). Let (X, d) be a metric space. A map $f : V \subset X \to V$ is said to be chaotic on V in the sense of Devaney if

(i) the periodic points of f in V are dense in V;

(ii) f is topologically transitive in V;

(iii) f has sensitive dependence on initial conditions in V.

Now, we give the definition of hyperbolic fixed point.

Definition 2.3 (see [25]). Let X be a Banach space, $U \subset X$ be a set, and $F : U \subset X \to X$ be a map. Assume that z is a fixed point of F, F is continuously differentiable in some neighborhood of z, and denote $DF(z)$ the Fréchet derivative of F at z. The fixed point z is called hyperbolic if $\sigma(DF(z)) \cap \{\lambda \in C : |\lambda| = 1\} = \emptyset$, where $\sigma(A)$ denotes the spectrum of a linear operator A. The hyperbolic fixed point z is called a saddle point if $\sigma(DF(z)) \cap \{\lambda \in C : |\lambda| > 1\} \neq \emptyset$ and $\sigma(DF(z)) \cap \{\lambda \in C : |\lambda| < 1\} \neq \emptyset$.

If F is invertible, then for any $p_0 \in X$, the set $O_F^+(p_0) := \{F^k(p_0) : k \geq 0\}$ is said to be the forward orbit of F from p_0, the set $O_F^-(p_0) := \{F^k(p_0) : k \leq 0\}$ is said to be the backward orbit of F from p_0. Since it is not required that F is invertible in this paper, a backward orbit of p_0 is a set $O_F^-(p_0) = \{p_j : j \leq 0\}$ with $p_{j+1} = F(p_j)$, $j \leq -1$, which may not exist, or exist but may not be unique. A whole orbit of p_0 is the union $O_F^+(p_0) \cup O_F^-(p_0)$, denoted by $O_F(p_0)$ in the case that it has a backward orbit $O_F^-(p_0)$. The stable set $W^s(z, F)$ and the unstable set $W^u(z, F)$ of a hyperbolic fixed point z of F are defined by

$$\begin{aligned} W^s(z,F) &:= \Big\{p_0 \in X\text{: the forward orbit } \{p_j\}_{j=0}^{+\infty} \text{ of } F \text{ from } p_0 \text{ such that,} \\ &\qquad p_j \longrightarrow z \text{ as } j \longrightarrow +\infty\Big\}, \\ W^u(z,F) &:= \Big\{p_0 \in X\text{: there exists a backward orbit } \{p_{-j}\}_{j=0}^{+\infty} \text{ of } F \\ &\qquad \text{from } p_0 \text{ such that } p_{-j} \longrightarrow z \text{ as } j \longrightarrow +\infty\Big\}, \end{aligned} \tag{2.2}$$

respectively. The local stable and unstable sets are defined by

$$W^s_{\text{loc}}(z, U, F) := W^s(z, F) \cap U, \qquad W^u_{\text{loc}}(z, U, F) := W^u(z, F) \cap U, \tag{2.3}$$

respectively, where U is some neighborhood of z. If $U = B(z, r)$ or $\overline{B}(z, r)$ for some $r > 0$, then the corresponding local stable and unstable sets of F are denoted by $W^s_{\text{loc}}(z, r, F)$ and $W^u_{\text{loc}}(z, r, F)$, respectively. By the Stable Manifold Theorem [26], if F is continuously differentiable in some neighborhood of a saddle point z, then there exists a neighborhood U of z such that the corresponding local stable and unstable set of z are submanifolds of X, respectively.

In the following, we first give the definition that two manifolds intersect transversally and then give the definition of transversal homoclinic orbit for continuous maps.

Definition 2.4 (see [25]). Two submanifolds V and W in a manifold M are transverse (in M) provided for any point $q \in V \cap W$, we have that $T_qV + T_qW = T_qM$, where T_qV and T_qW denote the tangent spaces of V and W at q, respectively, and "+" means the sum of the two subspaces (this allows for the possibility that $V \cap W = \emptyset$).

Remark 2.5. If $M = \mathbf{R}^n$, then V and W in M are transverse (in M) provided for any point $q \in V \cap W$, we have that $T_qV + T_qW = \mathbf{R}^n$. Obviously, if $\dim T_qV + \dim T_qW = n$, then the sum of the two subspaces T_qV and T_qW is a direct one, denoted by $\oplus$.

Definition 2.6 (see [12]). Let X be a Banach space, $F : X \to X$ be a map, and $z \in X$ be a saddle point of F.

(i) An orbit $O_F(p_0) = \{p_j\}_{j=-\infty}^{+\infty}$ is said to be a homoclinic orbit (asymptotic) to z if $p_0 \neq z$ and $\lim_{j\to+\infty} p_j = \lim_{j\to-\infty} p_j = z$.

(ii) A homoclinic orbit $O_F(p_0) = \{p_j\}_{j=-\infty}^{+\infty}$ to z is said to be transversal if there exists an open neighborhood U of z such that $p_{-i} \in W_{\mathrm{loc}}^u(z, U, F)$ and $p_j \in W_{\mathrm{loc}}^s(z, U, F)$ for any sufficiently large integers i, $j \geq 0$, and F^{i+j} sends a disc in $W_{\mathrm{loc}}^u(z, U, F)$ containing p_{-i} diffeomorphically onto its image that is transversal to $W_{\mathrm{loc}}^s(z, U, F)$ at p_j.

The following lemma is taken from Theorems 3.1 and 5.2, Corollary 6.1, and the result in Section 7 of [12].

Lemma 2.7. *Let $F : Z \to Z$ be a map, where $Z = X \times Y$, and X and Y are Banach spaces.*

(i) *Let A and B be linear continuous maps in X and Y, respectively, with the absolute values of the spectrum of A less than 1 and the absolute values of the spectrum of B larger than 1, and $\|A\|, \|B^{-1}\| \leq \lambda_0$ for some constant $0 < \lambda_0 < 1$.*

(ii) *Assume that U is an open neighborhood of 0 in Z and $f_1 : U \to X, f_2 : U \to Y$ are $C^k (k \geq 1)$ maps with $f_i(0) = 0, Df_i(0) = 0, i = 1, 2$. Further, assume that Df_1, Df_2 are uniformly continuous in U, and satisfies that for some constants $0 < \theta < 1 - \lambda_0$ and $\gamma > 0$, $\|Df_1(x, y)\|, \|Df_2(x, y)\| < \theta$ for all $(x, y) \in \overline{B}(0, \gamma) \subset U$.*

(iii) *Let $F : U \to Z$ be of the following form:*

$$F(x, y) = (Ax + f_1(x, y), By + f_2(x, y)), \tag{2.4}$$

and have the local stable and unstable manifolds $W_{\mathrm{loc}}^s(0, U, F) = \{(x, y) \mid (x, y) \in U, y = 0\}$ and $W_{\mathrm{loc}}^u(0, U, F) = \{(x, y) \mid (x, y) \in U, x = 0\} \neq \{0\}$.

(iv) *Assume that $O(p_0) = \{p_i\}_{i=-\infty}^{\infty}$ is a homoclinic orbit of F with $p_i \to 0$ as $i \to \pm\infty$, and there exists an integer $N > 0$ such that $p_{-N} \in W_{\mathrm{loc}}^u(0, U, F)$, $p_N \in W_{\mathrm{loc}}^s(0, U, F)$, and* (iv$_1$) F^{2N} *sends a disc* $O_1 \cap W^u_{\mathrm{loc}}(0, U, F)$ *centered at* p_{-N} *diffeomorphically onto* $O_2 = F^{2N}(O_1)$ *containing* p_N;

(iv$_2$) O_2 *intersects* $W^s_{\mathrm{loc}}(0, U, F)$ *transversally at* p_N.

Then $O(p_0)$ *is a transversal homoclinic orbit of* F. *Furthermore, there exists an integer* $k > 0$ *and a subset* Λ *in a neighborhood of* $O(p_0)$ *such that* F^k *on* Λ *is topologically conjugate to the full shift map on the doubly infinite sequence of two symbols. Consequently,* F *is chaotic in the sense of both Li-Yorke and Devaney, and its topological entropy* $h(F) \geq \log 2/k$.

Note that it is not required that F is a diffeomorphism, even F may not be continuous on the whole space Z in Lemma 2.7.

3. Distribution of Maps with Transversal Homoclinic Orbits

In this section, we first consider distribution of maps with transversal homoclinic orbits in a continuous self-map space, which consists of continuous maps that transform a closed, bounded, and convex set in a Banach space into itself. At the end of this section, we discuss distribution of chaotic maps in a continuous map space, in which a map may not transform its domain into itself.

Without special illustration, we always assume that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, and D_X and D_Y are bounded, convex, and open sets in X and Y, respectively. It is evident that $D = D_X \times D_Y$ is a bounded, convex, and open set in $Z = X \times Y$, where the norm $\|\cdot\|$ on Z is defined by $\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}$, for any $(x, y) \in Z$, where $x \in X, y \in Y$. Introduce the following map space:

$$C_0\left(\overline{D}, \overline{D}\right) := \left\{f : \overline{D} \longrightarrow \overline{D} \text{ is continuous and has a fixed point in } \overline{D}\right\}. \tag{3.1}$$

For any $f \in C_0(\overline{D}, \overline{D})$, let

$$\|f\| := \sup\left\{\|f(x)\| : x \in \overline{D}\right\}, \tag{3.2}$$

and for any $f, g \in C_0(\overline{D}, \overline{D})$, let

$$d(f, g) := \|f - g\|. \tag{3.3}$$

Then $(C_0(\overline{D}, \overline{D}), d)$ is a metric space. It may not be complete because a limit of a sequence of maps in $C_0(\overline{D}, \overline{D})$ is continuous and bounded, but may not have a fixed point in $\overline{D}$. But in the special case that Z is finite-dimensional, $(C_0(\overline{D}, \overline{D}), d)$ is a complete metric space by the Schauder fixed point theorem.

In this section, we first study distribution of maps with transversal homoclinic orbits in $C_0(\overline{D}, \overline{D})$.

For convenience, by $(x, y) \in Z$ denote $x \in X$ and $y \in Y$, by $\mathrm{Fix}(f)$ denote the set of all the fixed points of f. For $(x_1, y_1), (x_2, y_2) \in Z$ with $(x_1, y_1) \neq (x_2, y_2)$, by $l((x_1, y_1), (x_2, y_2))$ denote the straight half-line connecting (x_1, y_1) and (x_2, y_2):

$$l((x_1, y_1), (x_2, y_2)) := \{u = (x_1, y_1) + t((x_2, y_2) - (x_1, y_1)) : t \geq 0\}. \tag{3.4}$$

Lemma 3.1 (see [22, Lemma 3.1]). *For every map $f \in C_0(\overline{D}, \overline{D})$ and any $\varepsilon > 0$, there exists a map $g \in C_0(\overline{D}, \overline{D})$ such that $d(f, g) < \varepsilon$, $\mathrm{Fix}(g) \cap D \neq \emptyset$, and g is continuously differentiable in some neighborhood of some point $x^* \in \mathrm{Fix}(g) \cap D$.*

Lemma 3.2. *For every map $f \in C_0(\overline{D}, \overline{D})$ and every $\varepsilon > 0$, there exists a map $F \in C_0(\overline{D}, \overline{D})$ with $d(f, F) < \varepsilon$ such that F has a transversal homoclinic orbit in D.*

Proof. Fix any $f \in C_0(\overline{D}, \overline{D})$. By Lemma 3.1, it suffices to consider the case that f has a fixed point $z = (z_1, z_2) \in D$ with $z_1 \in X, z_2 \in Y$, and is continuously differentiable in some neighborhood of z.

For any $\varepsilon > 0$, there exists a positive constant $r_0 < \varepsilon/4$ with $\overline{B}(z, r_0) \subset D$ such that

$$\|f(x, y) - z\| < \frac{\varepsilon}{4}, \quad (x, y) \in \overline{B}(z, r_0). \tag{3.5}$$

Let $r = r_0/4$, take two constants a, b with $r < a < b < r_0/3$ and take two points $p_1 = (x_0, z_2) \in B(z, r), p_0 = (z_1, y_0) \in B(z, b) \setminus \overline{B}(z, a)$, where $x_0 \in X, y_0 \in Y$.

The rest of the proof is divided into three steps.

Step 1. Construct a map F that is locally controlled near z.

Define

$$F(x, y) = \begin{cases} (\lambda(x - z_1) + z_1, \mu(y - z_2) + z_2), & (x, y) \in \overline{B}(z, r), \\ (x + (x_0 - z_1), y - (y_0 - z_2)), & (x, y) \in \overline{B}(z, b) \setminus B(z, a), \end{cases} \tag{3.6}$$

where $x \in X$, $y \in Y$, and λ and μ are real parameters and satisfy

$$|\lambda| < 1, \qquad \frac{b}{r} < |\mu| < \frac{r_0}{r}. \tag{3.7}$$

Note that $|\mu| > 1$ since $r < b$.

For any $(x, y) \in B(z, a) \setminus \overline{B}(z, r)$, $F(x, y)$ is defined as follows. Let (x_1', y_1') and (x_2', y_2') be the intersection points of the straight line $l(z, (x, y))$ with $\partial\overline{B}(z, r)$ and $\partial\overline{B}(z, a)$, respectively, (see Figure 1). Set

$$F(x, y) = F(x_1', y_1') + t(x, y)(F(x_2', y_2') - F(x_1', y_1')), \tag{3.8}$$

where $t(x, y) \in (0, 1)$ is determined as follows;

$$(x, y) = (x_1', y_1') + t(x, y)((x_2', y_2') - (x_1', y_1')). \tag{3.9}$$

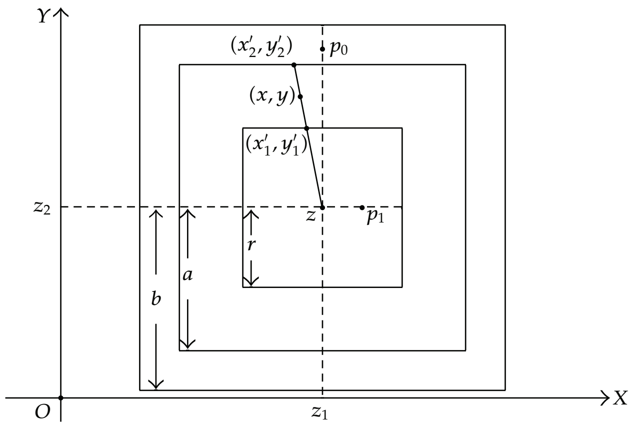

Figure 1

It is noted that when (x, y) continuously varies in $B(z, a)\setminus\overline{B}(z, r)$, so do the intersection points (x_1', y_1') and (x_2', y_2'). Consequently, $t(x, y)$ and then $F(x, y)$ are continuous in $B(z, a)\setminus\overline{B}(z, r)$.

Next, define $F(x, y) = f(x, y)$ for $(x, y) \in \overline{D}\setminus B(z, r_0)$. Finally, for any $(x, y) \in B(z, r_0)\setminus\overline{B}(z, b)$, suppose that (x_1'', y_1'') and (x_2'', y_2'') are the intersection points of the straight line $l(z, (x, y))$ with $\partial\overline{B}(z, b)$ and $\partial\overline{B}(z, r_0)$, respectively. Define $F(x, y)$ as that in (3.8), where $t(x, y)$ is determined by (3.9) with (x_1', y_1') and (x_2', y_2') replaced by (x_1'', y_1'') and (x_2'', y_2''), respectively. Hence, $F(x, y)$ is continuous in $B(z, r_0)\setminus\overline{B}(z, b)$.

Obviously, z is a saddle fixed point of F, and

$$\begin{gathered}\{(x, y) : x = z_1, \|y - z_2\|_Y < b\} \subset W^u_{\text{loc}}(z, b, F), \\ \{(x, y) : y = z_2, \|x - z_1\|_X < r\} \subset W^s_{\text{loc}}(z, r, F).\end{gathered} \tag{3.10}$$

Step 2. $F \in C_0(\overline{D}, \overline{D})$ and satisfies that $d(f, F) < \varepsilon$.

From the definition of F, it is easy to know that F is continuous on $\overline{D}$ and has a fixed point $z \in \overline{D}$, that is, $F \in C_0(\overline{D}, \overline{D})$.

Next, we will prove that $d(f, F) < \varepsilon$. For $(x, y) \in \overline{D}\setminus B(z, r_0)$, $\|F(x, y) - f(x, y)\| = 0 < \varepsilon$. For $(x, y) \in \overline{B}(z, r)$, it follows from (3.5) and (3.6) that

$$\|F(x, y) - f(x, y)\| \le \|F(x, y) - z\| + \|f(x, y) - z\| \le r_0 + \frac{\varepsilon}{4} < \varepsilon. \tag{3.11}$$

For $(x, y) \in \overline{B}(z, a)\setminus B(z, r)$, it follows from (3.5), (3.6), and (3.8) that

$$\begin{aligned}\|F(x, y) - f(x, y)\| &= \|F(x_1', y_1') + t(x, y)(F(x_2', y_2') - F(x_1', y_1')) - f(x, y)\| \\ &\le |1 - t(x, y)|\|F(x_1', y_1') - z\| + |t(x, y)|\|F(x_2', y_2') - z\| + \|f(x, y) - z\| \\ &\le r_0 + 2b + \frac{\varepsilon}{4} < \varepsilon.\end{aligned} \tag{3.12}$$

For $(x,y) \in \overline{B}(z,b) \setminus B(z,a)$, from (3.5) and (3.6), one has

$$\|F(x,y) - f(x,y)\| \le \|F(x,y) - z\| + \|f(x,y) - z\| < 2b + \frac{\varepsilon}{4} < \varepsilon. \tag{3.13}$$

For $(x,y) \in \overline{B}(z,r_0) \setminus B(z,b)$, from (3.5), (3.6), and (3.8), one has

$$\begin{aligned}\|F(x,y) - f(x,y)\| &= \|F(x_1'',y_1'') + t(x,y)(F(x_2'',y_2'') - F(x_1'',y_1'')) - f(x,y)\| \\ &\le |1 - t(x,y)|\|F(x_1'',y_1'') - z\| + |t(x,y)|\|F(x_2'',y_2'') - z\| + \|f(x,y) - z\| \\ &\le r_0 + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.\end{aligned} \tag{3.14}$$

Therefore, from the above discussion, $d(f,F) = \|F - f\| < \varepsilon$.

Step 3. F has a transversal homoclinic orbit in D.

It follows from (3.6) that $F(p_{-1}) = p_0$ and $F(p_0) = p_1$, where $p_{-1} = (z_1, \mu^{-1}(y_0 - z_2) + z_2) \in B(z,r)$ by (3.7). So, by (3.10) one has that $p_{-1} \in W^u_{\text{loc}}(z,r,F)$, $p_1 \in W^s_{\text{loc}}(z,r,F)$. Hence, $O(p_0) = \{p_j\}_{j=-\infty}^{\infty}$ satisfies that $\lim_{j\to\infty} p_j = \lim_{j\to-\infty} p_j = z$. Thus, $O(p_0) \subset D$ and is a homoclinic orbit of F.

Set a positive constant δ satisfying

$$\delta < \min\left\{b - \|y_0 - z_2\|_Y, \|y_0 - z_2\|_Y - a, |\mu|^{-1} b, r - |\mu|^{-1} b\right\}. \tag{3.15}$$

Then $0 < \delta < r$ by (3.7), and consequently it follows from (3.10) that the disc

$$O_0 := \{(x,y) : \ x = z_1, \|y - y_0\|_Y \le \delta\} \subset \left(B(z,b) \setminus \overline{B}(z,a)\right) \cap W^u_{\text{loc}}(z,b,F). \tag{3.16}$$

Further, set the discs

$$\begin{gathered} O_{-1} := \left\{(x,y) : \ x = z_1, \left\|y - \left(\mu^{-1}(y_0 - z_2) + z_2\right)\right\|_Y \le |\mu|^{-1}\delta\right\}, \\ O_1 := \{(x,y) : \ x = x_0, \|y - z_2\|_Y \le \delta\}. \end{gathered} \tag{3.17}$$

Then $O_{-1} \subset W^u_{\text{loc}}(z,r,F)$, $O_0 = F(O_{-1})$, and $O_1 = F(O_0)$ (see Figure 2).

It is evident that O_1 intersects $W^s_{\text{loc}}(z,r,F)$ transversally at point p_1. In addition, by the definition of F in $\overline{B}(z,b)$, one can get that $F^2 : O_{-1} \to O_1$ is a diffeomorphism. Therefore, $O(p_0)$ is a transversal homoclinic orbit asymptotic to z of F by Definition 2.6, where $N = 1$.

The entire proof is complete. □

Theorem 3.3. *Let X and Y be Banach spaces, $Z = X \times Y$, and D be a bounded, convex, and open set in Z. Then, for every map $f \in C_0(\overline{D}, \overline{D})$ and for any $\varepsilon > 0$, there exists a map $F \in C_0(\overline{D}, \overline{D})$ satisfying*

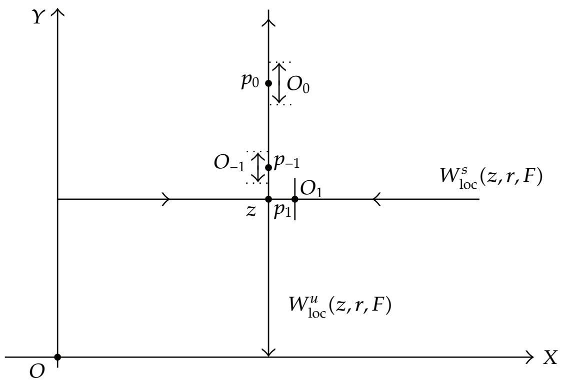

Figure 2

(1) $d(f, F) < \varepsilon$;

(2) *F has a transversal homoclinic orbit in D;*

(3) *F is chaotic in the sense of both Li-Yorke and Devaney;*

(4) *the topological entropy* $h(F) > 0$.

Proof. Let F be defined as in Lemma 3.2. Then (1) and (2) hold by Lemma 3.2.

Let F, r, λ, μ, and U be specified in the proof of Lemma 3.2. Without loss of generality, suppose that the fixed point z of F is the origin.

Set $U = B(0, r)$, $A = \lambda I, B = \mu I$, $f_1|_{\overline{U}} = 0, f_2|_{\overline{U}} = 0$. So A and B satisfy assumption (i) in Lemma 2.7, where $\lambda_0 = \max\{|\lambda|, |\mu|^{-1}\} < 1$. Take $\theta = (1 - \lambda_0)/2$. Then $\|Df_1(x, y)\| = \|Df_2(x, y)\| = 0 < \theta < 1 - \lambda_0$ for $(x, y) \in \overline{B}(0, r)$. Further,

$$W^u_{\text{loc}}(0, r, F) = \{(x, y) : x = 0, \|y\|_Y < r\},$$
$$W^s_{\text{loc}}(0, r, F) = \{(x, y) : y = 0, \|x\|_X < r\}. \tag{3.18}$$

Hence, F satisfies assumptions (ii) and (iii) in Lemma 2.7 with $\gamma = r$.

By the discussions in Step 3 in the proof of Lemma 3.2, F satisfies assumption (iv) in Lemma 2.7, where $O(p_0)$, O_{-1}, O_0, and O_1 are the same as those in the proof of Lemma 3.2 and $N = 1$. So, all the assumptions in Lemma 2.7 are satisfied. Consequently, (3) and (4) hold by Lemma 2.7. The proof is complete. □

When it is not required that a map transforms its domain D into itself, the convexity of domain $\overline{D}$ can be removed and all the corresponding results to Lemmas 3.1 and 3.2 and Theorem 3.3 still hold. In detail, let S be a bounded open set in Z and

$$C_0\left(\overline{S}, Z\right) := \left\{f : \overline{S} \longrightarrow Z \text{ is continuous and bounded, and has a fixed point in } \overline{S}\right\}. \tag{3.19}$$

Then $(C_0(\overline{S}, Z), d)$ is a metric space, where d is defined the same as that in (3.3). The results of Lemma 3.2 and Theorem 3.3 hold, where $C_0(\overline{D}, \overline{D})$ is replaced by $C_0(\overline{S}, Z)$. Their proofs are similar.

Now, we only present the detailed result corresponding to Theorem 3.3.

Theorem 3.4. *Let X and Y be Banach spaces, $Z = X \times Y$, and S be a bounded open set in Z. Then, for every map $f \in C_0(\overline{S}, Z)$ and for any $\varepsilon > 0$, there exists a map $F \in C_0(\overline{S}, Z)$ satisfying the following*

(1) $d(f, F) < \varepsilon$;

(2) *F has a transversal homoclinic orbit in S;*

(3) *F is chaotic in the sense of both Li-Yorke and Devaney;*

(4) *the topological entropy $h(F) > 0$.*

Remark 3.5. A general Banach space Z may not be discomposed into a product of two Banach spaces with dimension greater than or equal to 1. However, it is true for $Z = \mathbf{R}^n$ with $n \geq 2$. So Theorem 3.3 holds for each n-dimensional space $\mathbf{R}^n$ with $n \geq 2$. In addition, if D is a bounded and convex set in $\mathbf{R}^n$, every continuous map $f : \overline{D} \to \overline{D}$ has a fixed point in $\overline{D}$ by the Schauder fixed point theorem. In this case one has that

$$C_0\left(\overline{D}, \overline{D}\right) = C\left(\overline{D}, \overline{D}\right) = \left\{f : \overline{D} \longrightarrow \overline{D} \text{ is continuous}\right\}. \tag{3.20}$$

Remark 3.6. (1) As we all know, under C^1 perturbation, the hyperbolicity of a map is preserved. But it is obvious that the conclusion does not hold in the C^0 sense.

(2) In the C^0 topology, Theorems 3.3 and 3.4 show the density of distributions of maps with transversal homoclinic orbits, and consequently in the sense of both Li-Yorke and Devaney. However, it is not true in the C^1 topology. For example, consider the map

$$f(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \quad (x, y) \in I^2 = [0, 1] \times [0, 1]. \tag{3.21}$$

Clearly, $f \in C_0(I^2, I^2)$ and $z = 0$ is a globally asymptotically stable fixed point of f in I^2. By Theorem 3.3, for each $\varepsilon > 0$, there exists a map $F \in C_0(I^2, I^2)$ with $\|F - f\| < \varepsilon$ such that F is chaotic in the sense of both Li-Yorke and Devaney. But, in the C^1 topology, for each positive constant $\varepsilon < 1/2$ and for every map $F \in C^1(I^2, I^2)$ with

$$\|F - f\|_{C^1} := \max\{\|F - f\|, \|F' - f'\|\} < \varepsilon, \tag{3.22}$$

F is globally asymptotically stable in I^2, and so is not chaotic in any sense.

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References

[1] S. Smale, "The work of Curtis T. McMullen," in *Proceedings of the International Congress of Mathematicians*, vol. 1, pp. 127–132, 1998.

[2] M. V. Jakobson, "Smooth mappings of the circle into itself," *Sbornik: Mathematics*, vol. 85, no. 2, pp. 163–188, 1971.

[3] A. Blokh and M. Misiurewicz, "Typical limit sets of critical points for smooth interval maps," *Ergodic Theory and Dynamical Systems*, vol. 20, no. 1, pp. 15–45, 2000.
[4] W. Shen, "On the metric properties of multimodal interval maps and C^2 density of Axiom A," *Inventiones Mathematicae*, vol. 156, no. 2, pp. 301–403, 2004.
[5] O. Kozlovski, W. Shen, and S. van Strien, "Density of hyperbolicity in dimension one," *Annals of Mathematics. Second Series*, vol. 166, no. 1, pp. 145–182, 2007.
[6] J. Palis, "A global view of dynamics and a conjecture on the denseness of finitude of attractors," *Astérisque*, no. 261, pp. 335–347, 2000.
[7] J. Palis, "Open questions leading to a global perspective in dynamics," *Nonlinearity*, vol. 21, no. 4, pp. T37–T43, 2008.
[8] E. R. Pujals and M. Sambarino, "Homoclinic tangencies and hyperbolicity for surface diffeomorphisms," *Annals of Mathematics*, vol. 151, no. 3, pp. 961–1023, 2000.
[9] S. Crovisier, "Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems," *Annals of Mathematics*, vol. 172, no. 3, pp. 1641–1677, 2010.
[10] S. Crovisier and E. R. Pujal, "Homoclinic bifurcations versus hyperbolicity," In preparation.
[11] S. Smale, "Differentiable dynamical systems," *Bulletin of the American Mathematical Society*, vol. 73, pp. 747–817, 1967.
[12] J. K. Hale and X.-B. Lin, "Symbolic dynamics and nonlinear semiflows," *Annali di Matematica Pura ed Applicata*, vol. 144, pp. 229–259, 1986.
[13] P. E. Kloeden, "Chaotic difference equations are dense," *Bulletin of the Australian Mathematical Society*, vol. 15, no. 3, pp. 371–379, 1976.
[14] G. J. Butler and G. Pianigiani, "Periodic points and chaotic functions in the unit interval," *Bulletin of the Australian Mathematical Society*, vol. 18, no. 2, pp. 255–265, 1978.
[15] H. W. Siegberg, "Chaotic difference equations: generic aspects," *Transactions of the American Mathematical Society*, vol. 279, no. 1, pp. 205–213, 1983.
[16] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, vol. 1513 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1992.
[17] Y. Shi and G. Chen, "Chaos of discrete dynamical systems in complete metric spaces," *Chaos, Solitons and Fractals*, vol. 22, no. 3, pp. 555–571, 2004.
[18] Y. Shi, H. Ju, and G. Chen, "Coupled-expanding maps and one-sided symbolic dynamical systems," *Chaos, Solitons and Fractals*, vol. 39, no. 5, pp. 2138–2149, 2009.
[19] F. R. Marotto, "Snap-back repellers imply chaos in R^n," *Journal of Mathematical Analysis and Applications*, vol. 63, no. 1, pp. 199–223, 1978.
[20] Y. Shi and G. Chen, "Discrete chaos in Banach spaces," *Science in China. Series A. Mathematics*, vol. 48, no. 2, pp. 222–238, 2005.
[21] Y. Shi and P. Yu, "Chaos induced by regular snap-back repellers," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1480–1494, 2008.
[22] Y. Shi and Q. Xing, "Dense distribution of chaotic maps in continuous map spaces," Submitted.
[23] Z. Zhou, *Symbolic Dynamics*, Shanghai Scientific and Technological Education Publishing House, Shanghai, China, 1997.
[24] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley Studies in Nonlinearity, Addison-Wesley, Redwood City, Calif, USA, 2nd edition, 1989.
[25] C. Robinson, *Dynamical Systems*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 2nd edition, 1999.
[26] J. Palis, Jr. and W. De Melo, *Geometric Theory of Dynamical Systems*, Springer, New York, NY, USA, 1982.