Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 513128, 12 pages
doi:10.1155/2011/513128

Research Article

Nearly Jordan $*$-Homomorphisms between Unital C^*-Algebras

A. Ebadian,[1] S. Kaboli Gharetapeh,[1] and M. Eshaghi Gordji[2, 3]

[1] *Department of Mathematics, Urmia University, Urmia, Iran*
[2] *Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran*
[3] *Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Semnan, Iran*

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com

Received 26 February 2011; Revised 7 April 2011; Accepted 10 April 2011

Academic Editor: Irena Lasiecka

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Let A, B be two unital C^*-algebras. We prove that every almost unital almost linear mapping $h : A \to B$ which satisfies $h(3^n uy+3^n yu) = h(3^n u)h(y)+h(y)h(3^n u)$ for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \ldots$, is a Jordan homomorphism. Also, for a unital C^*-algebra A of real rank zero, every almost unital almost linear continuous mapping $h : A \to B$ is a Jordan homomorphism when $h(3^n uy + 3^n yu) = h(3^n u)h(y) + h(y)h(3^n u)$ holds for all $u \in I_1$ (A_{sa}), all $y \in A$, and all $n = 0, 1, 2, \ldots$. Furthermore, we investigate the Hyers- Ulam-Aoki-Rassias stability of Jordan $*$-homomorphisms between unital C^*-algebras by using the fixed points methods.

1. Introduction

The stability of functional equations was first introduced by Ulam [1] in 1940. More precisely, he proposed the following problem: given a group G_1, a metric group (G_2, d) and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$?. As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [2] gave a partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Hyers theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. This phenomenon of stability is called the Hyers-Ulam-Aoki-Rassias stability.

J. M. Rassias [5–7] established the stability of linear and nonlinear mappings with new control functions.

During the last decades, several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [8–10].

Bourgin is the first mathematician dealing with the stability of ring homomorphisms. The topic of approximate ring homomorphisms was studied by a number of mathematicians, see [11–19] and references therein.

Jun and Lee [20] proved the following: Let X and Y be Banach spaces. Denote by $\phi : X-\{0\}\times Y-\{0\} \to [0,\infty)$ a function such that $\widetilde{\phi}(x,y) = \sum_{n=0}^{\infty} 3^{-n}\phi(3^n x, 3^n y) < \infty$ for all $x, y \in X - \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \phi(x,y) \tag{1.1}$$

for all $x, y \in X - \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\| f(x) - f(0) - T(x) \| \le \frac{1}{3}\left(\widetilde{\phi}(x,-x) + \widetilde{\phi}(-x,3x)\right) \tag{1.2}$$

for all $x \in X - \{0\}$.

Recently, C. Park and W. Park [21] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra. Johnson (Theorem 7.2 of [22]) also investigated almost algebra $*$-homomorphisms between Banach $*$-algebras: Suppose that U and B are Banach $*$-algebras which satisfy the conditions of (Theorem 3.1 of [22]). Then for each positive ϵ and K, there is a positive δ such that if $T \in L(U, B)$ with $\|T\| < K, \|T^{\vee}\| < \delta$ and $\|T(x^*)^* - T(x)\| < \delta \ \|x\|(x \in U)$, then there is a $*$-homomorphism $T' : U \to B$ with $\|T - T'\| < \epsilon$. Here $L(U, B)$ is the space of bounded linear maps from U into B, and $T^{\vee}(x,y) = T(xy) - T(x)T(y)(x, y \in U)$. See [22] for details. Throughout this paper, let A be a unital C^*-algebra with unit e, and B a unital C^*-algebra. Let $U(A)$ be the set of unitary elements in A, $A_{sa} := \{x \in Ax = x^*\}$, and $I_1(A_{sa}) = \{v \in A_{sa} \mid \|v\| = 1, v \in Inv(A)\}$. In this paper, we prove that every almost unital almost linear mapping $h : A \to B$ is a Jordan homomorphism when $h(3^n uy + 3^n yu) = h(3^n u)h(y) + h(y)h(3^n u)$ holds for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \ldots$, and that for a unital C^*-algebra A of real rank zero (see [23]), every almost unital almost linear continuous mapping $h : A \to B$ is a Jordan homomorphism when $h(3^n uy + 3^n yu) = h(3^n u)h(y) + h(y)h(3^n u)$ holds for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2 \ldots$. Furthermore, we investigate the Hyers-Ulam-Aoki-Rassias stability of Jordan $*$-homomorphisms between unital C^*-algebras by using the fixed point methods.

Note that a unital C^*-algebra is of real rank zero, if the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [23]). We denote the algebric center of algebra A by $Z(A)$.

2. Jordan $*$-Homomorphisms on Unital C^*-Algebras

By a following similar way as in [24], we obtain the next theorem.

Theorem 2.1. *Let $f : A \to B$ be a mapping such that $f(0) = 0$ and that*

$$f(3^n uy + 3^n yu) = f(3^n u)f(y) + f(y)f(3^n u) \tag{2.1}$$

for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \ldots$. If there exists a function $\phi : (A - \{0\})^2 \times A \to [0, \infty)$ such that $\widetilde{\phi}(x, y, z) = \sum_{n=0}^{\infty} 3^{-n}\phi(3^n x, 3^n y, 3^n z) < \infty$ for all $x, y \in A - \{0\}$ and all $z \in A$ and that

$$\left\| 2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y) + f(u^*) - f(u)^* \right\| \leq \phi(x, y, u), \tag{2.2}$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A, u \in (U(A) \cup \{0\})$. If $\lim_n (f(3^n e)/3^n) \in U(B) \cap Z(B)$, then the mapping $f : A \to B$ is a Jordan $*$-homomorphism.

Proof. Put $u = 0$, $\mu = 1$ in (2.2), it follows from of [20, Theorem 1] that there exists a unique additive mapping $T : A \to B$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{3}\left(\widetilde{\phi}(x, -x, 0) + \widetilde{\phi}(-x, 3x, 0)\right) \tag{2.3}$$

for all $x \in A - \{0\}$. This mapping is given by

$$T(x) = \lim_n \frac{f(3^n x)}{3^n} \tag{2.4}$$

for all $x \in A$. By the same reasoning as the proof of [24, Theorem 1], T is $\mathbb{C}$-linear and $*$-preserving. It follows from (2.1) that

$$T(uy + yu) = \lim_n \frac{f(3^n uy + 3^n yu)}{3^n} = \lim_n \left[\frac{f(3^n u)}{3^n} f(y) + f(y)\frac{f(3^n u)}{3^n}\right] = T(u)f(y) + f(y)T(u) \tag{2.5}$$

for all $u \in U(A)$, all $y \in A$. Since T is additive, then by (2.5), we have

$$3^n T(uy + yu) = T(u(3^n y) + (3^n y)u) = T(u)f(3^n y) + f(3^n y)T(u) \tag{2.6}$$

for all $u \in U(A)$ and all $y \in A$. Hence,

$$T(uy + yu) = \lim_n \left[T(u)\frac{f(3^n y)}{3^n} + \frac{f(3^n y)}{3^n}T(u)\right] = T(u)T(y) + T(y)T(u) \tag{2.7}$$

for all $u \in U(A)$ and all $y \in A$. By the assumption, we have

$$T(e) = \lim_n \frac{f(3^n e)}{3^n} \in U(B) \bigcap Z(B) \tag{2.8}$$

hence, it follows by (2.5) and (2.7) that

$$2T(e)T(y) = T(e)T(y) + T(y)T(e) = T(ye + ey) = T(e)f(y) + f(y)T(e) = 2T(e)f(y) \tag{2.9}$$

for all $y \in A$. Since $T(e)$ is invertible, then $T(y) = f(y)$ for all $y \in A$. We have to show that f is Jordan homomorphism. To this end, let $x \in A$. By Theorem 4.1.7 of [25], x is a finite linear combination of unitary elements, that is, $x = \sum_{j=1}^{n} c_j u_j$ ($c_j \in \mathbb{C}, u_j \in U(A)$), and then it follows from (2.7) that

$$\begin{aligned} f(xy + yx) = T(xy + yx) &= T\left(\sum_{j=1}^{n} c_j u_j y + \sum_{j=1}^{n} c_j y u_j\right) = \sum_{j=1}^{n} c_j T(u_j y + y u_j) \\ &= \sum_{j=1}^{n} c_j (T(u_j y) + T(y u_j)) = \sum_{j=1}^{n} c_j (T(u_j)T(y) + T(y)T(u_j)) \\ &= T\left(\sum_{j=1}^{n} c_j u_j\right) T(y) + T(y) T\left(\sum_{j=1}^{n} c_j u_j\right) = T(x)T(y) + T(y)T(x) \\ &= f(x)f(y) + f(y)f(x) \end{aligned} \tag{2.10}$$

for all $y \in A$. And this completes the proof of theorem. □

Corollary 2.2. *Let $p \in (0,1)$, $\theta \in [0,\infty)$ be real numbers. Let $f : A \to B$ be a mapping such that $f(0) = 0$ and that*

$$f(3^n uy + 3^n yu) = f(3^n u)f(y) + f(y)f(3^n u) \tag{2.11}$$

for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \ldots$. Suppose that

$$\left\| 2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y) + f(z^*) - f(z)^* \right\| \leq \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \tag{2.12}$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. If $\lim_n (f(3^n e)/3^n) \in U(B) \cap Z(B)$, then the mapping $f : A \to B$ is a Jordan $*$-homomorphism.

Proof. Setting $\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in A$. Then by Theorem 2.1, we get the desired result. □

Theorem 2.3. *Let A be a C^*-algebra of real rank zero. Let $f : A \to B$ be a continuous mapping such that $f(0) = 0$ and that*

$$f(3^n uy + 3^n yu) = f(3^n u)f(y) + f(y)f(3^n u) \tag{2.13}$$

for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \ldots$. Suppose that there exists a function $\phi : (A - \{0\})^2 \times A \to [0, \infty)$ satisfying (2.2) and $\widetilde{\phi}(x, y, z) < \infty$ for all $x, y \in A - \{0\}$ and all $z \in A$. If $\lim_n (f(3^n e)/3^n) \in U(B) \cap Z(B)$, then the mapping $f : A \to B$ is a Jordan $*$-homomorphism.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique involutive $\mathbb{C}$-linear mapping $T : A \to B$ satisfying (2.3). It follows from (2.13) that

$$T(uy + yu) = \lim_n \frac{f(3^n uy + 3^n yu)}{3^n} = \lim_n \left[\frac{f(3^n u)}{3^n} f(y) + f(y) \frac{f(3^n u)}{3^n}\right] = T(u)f(y) + f(y)T(u) \tag{2.14}$$

for all $u \in I_1(A_{\text{sa}})$, and all $y \in A$. By additivity of T and (2.14), we obtain that

$$3^n T(uy + yu) = T(u(3^n y) + (3^n y)u) = T(u)f(3^n y) + f(3^n y)T(u) \tag{2.15}$$

for all $u \in I_1(A_{\text{sa}})$ and all $y \in A$. Hence,

$$T(uy + yu) = \lim_n \left[T(u)\frac{f(3^n y)}{3^n} + \frac{f(3^n y)}{3^n}T(u)\right] = T(u)T(y) + T(y)T(u) \tag{2.16}$$

for all $u \in I_1(A_{\text{sa}})$ and all $y \in A$. By the assumption, we have

$$T(e) = \lim_n \frac{f(3^n e)}{3^n} \in U(B) \cap Z(B). \tag{2.17}$$

Similar to the proof of Theorem 2.1, it follows from (2.14) and (2.16) that $T = f$ on A. So T is continuous. On the other hand, A is real rank zero. One can easily show that $I_1(A_{\text{sa}})$ is dense in $\{x \in A_{\text{sa}} : \|x\| = 1\}$. Let $v \in \{x \in A_{\text{sa}} : \|x\| = 1\}$. Then there exists a sequence $\{z_n\}$ in $I_1(A_{\text{sa}})$ such that $\lim_n z_n = v$. Since T is continuous, it follows from (2.16) that

$$\begin{aligned} T(vy + yv) &= T\left(\lim_n (z_n y + y z_n)\right) = \lim_n T(z_n y + y z_n) \\ &= \lim_n T(z_n)T(y) + \lim_n T(y)T(z_n) \\ &= T\left(\lim_n z_n\right)T(y) + T(y)T\left(\lim_n z_n\right) \\ &= T(v)T(y) + T(y)T(v) \end{aligned} \tag{2.18}$$

for all $y \in A$. Now, let $x \in A$. Then we have $x = x_1 + ix_2$, where $x_1 := (x + x^*)/2$ and $x_2 = (x - x^*)/2i$ are self adjoint.

First consider $x_1 = 0,\ x_2 \neq 0$. Since T is $\mathbb{C}$-linear, it follows from (2.18) that

$$\begin{aligned} f(xy+yx) &= T(xy+yx) = T(ix_2y + y(ix_2)) = T\left(i\|x_2\|\frac{x_2}{\|x_2\|}y + y\left(i\|x_2\|\frac{x_2}{\|x_2\|}\right)\right) \\ &= T\left(i\|x_2\|\frac{x_2}{\|x_2\|}\right)T(y) + T(y)T\left(i\|x_2\|\frac{x_2}{\|x_2\|}\right) \\ &= T(ix_2)T(y) + T(y)T(ix_2) = T(x)T(y) + T(y)T(x) \\ &= f(x)f(y) + f(y)f(x) \end{aligned} \tag{2.19}$$

for all $y \in A$.

If $x_2 = 0$, $x_1 \neq 0$, then by (2.18), we have

$$\begin{aligned} f(xy+yx) &= T(xy+yx) = T(x_1y + y(x_1)) = T\left(\|x_1\|\frac{x_1}{\|x_1\|}y + y\left(\|x_1\|\frac{x_1}{\|x_1\|}\right)\right) \\ &= T\left(\|x_1\|\frac{x_1}{\|x_1\|}\right)T(y) + T(y)T\left(\|x_1\|\frac{x_1}{\|x_1\|}\right) = T(x_1)T(y) + T(y)T(x_1) \\ &= T(x)T(y) + T(y)T(x) = f(x)f(y) + f(y)f(x) \end{aligned} \tag{2.20}$$

for all $y \in A$.

Finally, consider the case that $x_1 \neq 0$, $x_2 \neq 0$. Then it follows from (2.18) that

$$\begin{aligned} f(xy+yx) &= T(xy+yx) = T(x_1y + ix_2y + yx_1 + y(ix_2)) \\ &= T\left(\|x_1\|\frac{x_1}{\|x_1\|}y + y\left(\|x_1\|\frac{x_1}{\|x_1\|}\right)\right) + T\left(i\|x_2\|\frac{x_2}{\|x_2\|}y + y\left(i\|x_2\|\frac{x_2}{\|x_1\|}\right)\right) \\ &= \|x_1\|\left[T\left(\frac{x_1}{\|x_1\|}\right)T(y) + T(y)T\left(\frac{x_1}{\|x_1\|}\right)\right] + i\|x_2\|\left[T\left(\frac{x_2}{\|x_2\|}\right)T(y) + T(y)T\left(\frac{x_2}{\|x_1\|}\right)\right] \\ &= [\mathrm{T}(x_1) + T(ix_2)]T(y) + T(y)[T(x_1) + T(ix_2)] \\ &= T(x)T(y) + T(y)T(x) = f(x)f(y) + f(y)f(x) \end{aligned} \tag{2.21}$$

for all $y \in A$. Hence, $f(xy + yx) = f(x)f(y) + f(y)f(x)$ for all $x, y \in A$, and f is Jordan $*$-homomorphism. □

Corollary 2.4. *Let A be a C^*-algebra of rank zero. Let $p \in (0,1)$, $\theta \in [0,\infty)$ be real numbers. Let $f : A \to B$ be a mapping such that $f(0) = 0$ and that*

$$f(3^n uy + 3^n yu) = f(3^n u)f(y) + f(y)f(3^n u) \tag{2.22}$$

for all $u \in I_1(A_{sa})$, *all* $y \in A$, *and all* $n = 0, 1, 2, \ldots$. *Suppose that*

$$\left\| 2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y) + f(z^*) - f(z)^* \right\| \leq \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \tag{2.23}$$

for all $\mu \in \mathbb{T}$ *and all* $x, y, z \in A$. *If* $\lim_n f(3^n e)/3^n \in U(B) \cap Z(B)$, *then the mapping* $f : A \to B$ *is a Jordan* $*$*-homomorphism.*

Proof. Setting $\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in A$. Then by Theorem 2.3, we get the desired result. ☐

3. Stability of Jordan $*$-Homomorphisms: A Fixed Point Approach

We investigate the generalized Hyers-Ulam-Aoki-Rassias stability of Jordan $*$-homomorphisms on unital C^*-algebras by using the alternative fixed point.

Recently, Cădariu and Radu applied the fixed point method to the investigation of the functional equations. (See also [18, 26–43]).

Theorem 3.1. *Let* $f : A \to B$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\phi : A^5 \to [0, \infty)$ *satisfying*

$$\begin{aligned} &\left\| f\left(\frac{\mu x + \mu y + \mu z}{3}\right) + f\left(\frac{\mu x - 2\mu y + \mu z}{3}\right) + f\left(\frac{\mu x + \mu y - 2\mu z}{3}\right) - \mu f(x) + f(uv + uv) \right. \\ &\quad \left. - f(v)f(u) - f(u)f(v) + f(w^*) - f(w)^* \right\| \leq \phi(x, y, z, u, v, w), \end{aligned} \tag{3.1}$$

for all $\mu \in \mathbb{T}$, *and all* $x, y, z, u, v \in A, w \in U(A) \cup \{0\}$. *If there exists an* $L < 1$ *such that* $\phi(x, y, z, u, v, w) \leq 3L\phi(x/3, y/3, z/3, u/3, v/3, w/3)$ *for all* $x, y, z, u, v, w \in A$, *then there exists a unique Jordan* $*$*-homomorphism* $h : A \to B$ *such that*

$$\|f(x) - h(x)\| \leq \frac{L}{1 - L}\phi(x, 0, 0, 0, 0, 0) \tag{3.2}$$

for all $x \in A$.

Proof. It follows from $\phi(x, y, z, u, v, w) \leq 3L\phi(x/3, y/3, z/3, u/3, v/3, w/3)$ that

$$\lim_j 3^{-j}\phi\left(3^j x, 3^j y, 3^j z, 3^j u, 3^j v, 3^j w\right) = 0 \tag{3.3}$$

for all $x, y, z, u, v, w \in A$.

Put $y = z = w = u = 0$ and $\mu = 1$ in (3.1) to obtain

$$\left\| 3f\left(\frac{x}{3}\right) - f(x) \right\| \leq \phi(x, 0, 0, 0, 0, 0) \tag{3.4}$$

for all $x \in A$. Hence,

$$\left\| \frac{1}{3} f(3x) - f(x) \right\| \leq \frac{1}{3} \phi(3x, 0, 0, 0, 0, 0) \leq L\phi(x, 0, 0, 0, 0, 0) \tag{3.5}$$

for all $x \in A$.

Consider the set $X := \{g \mid g : A \to B\}$ and introduce the generalized metric on X:

$$d(h, g) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0, 0, 0, 0, 0)\ \forall x \in A\}. \tag{3.6}$$

It is easy to show that (X, d) is complete. Now we define the linear mapping $J : X \to X$ by

$$J(h)(x) = \frac{1}{3} h(3x) \tag{3.7}$$

for all $x \in A$. By Theorem 3.1 of [44],

$$d(J(g), J(h)) \leq Ld(g, h) \tag{3.8}$$

for all $g, h \in X$.

It follows from (3.5) that

$$d(f, J(f)) \leq L. \tag{3.9}$$

Now, from the fixed point alternative [45], J has a unique fixed point in the set $X_1 := \{h \in X : d(f, h) < \infty\}$. Let h be the fixed point of J. h is the unique mapping with

$$h(3x) = 3h(x) \tag{3.10}$$

for all $x \in A$ satisfying there exists $C \in (0, \infty)$ such that

$$\|h(x) - f(x)\| \leq C\phi(x, 0, 0, 0, 0, 0) \tag{3.11}$$

for all $x \in A$. On the other hand, we have $\lim_n d(J^n(f), h) = 0$. It follows that

$$\lim_n \frac{1}{3^n} f(3^n x) = h(x) \tag{3.12}$$

for all $x \in A$. It follows from $d(f, h) \leq (1/(1-L))d(f, J(f))$, that

$$d(f, h) \leq \frac{L}{1-L}. \tag{3.13}$$

This implies the inequality (3.2). It follows from (3.1), (3.3), and (3.12) that

$$\begin{aligned}
&\left\|h\left(\frac{x+y+z}{3}\right)+h\left(\frac{x-2y+z}{3}\right)+h\left(\frac{x+y-2z}{3}\right)-h(x)\right\| \\
&= \lim_n \frac{1}{3^n}\left\|f\left(3^{n-1}(x+y+z)\right)+f\left(3^{n-1}(x-2y+z)\right)+f\left(3^{n-1}(x+y-2z)\right)-f(3^n x)\right\| \\
&\le \lim_n \frac{1}{3^n}\phi(3^n x, 3^n y,, 3^n z, 0, 0, 0) \\
&= 0
\end{aligned} \tag{3.14}$$

for all $x, y, z \in A$. So

$$h\left(\frac{x+y+z}{3}\right)+h\left(\frac{x-2y+z}{3}\right)+h\left(\frac{x+y-2z}{3}\right)=h(x) \tag{3.15}$$

for all $x, y, z \in A$. Put $w = (x+y+z)/3$, $t = (x-2y+z)/3$ and $s = (x+y-2z)/3$ in above equation, we get $h(w+t+s) = h(w)+h(t)+h(s)$ for all $w, t, s \in A$. Hence, h is Cauchy additive. By putting $y = z = x$, $v = w = 0$ in (3.1), we have

$$\|f(\mu x) - \mu f(x)\| \le \phi(x, x,, x, 0, 0, 0) \tag{3.16}$$

for all $\mu \in \mathbb{T}$ and all $x \in A$. It follows that

$$\|h(\mu x) - \mu h(x)\| = \lim_m \frac{1}{3^m}\|f(\mu 3^m x) - \mu f(3^m x)\| \le \lim_m \frac{1}{3^m}\phi(3^m x, 3^m x, 3^m x, 0, 0, 0) = 0 \tag{3.17}$$

for all $\mu \in \mathbb{T}$, and all $x \in A$. One can show that the mapping $h : A \to B$ is $\mathbb{C}$-linear. By putting $x = y = z = u = v = 0$ in (3.1), it follows that

$$\begin{aligned}
\|h(w^*) - (h(w))^*\| &= \lim_m \left\|\frac{1}{3^m} f\left((3^m w)^*\right) - \frac{1}{3^m}\left(f(3^m w)\right)^*\right\| \\
&\le \lim_m \frac{1}{3^m}\phi(0, 0, 0, 0, 0, 3^m w) \\
&= 0
\end{aligned} \tag{3.18}$$

for all $w \in U(A)$. By the same reasoning as the proof of Theorem 2.1, we can show that $h : A \to B$ is $*$-preserving.

Since h is $\mathbb{C}$-linear, by putting $x = y = z = w = 0$ in (3.1), it follows that

$$\begin{aligned}
&\|h(uv+vu) - h(u)h(v) - h(v)h(u)\| \\
&= \lim_m \left\| \frac{1}{9^m} f(9^m(uv+vu)) - \frac{1}{9^m}\left[f(3^m u)f(3^m v) + f(3^m v)f(3^m u)\right] \right\| \\
&\leq \lim_m \frac{1}{9^m}\phi(0,0,0,3^m \mathrm{u}, 3^m v, 0) \leq \lim_m \frac{1}{3^m}\phi(0,0,0,3^m u, 3^m v, 0) \\
&= 0
\end{aligned} \tag{3.19}$$

for all $u, v \in A$. Thus $h : A \to B$ is Jordan $*$-homomorphism satisfying (3.2), as desired. □

We prove the following Hyers-Ulam-Aoki-Rassias stability problem for Jordan $*$-homomorphisms on unital C^*-algebras.

Corollary 3.2. *Let $p \in (0,1)$, $\theta \in [0,\infty)$ be real numbers. Suppose $f : A \to A$ satisfies*

$$\begin{aligned}
&\left\| f\left(\frac{\mu x + \mu y + \mu z}{3}\right) + f\left(\frac{\mu x - 2\mu y + \mu z}{3}\right) + f\left(\frac{\mu x + \mu y - 2\mu z}{3}\right) - \mu f(x) + f(uv+uv) \right. \\
&\quad \left. - f(v)f(u) - f(u)f(v) + f(w^*) - f(w)^* \right\| \\
&\quad \leq \theta\left(\|x\|^p + \|y\|^p + \|z\|^p + \|u\|^p + \|v\|^p + \|w\|^p\right),
\end{aligned} \tag{3.20}$$

for all $\mu \in \mathbb{T}$ and all $x, y, z, u, v \in A$, $w \in U(A) \cup \{0\}$. Then there exists a unique Jordan $*$-homomorphism $h : A \to B$ such that such that

$$\|f(x) - h(x)\| \leq \frac{3^p\theta}{3 - 3^p}\|x\|^p \tag{3.21}$$

for all $x \in A$.

Proof. Setting $\phi(x,y,z,u,v,w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|u\|^p + \|v\|^p + \|w\|^p)$ all $x, y, z, u, v, w \in A$. Then by $L = 3^{p-1}$ in Theorem 3.2, one can prove the result. □

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