Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 456216, 15 pages
doi:10.1155/2011/456216

Research Article

Note on Some Nonlinear Integral Inequalities and Applications to Differential Equations

Khaled Boukerrioua

Department of Mathematics, University of 08 mai 1945, BP. 401, Guelma 24000, Algeria

Correspondence should be addressed to Khaled Boukerrioua, khaledv2004@yahoo.fr

Received 29 June 2011; Accepted 12 September 2011

Academic Editor: Elena Braverman

Using ideas from Boukerrioua and Guezane-Lakoud (2008), some nonlinear integral inequalities are established.

1. Introduction

Integral inequalities provide a very useful and handy device for the study of qualitative as well as quantitative properties of solutions of differential equations. The Gronwall-Bellman type (see, e.g., [1–4]) is particularly useful in that they provide explicit bounds for the unknown functions. One of the most useful inequalities in the development of the theory of diferential equations is given in the following theorem.

Theorem 1.1 (see [3]). *If $u(t)$ and $f(t)$ are non-nonnegative continuous functions on $[0,\infty[$ satisfying*

$$u^2(t) \leq c^2 + 2\int_0^t f(s)u(s)ds, \tag{1.1}$$

for some constant $c \geq 0$, then

$$u(t) \leq c + \int_0^t f(s)ds, \quad t \in [0,\infty[. \tag{1.2}$$

The importance of this inequality lies in its successful utilization of the situation for which the other available inequalities do not apply directly. It has been frequently used to obtain global existence, uniqueness, stability, boundedness, and other properties of the solution for wide classes of nonlinear differential equations. The aim of this paper is to give other results on nonlinear integral inequalities and their applications.

2. Main Results

In this section, we begin by giving some material necessary for our study. We denote by $\mathbb{R}$, the set of real numbers and $\mathbb{R}_+$ the nonnegative real numbers.

Lemma 2.1. *For $x \in \mathbb{R}_+,\ y \in \mathbb{R}_+,\ 1/p + 1/q = 1$, one has*

$$x^{1/p} y^{1/q} \leq \frac{x}{p} + \frac{y}{q}. \tag{2.1}$$

Lemma 2.2 (see [1]). *Let $b(t)$ and $f(t)$ be continuous functions for $t \geq \alpha$, let $v(t)$ be a differentiable function for $t \geq \alpha$ and suppose*

$$\begin{gathered} v'(t) \leq b(t)v(t) + f(t), \quad t \geq \alpha, \\ v(\alpha) \leq v_0. \end{gathered} \tag{2.2}$$

Then for $t \geq \alpha$,

$$v(t) \leq v_0 \exp \int_\alpha^t b(s)ds + \int_\alpha^t f(s)\left(\exp \int_s^t b(\sigma)d\sigma\right)ds. \tag{2.3}$$

Now we state the main results of this work

Theorem 2.3. *Let u, a, b, h_i $(i = 1, \ldots, n)$ be real-valued nonnegative continuous functions and there exists a series of positive real numbers $p_1, p_2, \ldots, p_n$ and $u(t)$ satisfy the following integral inequality,*

$$u^p(t) \leq a(t) + b(t) \int_0^t \sum_{i=1}^{i=n} h_i(s) u^{p_i}(s) ds, \tag{2.4}$$

for $t \in \mathbb{R}_+$ then

$$u(t) \leq \left\{ a(t) + b(t) \int_0^t \sum_{i=1}^{i=n} h_i(s) \left(\frac{p_i}{p} a(s) + \frac{p - p_i}{p} \right) \times \exp\left(\int_s^t b(\sigma) \sum_{i=1}^{i=n} \frac{p_i}{p} h_i(\sigma) d\sigma \right) ds \right\}^{1/P}, \tag{2.5}$$

for $p \geq p^\star = \max\{p_i, i = 1, \ldots, n\}$.

Proof. Define a function $v(t)$ by

$$v(t) = \int_0^t \sum_{i=1}^{i=n} h_i(s) u^{p_i}(s) ds, \tag{2.6}$$

then $v(0) = 0$ and (2.4) can be written as

$$u^p(t) \le a(t) + b(t)v(t). \tag{2.7}$$

By (2.7) and Lemma 2.1, we get

$$u^{p_i}(t) = (u^p(t))^{p_i/p} \le (a(t) + b(t)v(t))^{p_i/p} \le \frac{p_i}{p}(a(t) + b(t)v(t)) + \frac{p - p_i}{p}. \tag{2.8}$$

Differentiating (2.6), we get

$$v'(t) = \sum_{i=1}^{i=n} h_i(t) u^{p_i}(t). \tag{2.9}$$

Using (2.8) and (2.9), it yields

$$\begin{aligned} v'(t) &\le \sum_{i=1}^{i=n} h_i(t) \left(\frac{p_i}{p}(a(t) + b(t)v(t)) + \frac{p - p_i}{p} \right) \\ &= A(t) + B(t)v(t), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} B(t) &= b(t) \sum_{i=1}^{i=n} \frac{p_i}{p} h_i(t), \\ A(t) &= \sum_{i=1}^{i=n} h_i(t) \left(\frac{p_i}{p} a(t) + \frac{p - p_i}{p} \right). \end{aligned} \tag{2.11}$$

By Lemma 2.2, we have

$$v(t) \le \int_0^t A(s) \exp\left[\int_s^t B(\sigma) d\sigma \right] ds. \tag{2.12}$$

Using (2.7) and (2.12), we get

$$u(t) \le \left\{ a(t) + b(t) \int_0^t \sum_{i=1}^{i=n} h_i(s) \left(\frac{p_i}{p} a(s) + \frac{p - p_i}{p} \right) \times \exp\left(\int_s^t b(\sigma) \sum_{i=1}^{i=n} \frac{p_i}{p} h_i(\sigma) d\sigma \right) ds \right\}^{1/P}. \tag{2.13}$$

This achieves the proof of the theorem. □

Remark 2.4. if we take $n = 2$, $p \geq 1$, $p_1 = p$, and $p_2 = 1$, then the inequality established in Theorem 2.3 become the inequality given in [4, Theorem $1(a_1)$].

Theorem 2.5. *Suppose that the hypothesis of Theorem 2.3 holds. Assume that the function $(a(t) + p^\star/p)/b(t)$ is nondecreasing and*

$$u^p(t) \leq a(t) + b(t)\int_0^t \sum_{i=1}^{i=n} h_i(s)u^{p_i}(s)ds, \tag{2.14}$$

for $t \in \mathbb{R}_+$ then

$$\begin{aligned} u(t) \leq {}& \left(a(t) + \frac{p^\star}{p}\right)^{1/p} \\ & \times \left(1 - \left(\frac{p^\star}{p} - 1\right)\int_0^t b(s)\sum_{i=1}^{i=n} h_i(t)\left(a(t) + \frac{p^\star}{p}\right)^{(p^\star/p)-1} ds\right)^{1/p-p^\star}, \end{aligned} \tag{2.15}$$

for $p_\star = \min\{p_i, i = 1, \ldots, n\} \leq p < p^\star$ and $t < \beta_{p,p^\star}$, where

$$\beta_{p,p^\star} = \sup\left\{t \in \mathbb{R}_+ : ((p^\star/p) - 1)\int_0^t b(s)\sum_{i=1}^{i=n} h_i(t)\left(a(t) + \frac{p^\star}{p}\right)^{(p^\star/p)-1} ds \leq 1\right\}. \tag{2.16}$$

Proof. For $p_\star \leq p < p^\star$

By (2.7) and the fact that $(p^\star/p) > 1$, one gets:

$$u^{p_i}(t) = (u^p(t))^{p_i/p} \leq (a(t) + b(t)v(t))^{(p_i/p)} \leq \left(a(t) + b(t)v(t) + p^\star/p\right)^{p^\star/p}. \tag{2.17}$$

Differentiating (2.6) and using (2.17), we obtain

$$v'(t) \leq \sum_{i=1}^{i=n} h_i(t)\left(a(t) + b(t)v(t) + \frac{p^\star}{p}\right)^{p^\star/p}, \tag{2.18}$$

then

$$v'(t) \leq \left(b(t)\sum_{i=1}^{i=n} h_i(t)\right)\left(\frac{a(t) + (p^\star/p)}{b(t)} + v(t)\right)\left(a(t) + \frac{p^\star}{p} + b(t)v(t)\right)^{(p^\star/p)-1}. \tag{2.19}$$

Since the function $(a(t) + p^\star/p)/b(t)$ is nondecreasing, for $0 \leq t \leq \tau$ then,

$$v'(t) \leq M(t)\left(\frac{\mathrm{a}(\tau) + (p^\star/p)}{b(\tau)} + v(t)\right), \tag{2.20}$$

where

$$M(t) = b(t)\sum_{i=1}^{i=n} h_i(t)\left(a(t) + \frac{p^\star}{p} + b(t)v(t)\right)^{(p^\star/p)-1}. \tag{2.21}$$

Consequently

$$v(t) + \frac{a(\tau) + (p^\star/p)}{b(\tau)} \le \frac{a(\tau) + (p^\star/p)}{b(\tau)} \exp\int_0^t M(s)ds. \tag{2.22}$$

For $\tau = t$, we can see that

$$a(t) + \frac{p^\star}{p} + b(t)v(t) \le \left(a(t) + \frac{p^\star}{p}\right)\exp\int_0^t M(s)ds, \tag{2.23}$$

then the function $M(t)$ can be estimated as

$$M(t) \le b(t)\sum_{i=1}^{i=n} h_i(t)\left(a(t) + \frac{p^\star}{p}\right)^{(p^\star/p)-1} \cdot \exp\int_0^t \left(\frac{p^\star}{p} - 1\right)M(s)ds. \tag{2.24}$$

Let

$$L(t) = \left(\frac{p^\star}{p} - 1\right)M(t). \tag{2.25}$$

Now we estimate the expression $L(t)\exp(-\int_0^t L(s)ds)$ by using (2.24) to get

$$L(t)\exp\left(\int_0^t -L(s)ds\right) \le \left(\frac{p^\star}{p} - 1\right)b(t)\sum_{i=1}^{i=n} h_i(t)\left(a(t) + \frac{p^\star}{p}\right)^{(p^\star/p)-1}. \tag{2.26}$$

Remarking that

$$\begin{aligned} L(t)\exp\left(\int_0^t -L(s)ds\right) &= \frac{d}{dt}\left(-\exp\left(\int_0^t -L(s)ds\right)\right) \\ &\le \left(\frac{p^\star}{p} - 1\right)b(t)\sum_{i=1}^{i=n} h_i(t)\left(a(t) + \frac{p^\star}{p}\right)^{(p^\star/p)-1}, \end{aligned} \tag{2.27}$$

we integrate (2.27) from 0 to t to get

$$\left(1 - \exp\int_0^t -L(s)ds\right) \le \int_0^t \left(\frac{p^\star}{p} - 1\right)b(t)\sum_{i=1}^{i=n} h_i(t)\left(a(t) + \frac{p^\star}{p}\right)^{(p^\star/p)-1} ds, \tag{2.28}$$

replacing $L(t)$ by its value in (2.28), we obtain

$$\left(1-\exp\int_0^t\left(1-\frac{p^\star}{p}\right)M(s)ds\right)\leq\int_0^t\left(\frac{p^\star}{p}-1\right)b(t)\sum_{i=1}^{i=n}h_i(t)\left(a(t)+\frac{p^\star}{p}\right)^{(p^\star/p)-1}ds, \tag{2.29}$$

then

$$\exp\int_0^t M(s)ds\leq\left\{1-\left(\left(\int_0^t\left(\frac{p^\star}{p}-1\right)b(t)\sum_{i=1}^{i=n}h_i(t)\left(a(t)+\frac{p^\star}{p}\right)^{(p^\star/p)-1}\right)ds\right\}^{1/(1-(p^\star/p))}. \tag{2.30}$$

Using (2.7), (2.23), and (2.30) we have,

$$u(t)\leq\left(a(t)+\frac{p^\star}{p}\right)^{1/p}\cdot\left(1-\left(\frac{p^\star}{p}-1\right)\int_0^t b(s)\sum_{i=1}^{i=n}h_i(t)\left(a(t)+\frac{p^\star}{p}\right)^{(p^\star/p)-1}ds\right)^{1/(p-p^\star)}. \tag{2.31}$$

This achieves the proof of the theorem. □

Theorem 2.6. *Suppose that the hypothesis of Theorem 2.3 holds. Assume that the function* $(a(t)+p_\star/p)/b(t)$*, is nondecreasing and*

$$u^p(t)\leq a(t)+b(t)\int_0^t\sum_{i=1}^{i=n}h_i(s)u^{p_i}(s)ds, \tag{2.32}$$

for $t\in\mathbb{R}_+$ *then*

$$u(t)\leq\left(a(t)+\frac{p_\star}{p}\right)^{1/p}\times\left(1-\left(\frac{p^\star}{p}-1\right)\int_0^t b(s)\sum_{i=1}^{i=n}h_i(t)\left(a(t)+\frac{p_\star}{p}\right)^{(p^\star/p)-1}ds\right)^{1/(p-p^\star)}, \tag{2.33}$$

for $p<p_\star$ *and* $t<\beta_{p,p_\star}$,

$$\beta_{p,p_\star}=\sup\left\{t\in\mathbb{R}_+:((p^\star/p)-1)\ \int_0^t b(s)\sum_{i=1}^{i=n}h_i(t)\left(a(t)+\frac{p_\star}{p}\right)^{(p^\star/p)-1}ds\leq1\right\}. \tag{2.34}$$

Proof. for $p<p_\star=\min\{p_i,i=1,\ldots,n\}$.

Using (2.7), the fact that the function $(a(t)+(p_\star/p))/b(t)$ is nondecreasing, and $p_\star/p\succ 1$, we have

$$u^{p_i}(t)=(u^p(t))^{p_i/p}\leq(a(t)+b(t)v(t))^{p_i/p}\leq\left(a(t)+b(t)v(t)+\frac{p_\star}{p}\right)^{p^\star/p}. \tag{2.35}$$

Differentiating (2.6) and using (2.35), we obtain

$$v'(t) \leq \sum_{i=1}^{i=n} h_i(t)\left(a(t) + b(t)v(t) + \frac{p_\star}{p}\right)^{p^\star/p}, \tag{2.36}$$

then

$$v'(t) \leq \left(b(t)\sum_{i=1}^{i=n} h_i(t)\right)\left(\frac{a(t) + (p_\star/p)}{b(t)} + v(t)\right)\left(a(t) + \frac{p_\star}{p} + b(t)v(t)\right)^{(p^\star/p)-1}, \tag{2.37}$$

from the proof of Theorem 2.5, we get the required inequality in (2.33). □

Remark 2.7. if we take $n = 2$, then the inequalities established in Theorems 2.5 and 2.6 become the inequalities given in [5, Theorem 1.2].

Theorem 2.8. *Suppose that the hypothesis of Theorem 2.3 holds and moreover the function $b(t)$ is decreasing. Let c be a real valued nonnegative continuous and nondecreasing function for $t \in \mathbb{R}_+$. If*

$$u^p(t) \leq c^p(t) + b(t)\int_0^t \sum_{i=1}^{i=n} h_i(s)u^{p_i}(s)ds, \tag{2.38}$$

then

(1)

$$u(t) \leq c(t)\left\{1 + b(t)\int_0^t \sum_{i=1}^{i=n} H_i(s)\exp\left(\int_s^t b(\sigma)\sum_{i=1}^{i=n}\frac{p_i}{p}H_i(\sigma)d\sigma\right)ds\right\}^{1/p}, \tag{2.39}$$

for $p \geq p^\star$, where

$$H_i(s) = h_i(s)c(s)^{p_i - p}. \tag{2.40}$$

(2)

$$u(t) \leq c(t)\left(1 + \frac{p^\star}{p}\right)^{1/p} \times \left(1 - \left(\frac{p^\star}{p} - 1\right)\int_0^t b(s)\sum_{i=1}^{i=n} H_i(s)\left(1 + \frac{p^\star}{p}\right)^{(p^\star/p)-1} ds\right)^{1/(p-p^\star)}, \tag{2.41}$$

for $p_\star \leq p < p^\star$ and $t < \beta_{p,p^\star}$ where

$$\beta_{p,p^\star} = \sup\left\{t \in \mathbb{R}_+ : \left((p^\star/p) - 1\right)\int_0^t b(s)\sum_{i=1}^{i=n} H_i(s)\left(1 + \frac{p^\star}{p}\right)^{(p^\star/p)-1} ds \leq 1\right\}. \tag{2.42}$$

(3)

$$u(t) \leq c(t)\left(1+\frac{p_\star}{p}\right)^{1/p} \times \left(1-\left(\frac{p^\star}{p}-1\right)\int_0^t b(s)\sum_{i=1}^{i=n} H_i(s)\left(1+\frac{p_\star}{p}\right)^{(p^\star/p)-1} ds\right)^{1/(p-p^\star)}, \tag{2.43}$$

for and $t < \beta_{p,p^\star}$ *where*

$$\beta_{p,p_\star} = \sup\left\{t \in \mathbb{R}_+ : ((p^\star/p)-1)\int_0^t b(s)\sum_{i=1}^{i=n} H_i(s)\left(1+\frac{p_\star}{p}\right)^{(p^\star/p)-1} ds \leq 1\right\}. \tag{2.44}$$

Proof. Since $c(t)$ is a nonnegative, continuous, and nondecreasing function, for $t \in \mathbb{R}_+$, from (2.38) we observe that

$$\left(\frac{u(t)}{c(t)}\right)^p \leq 1 + b(t)\int_0^t \sum_{i=1}^{i=n} h_i(s)c(s)^{p_i-p}\left(\frac{u(s)}{c(s)}\right)^{p_i} ds, \tag{2.45}$$

we put

$$w(t) = \frac{u(t)}{c(t)}, \qquad H_i(t) = h_i(t)c(t)^{p_i-p}, \tag{2.46}$$

then, we have

$$w^p(t) \leq 1 + b(t)\int_0^t \sum_{i=1}^{i=n} H_i(s)w^{p_i}(s)ds. \tag{2.47}$$

Then a direct application of the inequalities established in Theorems 2.3, 2.5, and 2.6 gives the required results. □

Theorem 2.9. *Suppose that the hypothesis of Theorem 2.3 holds. Assume that the functions* $(a(t) + p^\star/p)/b(t)$, $(a(t)+p_\star/p)/b(t)$ *are nondecreasing and let* $k(t,s)$ *and its derivative partial* $\partial/\partial s k(t,s)$ *be real-valued nonnegative continuous functions, for* $0 \leq s \leq t \leq \infty$. *If*

$$u^p(t) \leq a(t) + b(t)\int_0^t K(t,s)\sum_{i=1}^{i=n} h_i(s)u^{p_i}(s)ds, \tag{2.48}$$

then

(1)

$$u(t) \leq \left\{ a(t) + b(t) \int_0^t A(\sigma) \left(\exp \int_\sigma^t B(\tau)\, d\tau) d\sigma \right) \right\}^{1/p}, \tag{2.49}$$

where

$$A(t) = K(t,t) \left\{ \sum_{i=1}^{i=n} h_i(t) \left(\frac{p_i}{p} a(t) + \frac{p - p_i}{p} \right) \right\} + \int_0^t \frac{\partial}{\partial s} k(t,s) \left\{ \sum_{i=1}^{i=n} h_i(s) \left(\frac{p_i}{p} a(s) + \frac{p - p_i}{p} \right) \right\} ds, \tag{2.50}$$

$$B(t) = K(t,t) b(t) \sum_{i=1}^{i=n} \frac{p_i}{p} h_i(t) + \int_0^t \frac{\partial}{\partial s} k(t,s) b(s) \sum_{i=1}^{i=n} \frac{p_i}{p} h_i(s) ds, \tag{2.51}$$

for $p \geq p^\star$.

(2)

$$u(t) \leq \left(a(t) + \frac{p^\star}{p} \right)^{1/p} \left\{ 1 - \left(\frac{p^\star}{p} - 1 \right) \int_0^t l(s) \left(a(s) + \frac{p^\star}{p} \right)^{(p^\star/p)-1} ds \right\}^{1/(p-p^\star)}. \tag{2.52}$$

For $p_\star \leq p \leq p^\star$ *and* $t < \beta_{p,p^\star}$, *where*

$$\beta_{p,p_\star} = \sup \left\{ t \in R_+ : ((p^\star/p) - 1) \int_0^t l(s) \left(a(s) + \frac{p_\star}{p} \right)^{(p^\star/p)-1} ds \leq 1 \right\}. \tag{2.53}$$

(3)

$$u(t) \leq \left(a(t) + \frac{p_\star}{p} \right)^{1/p} \left\{ 1 - \left(\frac{p^\star}{p} - 1 \right) \int_0^t l(s) \left(a(s) + \frac{p_\star}{p} \right)^{(p^\star/p)-1} ds \right\}^{1/(p-p^\star)}. \tag{2.54}$$

For $p < p_\star$ *and* $t < \beta_{p,p_\star}$, *where*

$$\beta_{p,p_\star} = \sup \left\{ t \in R_+ : ((p^\star/p) - 1) \int_0^t l(s) \left(a(s) + \frac{p_\star}{p} \right)^{(p^\star/p)-1} ds \leq 1 \right\}, \tag{2.55}$$

$$l(t) = b(t) K(t,t) \sum_{i=1}^{i=n} h_i(t) + b(t)^{1-(p^\star/p)} \int_0^t \frac{\partial}{\partial s} k(t,s) b(s)^{p^\star/p} \sum_{i=1}^{i=n} h_i(s) ds. \tag{2.56}$$

Proof. Let

$$v(t) = \int_0^t K(t,s)\sum_{i=1}^{i=n} h_i(s)u^{p_i}(s)ds, \tag{2.57}$$

(1) for $p \geq p^{\star}$.

Differentiating (2.57) we get

$$v'(t) = K(t,t)\sum_{i=1}^{i=n} h_i(t)u^{p_i}(t) + \int_0^t \frac{\partial}{\partial s}k(t,s)\sum_{i=1}^{i=n} h_i(s)u^{p_i}(s)ds. \tag{2.58}$$

Using (2.8) and (2.58) and the fact that $v(t)$ is nondecreasing, we obtain, for $0 \leq s \leq t$,

$$\begin{aligned}
v'(t) \leq{}& \left(K(t,t)b(t)\sum_{i=1}^{i=n}\frac{p_i}{p}h_i(t) + \int_0^t \frac{\partial}{\partial s}k(t,s)b(s)\sum_{i=1}^{i=n}\frac{p_i}{p}h_i(s)ds \right)v(t) \\
&+ K(t,t)\left\{ \sum_{i=1}^{i=n} h_i(t)\left(\frac{p_i}{p}a(t) + \frac{p-p_i}{p} \right) \right\} \\
&+ \int_0^t \frac{\partial}{\partial s}k(t,s)\left\{ \sum_{i=1}^{i=n} h_i(s)\left(\frac{p_i}{p}a(s) + \frac{p-p_i}{p} \right) \right\}ds,
\end{aligned} \tag{2.59}$$

then

$$v'(t) \leq B(t)v(t) + A(t). \tag{2.60}$$

By Lemma 2.2, we have

$$v(t) \leq \int_0^t A(s)\left(\exp\int_s^t B(\sigma)d\sigma \right)ds, \tag{2.61}$$

where

$$\begin{aligned}
A(t) &= K(t,t)\left\{ \sum_{i=1}^{i=n} h_i(t)\left(\frac{p_i}{p}a(t) + \frac{p-p_i}{p} \right) \right\} + \int_0^t \frac{\partial}{\partial s}k(t,s)\left\{ \sum_{i=1}^{i=n} h_i(s)\left(\frac{p_i}{p}a(s) + \frac{p-p_i}{p} \right) \right\}ds, \\
B(t) &= K(t,t)b(t)\sum_{i=1}^{i=n}\left(\frac{p_i}{p} \right)h_i(t) + \int_0^t \frac{\partial}{\partial s}k(t,s)b(s)\sum_{i=1}^{i=n}\left(\frac{p_i}{p} \right)h_i(s)ds.
\end{aligned} \tag{2.62}$$

Finally using (2.61) in (2.7), we get the required inequality.

(2) For $p_\star \le p < p^\star$. Using (2.17) and (2.58), we get

$$\begin{aligned} v'(t) &\le K(t,t)\sum_{i=1}^{i=n} h_i(t)\left(a(t)+\frac{p^\star}{p}+b(t)v(t)\right)^{p^\star/p} \\ &\quad+\int_0^t \frac{\partial}{\partial s}k(t,s)\left(a(s)+\frac{p^\star}{p}+b(s)v(s)\right)^{p^\star/p} ds, \end{aligned} \tag{2.63}$$

$$\begin{aligned} v'(t) &\le b(t)K(t,t)\sum_{i=1}^{i=n} h_i(t)\left(\frac{a(t)+(p^\star/p)}{b(t)}+v(t)\right)\left(a(t)+\frac{p^\star}{p}+b(t)v(t)\right)^{(p^\star/p)-1}+\int_0^t \frac{\partial}{\partial s}k(t,s) \\ &\quad\times\left\{b(s)^{p^\star/p}\sum_{i=1}^{i=n} h_i(s)\left(\frac{a(s)+(p^\star/p)}{b(s)}+v(s)\right)\left(\frac{a(s)+(p^\star/p)}{b(s)}+v(s)\right)^{(p^\star/p)-1}\right\} ds. \end{aligned} \tag{2.64}$$

For $0 \le s \le \mathrm{t} \le \tau$, and the fact that $(a(t)+p^\star/p)/b(t)$ is nondecreasing, we have

$$\begin{aligned} \left(\frac{a(s)+(p^\star/p)}{b(s)}+v(s)\right) &\le \left(\frac{a(t)+(p^\star/p)}{b(t)}+v(t)\right) \le \left(\frac{a(\tau)+(p^\star/p)}{b(\tau)}+v(t)\right), \\ \left(\frac{a(s)+(p^\star/p)}{b(s)}+v(s)\right)^{(p^\star/p)-1} &\le \left(\frac{a(t)+(p^\star/p)}{b(t)}+v(t)\right)^{(p^\star/p)-1} \\ &= b(t)^{1-(p^\star/p)}\left(a(t)+\frac{p^\star}{p}+b(t)v(t)\right)^{(p^\star/p)-1}, \end{aligned} \tag{2.65}$$

then,

$$v'(t) \le M(t)\left[\frac{a(\tau)+(p^\star/p)}{b(\tau)}+v(t)\right], \tag{2.66}$$

where

$$\begin{aligned} M(t) &= \left(b(t)K(t,t)\sum_{i=1}^{i=n} h_i(t)+b(t)^{1-(p^\star/p)}\int_0^t \frac{\partial}{\partial s}k(t,s)b(s)^{p^\star/p}\sum_{i=1}^{i=n} h_i(s)ds\right) \\ &\quad\times\left(a(t)+p+b(t)v(t)\right)^{(p^\star/p)-1}. \end{aligned} \tag{2.67}$$

From the proof of Theorem 2.5, we get the required inequality.

(3) Using (2.35) and (2.58), we get

$$v'(t) \leq K(t,t)\sum_{i=1}^{i=n} h_i(t)\left(a(t)+\frac{p_\star}{p}+b(t)v(t)\right)^{p^\star/p}+\int_0^t \frac{\partial}{\partial s}k(t,s)\left(a(s)+\frac{p_\star}{p}+b(s)v(s)\right)^{p^\star/p} ds. \tag{2.68}$$

taking account the fact that $(a(t)+(p_\star/p))/b(t)$ is nondecreasing and from the proof of Theorem 2.6, we get the required inequality. □

Remark 2.10. if we take $n = 2$, $p \geq 1$, $p_1 = p$, $p_2 = 1$, then the inequality established in Theorem 2.9 (part 1) becomes the inequality given in [4, Theorem $1(a_3)$].

3. Further Results

In this section, we investigate some Gronwall-type inequalities.

Theorem 3.1. *Assume that $u(t)$ and $f(t)$ are non-nonnegative continuous functions on $[0,\infty[$ and $c \geq 0$ is a constant. If $k(t,s)$ is defined as in Theorem 2.9, then*

$$u^p(t) \leq c+\int_{t_0}^t f(\tau)\left[u^q(\tau)+\int_{t_0}^{\tau} k(\tau,s)u^r(s)ds\right]d\tau, \tag{3.1}$$

implies

$$u(t) \leq \left\{c+\int_{t_0}^t f(\tau)\left[\left(\frac{q}{p}c+\frac{p-q}{p}\right)\exp\int_{t_0}^t A^*(s)ds+\int_{t_0}^t B^*(s)\left(\exp\int_s^t A^*(\sigma)d\sigma\right)ds\right]d\tau\right\}^{1/p} \tag{3.2}$$

where

$$\begin{aligned} A^*(t) &= \left\{\frac{q}{p}f(t)+\frac{r}{q}\left(K(t,s)+\int_{t_0}^t \frac{\partial}{\partial s}K(t,s)ds\right)\right\}, \\ B^*(t) &= \frac{p-r}{p}\left\{K(t,s)+\int_{t_0}^t \frac{\partial}{\partial s}K(t,s)ds\right\}, \end{aligned} \tag{3.3}$$

where $p \neq 0$, $0 \leq q \leq p$ and $0 \leq r \leq p$.

Proof. Define a function $z(t)$ by the right side of (3.1) then

$$\begin{aligned} z'(t) &= f(t)\left[u^q(t) + \int_{t_0}^{t} K(t,s)u^r(s)ds\right] \\ &\le f(t)\left[\frac{q}{p}Z(t) + \frac{p-q}{p} + \int_{t_0}^{t} k(t,s)\left(\frac{r}{p}Z(t) + \frac{p-r}{p}\right)ds\right]. \end{aligned} \tag{3.4}$$

Define a function $v(t)$ by

$$v(t) = \frac{q}{p}Z(t) + \frac{p-q}{p} + \int_{t_0}^{t} K(t,s)\left(\frac{r}{p}Z(s) + \frac{p-r}{p}\right)ds. \tag{3.5}$$

Then

$$\begin{gathered} v(t_0) = \frac{q}{p}c + \frac{p-q}{p}, \\ \frac{q}{p}Z(t) \le v(t), \\ z'(t) \le f(t)v(t), \end{gathered} \tag{3.6}$$

and $v(t)$ is nondecreasing for $t \in \mathbb{R}_+$.

Then,

$$v'(t) = \frac{q}{p}z'(t) + K(t,s)\left(\frac{r}{p}Z(t) + \frac{p-r}{p}\right) + \int_{t_0}^{t} \frac{\partial}{\partial s}k(t,s)\left(\frac{r}{p}Z(s) + \frac{p-r}{p}\right)ds, \tag{3.7}$$

its follow from (3.4), (3.6), and (3.7) that

$$v'(t) \le \left[\left\{\frac{q}{p}f(t) + \frac{r}{q}\left(K(t,s) + \int_{t_0}^{t} \frac{\partial}{\partial s}K(t,s)\ ds\right)\right\}v(t) + \frac{p-r}{p}\left\{K(t,s) + \int_{t_0}^{t} \frac{\partial}{\partial s}K(t,s)ds\right\}\right], \tag{3.8}$$

then (3.8) can be written as

$$v'(t) \le A^*(t)v(t) + B^*(t), \tag{3.9}$$

where

$$\begin{aligned} A^*(t) &= \left\{\frac{q}{p}f(t) + \frac{r}{q}\left(K(t,s) + \int_{t_0}^{t} \frac{\partial}{\partial s}K(t,s)ds\right)\right\}, \\ B^*(t) &= \frac{p-r}{p}\left\{K(t,s) + \int_{0}^{t} \frac{\partial}{\partial s}K(t,s)ds\right\}, \end{aligned} \tag{3.10}$$

by Lemma 2.2 we obtain

$$v(t) \le \left(\frac{q}{p}c + \frac{p-q}{p}\right)\left(\exp\int_{t_0}^{t} A^*(s)ds\right) + \int_{t_0}^{t} B^*(s)\left(\exp\int_{s}^{t} A^*(\sigma)d\sigma\right)ds, \tag{3.11}$$

from (3.6) and (3.11), it follows that

$$z'(t) \le f(t)\left[\left(\frac{q}{p}c + \frac{p-q}{p}\right)\exp\int_{t_0}^{t} A^*(s)ds + \int_{t_0}^{t} B^*(s)\left(\exp\int_{s}^{t} A^*(\sigma)d\sigma\right)ds\right], \tag{3.12}$$

integrating (3.12), we obtain

$$z(t) \le c + \int_{t_0}^{t} f(\tau)\left[\left(\frac{q}{p}c + \frac{p-q}{p}\right)\exp\int_{t_0}^{t} A^*(s)ds + \int_{t_0}^{t} B^*(s)\left(\exp\int_{s}^{t} A^*(\sigma)d\sigma\right)ds\right]d\tau, \tag{3.13}$$

but

$$u(t) \le (z(t))^{1/p}, \tag{3.14}$$

then the result required is found. □

4. Application

In this section we present some applications of Theorems 2.3, 2.5, 2.6 and 3.1 to investigate certain properties of solutions of differential equation.

Example 4.1. We consider a nonlinear differential equation

$$\begin{aligned} (u^p)'(t) &= H(t, u(t)), \\ u(0) &= u_0. \end{aligned} \tag{4.1}$$

Assume that p, p_i $(i = 1, \dots, n) \ge 0$, are fixed real numbers, u_0 is a real constant, and $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$,

$$|H(t, u(t))| \le l(t) + \sum_{i=1}^{i=n} h_i(t) u^{p_i}(t), \tag{4.2}$$

$h_i, l : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions.

Integrating (4.1), from 0 to t and using (4.2) we obtain

$$|u(t)|^p \le a(t) + \int_0^t \sum_{i=1}^{i=n} h_i(s) |u^{p_i}(s)|(s) ds, \tag{4.3}$$

where $a(t) = |u_0|^p + \int_0^t l(s)ds$. By applying Theorems 2.3, 2.5, and 2.6, we estimate the solution $u(t)$ of the equation, that gives us a bound of the solution.

Example 4.2. Consider the following initial value problem:

$$(u^p(t))' = f(t)\left[u^q(t) + \int_{t_0}^{t} K(t,s)u^r(s)ds\right], \quad u(t_0) = c, \tag{4.4}$$

where $f(t)$ and $k(t,s)$ are as defined in Theorem 3.1, and $p \neq 0$, $0 \leq q \leq p$ and $0 \leq r \leq p$, and c is a constant.

Theorem 4.3. *Assume $u(t)$ is a solution of* (4.4), *then*

$$|u(t)| \leq \left\{ |c|^p + \int_{t_0}^{t} f(\tau)\left[\left(\frac{q}{p}c + \frac{p-q}{p}\right)\exp\int_{t_0}^{t} A^*(s)ds + \int_{t_0}^{t} B^*(s)\left(\exp\int_{s}^{t} A^*(\sigma)d\sigma\right)ds\right]d\tau\right\}^{1/p} \tag{4.5}$$

where $A^*(t)$ and $B^*(t)$ are defined in (3.10).

Proof. The solution $u(t)$of (4.4) satisfies the following equivalent equation:

$$u^p(t) = c^p + \int_{t_0}^{t} f(\tau)\left[u^q(\tau) + \int_{t_0}^{t} k(\tau,s)u^r(s)ds\right]d\tau. \tag{4.6}$$

It follows from (4.6) that

$$|u(t)|^p \leq |c|^p + \int_{t_0}^{t} f(\tau)\left[|u(\tau)|^q + \int_{t_0}^{t} k(\tau,s)|u(\tau)|^r ds\right]d\tau. \tag{4.7}$$

Using Theorem 3.1, we obtain (4.5). □

References

[1] D. Baĭnov and P. Simeonov, *Integral Inequalities and Applications*, vol. 57 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[2] B. G. Pachpatte, "On some new inequalities related to certain inequalities in the theory of differential equations," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 1, pp. 128–144, 1995.

[3] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, vol. 197 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1998.

[4] B. G. Pachpatte, "On some new inequalities related to a certain inequality arising in the theory of differential equations," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 2, pp. 736–751, 2000.

[5] K. Boukerrioua and A. Guezane-Lakoud, "Some nonlinear integral inequalities arising in differential equations," *Electronic Journal of Differential Equations*, vol. 2008, no. 80, pp. 1–6, 2008.