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Research Article

Random Attractors for the Stochastic Discrete Long Wave-Short Wave Resonance Equations

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We prove the existence of the random attractor for the stochastic discrete long wave-short wave resonance equations in an infinite lattice. We prove the asymptotic compactness of the random dynamical system and obtain the random attractor.

1. Introduction

There has been considerable progress in the study of infinite-dimensinal dynamical systems in the past few decades (see [1–5]). Recently, the dynamics of infinite lattice systems has attracted a great deal of attention from mathematicians and physicists; see [6–11] and the references therein. Various properties of solutions for lattice dynamical systems (LDSs) have been extensively investigated. For example, the long-time behavior of LDSs was studied in [5, 10]. Lattice dynamical systems play an important role in their potential application such as biology, chemical reaction, pattern recognition and image processing, electrical engineering, and laser systems. However, a system in reality is usually affected by external perturbations within many cases that are of great uncertainty or random influence. These random effects are introduced not only to compensate for the defects in some deterministic models but also to explain the intrinsic phenomena. Therefore, there is much work concerning stochastic lattice dynamical systems. The study of random attractors gained considerable attention during the past decades; see [12] for a comprehensive survey. Bates et al. [13] first investigated the existence of global random attractor for a kind of first-order dynamic systems driven by white noise on lattice $\mathbb{Z}$; then, Lv and Sun [14] extended the results of Bates to the dimensional lattices. Stochastic complex Ginzburg-Landau equations, FitzHugh-Nagumo equation, and

KGS equations in an infinite lattice are studied by Lv and Sun [15], Huang [16], and Yan et al. [17], respectively.

The long wave-short wave (LS) resonance system is an important model in nonlinear science. Long wave-short wave resonance equations arise in the study of the interaction of surface waves with both gravity and capillary modes present and also in the analysis of internal waves as well as Rossby waves as in [18]. In the plasma physics they describe the resonance of the high-frequency electron plasma oscillation and the associated low-frequency ion density perturbation in [19].

Due to their rich physical and mathematical properties the long wave-short wave resonance equations have drawn much attention of many physicists and mathematicians. The LS system is as follows:

$$iu_t + u_{xx} - uv + i\alpha u = f(x,t), \quad v_t + \gamma|u|_x^2 + \beta v = g(x,t), \quad x \in \mathbb{R},\ t \geq 0, \tag{1.1}$$

where u denotes a complex-valued vector and v represents a real-valued function; $f(x,t)$ and $g(x,t)$ are given complex- and real-valued functions, respectively. The constants α, β are positive, and $\gamma \in \mathbb{R} \setminus \{0\}$ is real. For the LS equations, Boling Guo obtained the global solution in [20]. The existence of global attractor was studied in [21–23].

Throughout this paper, we set

$$\mathbb{L}^2 = \left\{ u = (u_n)_{n\in\mathbb{Z}}, u_n \in \mathbb{C} : \sum_{n\in\mathbb{Z}} |u_n|^2 < \infty \right\}, \qquad \ell^2 = \left\{ v = (v_n)_{n\in\mathbb{Z}},\ v_n \in \mathbb{R} : \sum_{n\in\mathbb{Z}} v_n^2 < \infty \right\}. \tag{1.2}$$

For brevity, we use H to denote Hilbert space $\mathbb{L}^2$ or ℓ^2 and equip H with the inner product and norm as

$$(u,v) = \sum_{n\in\mathbb{Z}} u_n \overline{v}_n, \quad \|u\|^2 = (u,u) = \sum_{n\in\mathbb{Z}} |u|^2, \quad \forall u, v \in H, \tag{1.3}$$

where $\overline{v}_n$ denotes the conjugate of v_n.

In this paper, we consider the following stochastic discrete LS equations

$$i\frac{du_n}{dt} - (Au)_n - v_n u_n + i\alpha u_n = f_n(t) + a_n u_n \frac{dw_n^1}{dt}, \quad n \in \mathbb{Z},\ t > 0, \tag{1.4}$$

$$\frac{dv_n}{dt} + \gamma\Big(B\big(|u|^2\big)\Big)_n + \beta v_n = g_n(t) + b_n \frac{dw_n^2}{dt}, \quad n \in \mathbb{Z},\ t > 0, \tag{1.5}$$

with the initial conditions

$$u_n(0) = u_{n0}, \quad v_n(0) = v_{n_0}, \quad n \in \mathbb{Z}, \tag{1.6}$$

where $|u|^2 = (|u_n|^2)_{n\in\mathbb{Z}}$, $u_n(t) \in \mathbb{C}$, $v_n(t) \in \mathbb{R}$ ($\mathbb{C}, \mathbb{R}$ are the sets of complex and real numbers, resp.), $a = (a_n)_{n\in\mathbb{Z}} \in \ell^2$ and $b = (b_n)_{n\in\mathbb{Z}} \in \ell^2$, $f(t) = (f_n(t))_{n\in\mathbb{Z}}$, $g(t) = (g_n(t))_{n\in\mathbb{Z}} \in C_B(\mathbb{R}, \ell^2)$, the space of bounded continuous functions from $\mathbb{R}$ into ℓ^2. $\{w_n^1(t) : n \in \mathbb{Z}\}$ and $\{w_n^2(t) : n \in \mathbb{Z}\}$

are two independent two-side real-valued standard Wiener process. $\mathbb{Z}$ is the integer set, i is the unit of the imaginary numbers such that $i^2 = -1$, and A, B are linear operators defined, respectively, by

$$(Au)_n = u_{n+1} + u_{n-1} - 2u_n, \quad n \in \mathbb{Z}, \ \forall u = (u_n)_{n\in\mathbb{Z}},$$
$$(Bu)_n = u_{n+1} - u_n, \quad n \in \mathbb{Z}, \ \forall u = (u_n)_{n\in\mathbb{Z}}. \tag{1.7}$$

In addition, simple computation shows that, for $u = (u_n)_{n\in\mathbb{Z}} \in H$, there holds

$$\|Bu\|^2 = \sum_{n\in\mathbb{Z}} |u_{n+1} - u_n|^2 \le 2\sum_{n\in\mathbb{Z}} \left(|u_{n+1}|^2 + |u_n|^2\right) = 4\|u\|^2. \tag{1.8}$$

This paper is organized as follows. In the next section, we recall some basic concepts and already know results to random dynamical system and random attractors. In Section 3, we prove the existence of the global random attractor for stochastic LS lattice dynamical systems (1.4)–(1.6).

2. Preliminaries

In this section, we first introduce the definitions of the random dynamical systems and random attractor, which are taken from [13]. Let $(H, \|\cdot\|_H)$ be a Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.

Definition 2.1. $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ are called metric dynamical systems; if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathbb{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, $\theta_0 = \mathbb{I}$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.2. A stochastic process $\phi(t, \omega)$ is called a continuous random dynamical system (RDS) over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ if ϕ is $(\mathbb{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathbb{B}(H), \mathbb{B}(H))$-measurable, and for all $\omega \in \Omega$

(i) the mapping $\phi : \mathbb{R}^+ \times \Omega \times H \to H$ is continuous;

(ii) $\phi(0, \omega) = \mathbb{I}$ on H;

(iii) $\phi(t + s, \omega, \chi) = \phi(t, \theta_s\omega, \phi(s, \omega, \chi))$ for all $t, s \ge 0$ and $\chi \in H$ (cocycle property).

Definition 2.3. A random bounded set $B(\omega) \subset H$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$ and all $\epsilon > 0$

$$\lim_{t\to\infty} e^{-\epsilon t} d(B(\theta_{-t}\omega)) = 0, \tag{2.1}$$

where $d(B) = \sup_{\chi\in B}\|\chi\|_H$.

Consider a continuous random dynamical system $\phi(t, w)$ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, and let $\mathbb{D}$ be the collection of all tempered random sets of H.

Definition 2.4. A random set $\mathbb{K}(\omega)$ is called an absorbing set in $\mathbb{D}$ if for all $B \in \mathbb{D}$ and a.e. $\omega \in \Omega$ there exist $t_B(\omega) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset \mathbb{K}(\omega), \quad t \ge t_B(\omega). \tag{2.2}$$

Definition 2.5. A random set $\mathbb{A}(\omega)$ is a random $\mathbb{D}$-attractor for RDS ϕ if

(i) $\mathbb{A}(\omega)$ is a random compact set, that is, $\omega \to d(\chi, \mathbb{A}(\omega))$ is measurable for every $\chi \in H$ and $\mathbb{A}(\omega)$ is compact for a.e. $\omega \in \Omega$;

(ii) $\mathbb{A}(\omega)$ is strictly invariant, that is, $\phi(t, \omega, \mathbb{A}(\omega)) = \mathbb{A}(\theta_t \omega)$, for all $t \geq 0$ and for a.e. $\omega \in \Omega$;

(iii) $\mathbb{A}(\omega)$ attracts all sets in $\mathbb{D}$, that is, for all $B \in \mathbb{D}$ and a.e. $\omega \in \Omega$ we have

$$\lim_{t \to \infty} d\big(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))\mathbb{A}(\omega)\big) = 0, \tag{2.3}$$

where $d(X, Y) = \sup_{\chi \in X} \inf_{y \in Y} \|\chi - y\|_H$, $X, Y \subset H$.

The collection $\mathbb{D}$ is called the domain of attraction of $\mathbb{A}$.

Definition 2.6. Let ϕ be an RDS on Hilbert space H. ϕ is called asymptotically compact if, for any bounded sequence $\{\chi_n\} \subset H$ and $t_n \to \infty$, the set $\{\phi(t_n, \theta_{-t_n}\omega, \chi_n)\}$ is precompact in H, for any $\omega \in \Omega$.

From [13], we have the following result.

Proposition 2.7. *Let $\mathbb{K} \in \mathbb{D}$ be an absorbing set for an asymptotically compact continuous RDS ϕ. Then ϕ has a unique global random $\mathbb{D}$-attractor*

$$\mathbb{A}(\omega) = \bigcap_{\kappa \geq t_{\mathbb{K}(\omega)}} \overline{\bigcup_{t \geq \kappa} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))} \tag{2.4}$$

which is compact in H.

Let $W^1(t) = \sum_{n \in \mathbb{Z}} a_n w_n^1(t) e_n$ and $W^2(t) = \sum_{n \in \mathbb{Z}} b_n w_n^2(t) e_n$, where $(a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}} \in \ell^2$. Here $\{e_n\}$ denotes the standard complete orthonormal system in ℓ^2, which means that the nth component of e_n is 1 and all other elements are 0. Then $W^1(\cdot)$ and $W^2(\cdot)$ are ℓ^2-valued Q-Wiener processes. It is obvious that $EW^1(t) = EW^2(t) = 0$. For details we refer to [24].

Now, we abstract (1.4)–(1.6) as stochastic ordinary differential equations with respect to time t in $E = \mathbb{L}^2 \times \ell^2$. Let $a = (a_n)_{n \in \mathbb{Z}}, b = (b_n)_{n \in \mathbb{Z}}, f = (f_n)_{n \in \mathbb{Z}}$, and $g = (g_n)_{n \in \mathbb{Z}}$. Then (1.4)–(1.6) can be written as the following integral equations:

$$u(t) = u_0 + \int_0^t \left[-iu(s)v(s) - \alpha u(s) - iAu(s) - if(s)\right] ds - i \int_0^t u(s) dW^1, \tag{2.5}$$

$$v(t) = v_0 + \int_0^t \left[-\beta v(s) - \gamma B\left(|u|^2\right) + g(s)\right] ds + W^2. \tag{2.6}$$

Remark 2.8. The special form of multiplicative noise in (2.5) is more suitable than the white noise "adW" and the additive noise "$\sum_{j=1}^n a_j dW^j$", because it is more approximative to the perturbations of the short wave for this model.

For our purpose we introduce the probability space as

$$\Omega = \left\{\omega \in C\left(\mathbb{R}, \ell^2\right) : \omega(0) = 0\right\} \tag{2.7}$$

endowed with the compact open topology [12]. $\mathbb{P}$ is the corresponding Wiener measure, and $\mathcal{F}$ is the $\mathbb{P}$-completion of the Borel σ-algebra on Ω.

Let $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t),\ t \in \mathbb{R}$. Then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system with the filtration $\mathcal{F}_t := \vee_{s\le t}\mathcal{F}_s^t,\ t \in \mathbb{R}$, where $\mathcal{F}_s^t = \sigma\{W(t_2) - W(t_1) : s \le t_1 \le t_2 \le t\}$ is the smallest σ-algebra generated by the random variable $W(t_2) - W(t_1)$ for all t_1, t_2 such that $s \le t_1 \le t_2 \le t$; see [12] for more details.

3. The Existence of a Random Attractor

In this section, we study the dynamics of solutions for the stochastic LS (1.4)–(1.6). Then we apply Proposition 2.7 to prove the existence of a global random attractor for stochastic lattice LS equations.

Before proving the existence of global solution for (2.5)-(2.6), we need the following a priori estimate.

Lemma 3.1. *Suppose that* $f(t) = (f_n(t))_{n\in\mathbb{Z}} \in C_B(\mathbb{R}, \ell^2)$. *Then, the solution of* (1.4)–(1.6) *satisfies*

$$\|u(t, \omega; u_0)\|^2 \le e^{-\alpha t}\|u_0\|^2 + \frac{1}{\alpha}\|f\|^2, \quad t \ge 0, \tag{3.1}$$

for all $\omega \in \Omega$ *and* $\|f\| = \sup_{t\in\mathbb{R}}|f(t)|^2$.

Proof. We write (1.4) in the form of vector as

$$iu_t - Au - vu + i\alpha u = f + uW_t^1, \quad t > 0. \tag{3.2}$$

Taking the imaginary part of the inner product of (3.2) with u, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \alpha\|u\|^2 = -\operatorname{Im}(f, u) \le \frac{\alpha}{2}\|u\|^2 + \frac{1}{2\alpha}\|f\|^2. \tag{3.3}$$

So we have

$$\frac{d}{dt}\|u(t, \omega; u_0)\|^2 + \alpha\|u(t, \omega; u_0)\|^2 \le \frac{1}{\alpha}\|f\|^2. \tag{3.4}$$

By the Gronwall inequality, we get

$$\|u(t, \omega; u_0)\|^2 \le e^{-\alpha t}\|u_0\|^2 + \frac{1}{\alpha}\|f\|^2. \tag{3.5}$$

Thus, we derive (3.1). □

By Lemma 3.1, we know that $\|u\|^2$ is bounded in any bounded subset of $[0,\infty)$, that is, $\|u\|^2 \le e^{-\alpha t}\|u_0\|^2 + (1/\alpha)\|f\|^2$, $0 \le t \le T$, for any fixed constant $T > 0$.

In order to show the existence of global solutions of (2.5)-(2.6), we first change (2.5)-(2.6) into deterministic equations. First, due to special linear multiplicative noise, (2.5) can be reduced to an equation with random coefficients by a suitable change of variable. Consider the process $z(t) = e^{iW^1(t)}$, which satisfies the stochastic differential equation

$$dz(t) = -\frac{1}{2}z(t)dt + iz(t)dW^1. \tag{3.6}$$

The process $\tilde{u} = z(t)u(t)$ follows the random differential equation,

$$i\frac{d\tilde{u}}{dt} - A\tilde{u} + i\left(\alpha + \frac{1}{2}\right)\tilde{u} - v\tilde{u} - f(t)z(t) = 0. \tag{3.7}$$

We denote $\tilde{v} = v(t) - W^2(t)$, then (2.5)-(2.6) can be changed into the following equations:

$$\begin{aligned}
\tilde{u}(t) &= \tilde{u}_0 + \int_0^t \left[-i\tilde{u}(s)v(s) - \left(\alpha + \frac{1}{2}\right)\tilde{u}(s) - iA\tilde{u}(s) - if(s)z(s)\right]ds, \\
\tilde{v}(t) &= \tilde{v}_0 + \int_0^t \left[-\beta\tilde{v}(s) - \gamma B\left(|u|^2\right) + g(s) - \beta W^2(s)\right]ds.
\end{aligned} \tag{3.8}$$

Remark 3.2. For the general multiplicative noise, we can also choose a suitable process and a change of variable to convert the stochastic equations into deterministic equations.

For each fixed $\omega \in \Omega$, (3.8) are deterministic equations, and we have the following result.

Theorem 3.3. *For any $T > 0$, (2.5)-(2.6) are well posed and admit a unique solution $(u(t), v(t)) \in \mathbb{L}^2(\Omega; C([0,T]; E))$. Moreover, the solution of (2.5)-(2.6) depends continuously on the initial data (u_0, v_0).*

Proof. By standard existence theorem for ODEs, it follows that (3.8) possess a local solution $(\tilde{u}(t), \tilde{v}(t)) \in C(0,T;E)$, where $[0, T_{\max}]$ is the maximal interval of existence of the solution of (3.8). Now, we prove that this local solution is a global solution. Let $\omega \in \Omega$; from (3.8) it follows that

$$\begin{aligned}
&\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2 \\
&\quad = \|\tilde{u}_0\|^2 + \|\tilde{v}_0\|^2 - 2\left(\alpha + \frac{1}{2}\right)\int_0^t \|\tilde{u}(s)\|^2 ds - 2\int_0^t \operatorname{Re}\left(ifz, \tilde{u}(s)\right)ds - 2\beta\int_0^t \|\tilde{v}(s)\|^2 ds \\
&\qquad - 2\gamma\int_0^t \left(B\left(|u|^2\right), \tilde{v}(s)\right)ds + 2\int_0^t (g(s), \tilde{v}(s))ds - 2\beta\int_0^t \left(W^2(s), \tilde{v}(s)\right)ds.
\end{aligned} \tag{3.9}$$

By the Young inequality and (1.8), direct computation shows that

$$
\begin{aligned}
-2\operatorname{Re}(ifz,\tilde{u}) &\le \frac{2}{\alpha}\|f\|^2\|z\|^2+\frac{\alpha}{2}\|\tilde{u}\|^2 \le \frac{1}{\alpha}\left(\|f\|^4+\left\|W^1\right\|^4\right)+\frac{\alpha}{2}\|\tilde{u}\|^2,\\
-2\gamma\left(B\left(|u|^2\right),\tilde{v}\right) &\le 2|\gamma|\left\|B\left(|u|^2\right)\right\|\|\tilde{v}\| \le |\gamma|\varepsilon\|\tilde{v}\|^2+\frac{4|\gamma|}{\varepsilon}\|u\|^4,\\
2(g,\tilde{v}) &\le |\gamma|\varepsilon\|\tilde{v}\|^2+\frac{1}{|\gamma|\varepsilon}\|g\|^2,\\
-2\beta\left(W^2,\tilde{v}\right) &\le 2\beta\left\|W^2\right\|\|\tilde{v}\| \le |\gamma|\varepsilon\|\tilde{v}\|^2+\frac{\beta}{|\gamma|\varepsilon}\left\|W^2\right\|^2.
\end{aligned}
\tag{3.10}
$$

Combining the above inequalities with Lemma 3.1, we obtain

$$
\begin{aligned}
\|\tilde{u}(t)\|^2+\|\tilde{v}(t)\|^2 &\le \|\tilde{u}_0\|^2+\|\tilde{v}_0\|^2-C_0\int_0^t\left(\|\tilde{u}(s)\|^2+\|\tilde{v}(s)\|^2\right)ds\\
&\quad + C_1\int_0^t\left(\|f\|^2+\|g\|^2+\left\|W^2\right\|^2+\left\|W^1\right\|^4\right)ds\\
&\le \|\tilde{u}_0\|^2+\|\tilde{v}_0\|^2+C_1\int_0^t\left(\|f\|^2+\|g\|^2+\left\|W^2\right\|^2+\left\|W^1\right\|^4\right)ds,
\end{aligned}
\tag{3.11}
$$

where C_0, C_1 are constants depending on α, β, γ, and ε is a sufficiently small positive number. By the Gaussian property of W^1 and W^2, (3.11) implied that (3.8) admit a global solution $(\tilde{u}(t), \tilde{v}(t)) \in \mathbb{L}^2(\Omega; C([0,T]; E))$. The proof of the lemma is completed. □

From the definition $(\theta_t)_{t\in\mathbb{R}}$, we know

$$
W(t+h,\omega) = W(t,\theta_h\omega) + W(h,\omega), \quad \forall t,h \in \mathbb{R},
\tag{3.12}
$$

and combining the above theorem we have the following result.

Theorem 3.4. *System* (2.5)-(2.6) *generates a continuous random dynamical system* $(\phi(t,\theta_{-t}\omega))_{t\ge 0}$ *over* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$.

The proof is similar to that of Theorem 3.2 in [13], so we omit it.

Now, we prove the existence of a random attractor for system (2.5)-(2.6). By Proposition 2.7, we first prove that RDS ϕ possesses a bounded absorbing set $\mathbb{K}(\omega)$. We introduce an Ornstein-Uhlenbeck process in ℓ^2 on the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ given by Wiener process:

$$
z^1(\theta_t\omega) = -\nu\int_{-\infty}^0 e^{\nu h}\theta_t\omega(h)dh, \quad t\in\mathbb{R}, \qquad z^2(\theta_t\omega) = -\mu\int_{-\infty}^0 e^{\mu h}\theta_t\omega(h)dh, \quad t\in\mathbb{R},
\tag{3.13}
$$

where ν and μ are positive. The above integral exists in the sense of for any path ω with a subexponential growth and z^1, z^2 solve the following Itô equations:

$$dz^1 + \nu z^1 dt = dW^1(t), \qquad dz^2 + \mu z^2 dt = dW^2(t). \tag{3.14}$$

In fact, the mapping $t \to z^i(\theta_t \omega),\ i = 1,2$, is the $O - U$ process. Furthermore, there exists a θ_t invariant set $\Omega' \subset \Omega$ of full $\mathbb{P}$ measure such that

(1) the mapping $t \to z^i(\theta_t \omega),\ i = 1,2$, is continuous for each $\omega \in \Omega'$,

(2) the random variables $\|z^i(\omega)\|,\ i = 1,2$, are tempered.

Lemma 3.5. *There exists a θ_t invariant set $\Omega' \subset \Omega$ of full $\mathbb{P}$ measure and an absorbing random set $\mathbb{K}(\omega), \omega \in \Omega'$, for the random dynamical system $(\phi(t, \theta_{-t}\omega))_{t\geq 0}$.*

Proof. We use the estimates in Theorem 3.3. By (3.11), we have

$$\frac{d\left(\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2\right)}{dt} \leq -C_0\left(\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2\right) + C_1 \rho(\theta_t \omega), \tag{3.15}$$

where $\rho(\theta_t \omega) = \|f\|^2 + \|g\|^2 + \|W^1\|^4 + \|W^2\|^2$.

By the Gronwall inequality, we have

$$\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2 \leq \left(\|\tilde{u}_0\|^2 + \|\tilde{v}_0\|^2\right) e^{-C_0 t} + \int_0^t e^{-C_0(t-s)} \rho(\theta_s \omega) ds. \tag{3.16}$$

Replace ω by $\theta_{-t}\omega$ in the above inequality to construct the radius of the absorbing set and define

$$\varrho^2(\omega) = 4 \lim_{t\to\infty} \int_0^t e^{-C_0(t-s)} \rho(\theta_{s-t}\omega) ds = 4 \lim_{t\to\infty} \int_{-t}^0 e^{-C_0 s} \rho(\theta_s \omega) ds. \tag{3.17}$$

Define

$$R^2(\omega) = \varrho^2(\omega) + \frac{4}{\alpha} \|f\|^2 \left\|W^1\right\|^4 + \left\|W^2\right\|^2. \tag{3.18}$$

Then, $\mathbb{K}(\omega) \triangleq \mathbb{K}(0, R(\omega))$ is a tempered ball by the property of W^1, W^2, and, for any $B \in \mathbb{D},\ \omega \in \Omega$. Here, $\mathbb{D}$ denotes the collection of all tempered random sets of Hilbert space H. The proof of the lemma is completed. □

Lemma 3.6. *Let $(u_0, v_0) \in \mathbb{K}(\omega)$, the absorbing set given in Lemma 3.5. Then, for every $\varepsilon > 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist $T(\varepsilon, \omega) > 0$ and $N(\varepsilon, \omega) > 0$ such that the solution (u, v) of system* (2.5)-(2.6) *satisfies*

$$\sum_{|n| > N(\varepsilon,\omega)} \left[|u_n(t, \theta_{-t}\omega)|^2 + |v_n(t, \theta_{-t}\omega)|^2\right] \leq \varepsilon, \quad \forall t \geq T(\varepsilon, \omega). \tag{3.19}$$

Proof. Let $\eta(x) \in C(\mathbb{R}^+, [0,1])$ be a cut-off function satisfying

$$\eta(x) = \begin{cases} 1, & \forall x \in [2, +\infty), \\ 0, & \forall x \in [0,1], \end{cases} \tag{3.20}$$

and $|\eta'(x)| \le \eta_0$ (a positive constant).

Taking the inner product of (3.8) with $(\eta(|n|/M)\tilde{u}_n)_{n\in\mathbb{Z}}$ and $(\eta(|n|/M)\tilde{v}_n)_{n\in\mathbb{Z}}$, respectively, we get

$$\begin{aligned} \frac{d}{dt}\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)|\tilde{u}_n|^2 &= -2\left(\alpha+\frac{1}{2}\right)\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)|\tilde{u}_n|^2 - 2\,\mathrm{Im}\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)(f_n z_n, \tilde{u}_n), \\ \frac{d}{dt}\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)|\tilde{v}_n|^2 &= -2\beta\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)|\tilde{v}_n|^2 - 2\gamma\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)\left(B\left(|u|^2\right)_n, \tilde{v}_n\right) + 2\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)(g_n, \tilde{v}_n) \\ &\quad - 2\beta\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)\left(W_n^2, \tilde{v}_n\right). \end{aligned} \tag{3.21}$$

We also use the estimates in Theorem 3.3. Similar to (3.11), it follows that

$$\begin{aligned} &\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)\left(|\tilde{u}_n|^2+|\tilde{v}_n|^2\right) \\ &\quad\le e^{-C_0(t-T_k)}\left(\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)|\tilde{u}_n(T_k,\omega)|^2 + \sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)|\tilde{v}_n(T_k,\omega)|^2\right) \\ &\qquad + C_1\int_{T_k}^{t} e^{C_0(s-t)}\sum_{|n|\ge M}\left(\left|z_n^1(\theta_s\omega)\right|^2 + \left|z_n^2(\theta_s\omega)\right|^2 + \left|f_n\right|^2 + \left|g_n\right|^2\right)ds. \end{aligned} \tag{3.22}$$

Replace ω by $\theta_{-t}\omega$ in (3.22). Then, we estimate each term on the right-hand of (3.22); it follows that

$$\begin{aligned} &e^{-C_0(t-T_k)}\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)\left(|\tilde{u}_n(T_k,\theta_{-t}\omega)|^2 + |\tilde{v}_n(T_k,\theta_{-t}\omega)|^2\right) \\ &\quad\le e^{-C_0(t-T_k)}\Bigg[\left(|u_0(\theta_{-t}\omega)|^2 + |v_0(\theta_{-t}\omega)|^2\right)e^{-C_0T_k} \\ &\qquad + \int_0^{T_k} e^{-C_0(T_k-s)}\left(\left|z^1(\theta_{s-t}\omega)\right|^4 + \left|z^2(\theta_{s-t}\omega)\right|^2\right)ds \\ &\qquad + e^{-C_0(t-T_k)}\int_0^{T_k} e^{-C_0(T_k-s)}\left(\left|f\right|^2 + \left|g\right|^2\right)ds\Bigg]. \end{aligned} \tag{3.23}$$

Since $\|z^i(\omega)\|$, $i = 1,2$, are tempered and $z^i(\theta_t\omega)$, $i = 1,2$, are continuous in t, there is a tempered function $r(\omega) > 0$ such that

$$\left\|z^1(\theta_t\omega)\right\|^4 + \left\|z^2(\theta_t\omega)\right\|^2 \le r(\theta_t\omega). \tag{3.24}$$

Combining (3.23) with (3.24), there is a $T_1(\varepsilon,\omega) > T_k$ such that

$$e^{-C_0(t-T_k)}\sum_{n\in\mathbb{Z}}\eta\left(\frac{|n|}{M}\right)\left(|\tilde{u}_n(T_k,\theta_{-t}\omega)|^2 + |\tilde{v}_n(T_k,\theta_{-t}\omega)|^2\right) \le \frac{\varepsilon}{3}. \tag{3.25}$$

Next, we estimate

$$C_1\int_{T_k}^{t} e^{C_0(s-t)}\sum_{|n|\ge M}\left(\left|z_n^1(\theta_{s-t}\omega)\right|^4 + \left|z_n^2(\theta_{s-t}\omega)\right|^2\right)ds. \tag{3.26}$$

Let $T^* \ge (1/C_0)\ln(6C_1r(\omega)/C_0\varepsilon)$ and $N_1(\varepsilon,\omega)$ be fixed positive constants. Then, for $t > T^* + T_k$ and $M > N_1(\varepsilon,\omega)$, by the Lebesgue theorem, we have

$$\begin{aligned}
&C_1\int_{T_k}^{t} e^{C_0(s-t)}\sum_{|n|\ge M}\left(\left|z_n^1(\theta_{s-t}\omega)\right|^4 + \left|z_n^2(\theta_{s-t}\omega)\right|^2\right)ds\\
&\quad= C_1\int_{-T^*}^{0} e^{C_0\xi}\sum_{|n|\ge M}\left(\left|z_n^1(\theta_{\xi}\omega)\right|^4 + \left|z_n^2(\theta_{\xi}\omega)\right|^2\right)ds\\
&\qquad+ C_1\int_{T_k-t}^{-T^*} e^{C_0\xi}\sum_{|n|\ge M}\left(\left|z_n^1(\theta_{\xi}\omega)\right|^4 + \left|z_n^2(\theta_{\xi}\omega)\right|^2\right)ds\\
&\quad\le C_1\int_{-T^*}^{0} e^{C_0\xi}\sum_{|n|\ge M}\left(\left|z_n^1(\theta_{\xi}\omega)\right|^4 + \left|z_n^2(\theta_{\xi}\omega)\right|^2\right)ds + \frac{C_1}{C_0}r(\omega)e^{C_0T^*}\\
&\quad\le \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.
\end{aligned} \tag{3.27}$$

Since $f(t) \in C_B(\mathbb{R},\ell^2)$ and $g(t) \in C_B(\mathbb{R},\ell^2)$, there exists $N_2(\varepsilon,\omega)$ such that for $M > N_2(\varepsilon,\omega)$

$$C_1\int_{T_k}^{t} e^{C_0(s-t)}\sum_{|n|\ge M}\left(|f_n|^2 + |g_n|^2\right)ds \le \frac{\varepsilon}{3}. \tag{3.28}$$

Therefore, let

$$\widetilde{T}(\varepsilon,\omega) = \max\{T_1(\varepsilon,\omega), T^*(\varepsilon,\omega)\}, \qquad \widetilde{N}(\varepsilon,\omega) = \max\{N_1(\varepsilon,\omega), N_2(\varepsilon,\omega)\}. \tag{3.29}$$

Then, for $t > \widetilde{T}(\varepsilon,\omega)$ and $M > \widetilde{N}(\varepsilon,\omega)$, we obtain

$$\sum_{|n|>M}\left(|\widetilde{u}_n(t,\theta_t\omega)|^2 + |\widetilde{v}_n(t,\theta_t\omega)|^2\right) \leq \varepsilon. \tag{3.30}$$

Direct computation shows that

$$\|u\|^2 + \|v\|^2 \leq 2\left(\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2\right) + 4\left\|z^2(\theta_t\omega)\right\|^2. \tag{3.31}$$

Therefore, we obtain

$$\sum_{|n|>M}\left(|u_n(t,\theta_t\omega)|^2 + |v_n(t,\theta_t\omega)|^2\right) \leq \varepsilon. \tag{3.32}$$

The proof of the lemma is completed. □

Lemma 3.7. *The random dynamical systems $(\phi(t,\theta_{-t}\omega))_{t\geq 0}$ are asymptotically compact.*

Proof. We use the method of [25]. Let $\omega \in \Omega$. Consider a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$. Since $\mathbb{K}(\omega)$ is a bounded absorbing set, for large n, $(u_n, v_n) = \phi(t_n, \theta_{-t_n}\omega)(u_0, v_0) \in \mathbb{K}(\omega)$, where $(u_0, v_0) \in \mathbb{K}(\omega)$. Then, there exist $(u,v) \in E$ and a sequence (u_n, v_n) (denoted by itself) such that

$$(u_n, v_n) \rightharpoonup (u,v) \quad \text{weakly in } E. \tag{3.33}$$

Next, we show that the above weak convergence is actually strong convergence in E.

From Lemma 3.6, for any $\varepsilon > 0$, there exist positive constants $N_3(\varepsilon,\omega)$ and $\widetilde{M}_1$ such that, for $n > \widetilde{M}_1$,

$$\sum_{i>N_3}\left(|u_{in}(t_n,\theta_{t_n}\omega)|^2 + |v_{in}(t_n,\theta_{t_n}\omega)|^2\right) \leq \frac{\varepsilon}{6}. \tag{3.34}$$

Since $(u,v) \in E$, there exists $N_4(\varepsilon,\omega) > 0$ such that

$$\sum_{|i|\geq N_4}\left(|u_i|^2 + |v_i|^2\right) \leq \frac{\varepsilon}{6}. \tag{3.35}$$

Let $\widetilde{N}(\varepsilon,\omega) = \max\{N_3(\varepsilon,\omega), N_4(\varepsilon,\omega)\}$, then, from (3.33), there exists $\widetilde{M}_2 > 0$ such that, for $n > \widetilde{M}_2$,

$$\sum_{|i|\leq\widetilde{N}}\left(|u_{in} - u|^2 + |v_{in} - v|^2\right) \leq \frac{\varepsilon}{3}. \tag{3.36}$$

By (3.34)–(3.36), we obtain that, for $n > \widetilde{M} = \max\{\widetilde{M}_1, \widetilde{M}_2\}$,

$$\begin{aligned}
&\|u_n(t_n, \theta_{-t_n}\omega) - u\|^2 + \|v_n(t_n, \theta_{-t_n}\omega) - v\|^2 \\
&\quad = \sum_{|i|\le\widetilde{N}} \left(|u_{in}(t_n, \theta_{-t_n}\omega) - u_i|^2 + |v_{in}(t_n, \theta_{-t_n}\omega) - v_i|^2\right) \\
&\qquad + \sum_{|i|>\widetilde{N}} \left(|u_{in}(t_n, \theta_{-t_n}\omega) - u_i|^2 + |v_{in}(t_n, \theta_{-t_n}\omega) - v_i|^2\right) \\
&\quad \le \sum_{|i|\le\widetilde{N}} \left(|u_{in}(t_n, \theta_{-t_n}\omega) - u_i|^2 + |v_{in}(t_n, \theta_{-t_n}\omega) - v_i|^2\right) \\
&\qquad + 2\sum_{|i|>\widetilde{N}} \left(\left|u_{in}(t_n, \theta_{-t_n}\omega)^2\right| + |v_{in}(t_n, \theta_{-t_n}\omega)|^2\right) + 2\sum_{|i|>\widetilde{N}} \left(|u_i|^2 + |v_i|^2\right) \\
&\quad \le \frac{\varepsilon}{3} + \frac{2\varepsilon}{6} + \frac{2\varepsilon}{6} \le \varepsilon.
\end{aligned} \tag{3.37}$$

The proof of the lemma is completed. □

Now, combining Lemmas 3.5, 3.7 with Proposition 2.7, we can easily obtain the following result.

Theorem 3.8. *The random dynamical systems $(\phi(t, \theta_{-t}\omega))_{t\ge0}$ possess a global random attractor in E.*

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