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*Research Article*

# Periodic Solutions for Autonomous $(q, p)$-Laplacian System with Impulsive Effects

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By using the variational method, some existence theorems are obtained for periodic solutions of autonomous $(q, p)$-Laplacian system with impulsive effects.

## 1. Introduction

Let $B = \{1, 2, \ldots, l\}$, $C = \{1, 2, \ldots, k\}$, $l, k \in \mathbb{N}$.

In this paper, we consider the following system:

$$\begin{gathered}
\frac{d}{dt}\Phi_q(\dot{u}_1(t)) = \nabla_{u_1} F(u_1(t), u_2(t)), \quad \text{a.e. } t \in [0, T], \\
\frac{d}{dt}\Phi_p(\dot{u}_2(t)) = \nabla_{u_2} F(u_1(t), u_2(t)), \quad \text{a.e. } t \in [0, T], \\
u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\
u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0, \\
\Delta\Phi_q(\dot{u}_1(t_j)) = \Phi_q\left(\dot{u}_1\left(t_j^+\right)\right) - \Phi_q\left(\dot{u}_1\left(t_j^-\right)\right) = \nabla I_j(u_1(t_j)), \quad j \in B, \\
\Delta\Phi_p(\dot{u}_2(s_m)) = \Phi_p(\dot{u}_2(s_m^+)) - \Phi_p(\dot{u}_2(s_m^-)) = \nabla K_m(u_2(s_m)), \quad m \in C,
\end{gathered} \tag{1.1}$$

where $p > 1$, $q > 1$, $T > 0$, $u(t) = (u_1(t), u_2(t)) = (u_1^1(t), u_1^2(t), \ldots, u_1^N(t), u_2^1(t), u_2^2(t), \ldots, u_2^N(t))^\tau$, $t_j (j = 1, 2, \ldots, l)$, and $s_m (m = 1, 2, \ldots, k)$ are the instants where the impulses occur and $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T$, $0 = s_0 < s_1 < s_2 < \cdots < s_k < s_{k+1} = T$, $I_j : \mathbb{R}^N \to \mathbb{R}$ $(j \in B)$, and $K_m : \mathbb{R}^N \to \mathbb{R}$ $(m \in C)$ are continuously differentiable

$$\Phi_\mu(z) = |z|^{\mu-2}z = \left(\sum_{i=1}^{N} z_i^2\right)^{(\mu-2)/2} \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, \quad \mu \in \mathbb{R},\ \mu > 1,$$
$$\nabla I_j(x) = \begin{pmatrix} \dfrac{\partial I_j}{\partial x_1} \\ \vdots \\ \dfrac{\partial I_j}{\partial x_N} \end{pmatrix}, \qquad \nabla K_m(x) = \begin{pmatrix} \dfrac{\partial K_m}{\partial x_1} \\ \vdots \\ \dfrac{\partial K_m}{\partial x_N} \end{pmatrix}, \tag{1.2}$$

and $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption.

(A) *$F(x)$ is continuously differentiable in $(x_1, x_2)$, and there exist $a_1, a_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$\begin{gathered} |F(x_1,x_2)| \le a_1(|x_1|) + a_2(|x_2|), \qquad |\nabla F(x_1,x_2)| \le a_1(|x_1|) + a_2(|x_2|), \\ \left|I_j(x_1)\right| \le a_1(|x_1|), \quad \left|\nabla I_j(x_1)\right| \le a_1(|x_1|), \quad j \in B, \\ |K_m(x_2)| \le a_2(|x_2|), \quad |\nabla K_m(x_2)| \le a_2(|x_2|), \quad m \in C, \end{gathered} \tag{1.3}$$

for all $x = (x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$.

When $p = q = 2$, $I_j \equiv 0$ $(j \in B)$, $K_m \equiv 0$ $(m \in C)$, and $F(u_1, u_2) = F_1(u_1)$, system (1.1) reduces to the following autonomous second-order Hamiltonian system:

$$\begin{gathered} \ddot{u}_1(t) = \nabla_{u_1} F_1(u_1(t)), \quad \text{a.e. } t \in [0,T], \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0. \end{gathered} \tag{1.4}$$

There have been lots of results about the existence of periodic solutions for system (1.4) and nonautonomous second order Hamiltonian system

$$\begin{gathered} \ddot{u}_1(t) = \nabla_{u_1} F_1(t, u_1(t)), \quad \text{a.e. } t \in [0,T], \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \end{gathered} \tag{1.5}$$

(e.g., see [1–21]). Many solvability conditions have been given, for instance, coercive condition, subquadratic condition, superquadratic condition, convex condition, and so on.

When $p = q = 2$, $\nabla I_j \not\equiv 0$ $(j \in B)$, $K_m \equiv 0$ $(m \in C)$, and $F(u_1, u_2) = F_1(u_1)$, system (1.1) reduces to the following autonomous second-order Hamiltonian system with impulsive effects:

$$\begin{gathered}
\ddot{u}_1(t) = \nabla_{u_1} F_1(u_1(t)), \quad \text{a.e. } t \in [0,T], \\
u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\
\dot{u}_1\left(t_j^+\right) - \dot{u}_1\left(t_j^-\right) = \nabla I_j(u_1(t_j)).
\end{gathered} \tag{1.6}$$

Recently, many authors studied the existence of periodic solutions for impulsive differential equations by using variational methods, and lots of interesting results have been obtained. For example, see [22–28]. Especially, nonautonomous second-order Hamiltonian system with impulsive effects is considered in [25, 26] by using the least action principle and the saddle point theorem.

When $I_j \equiv 0$ $(j \in B)$ and $K_m \equiv 0$ $(m \in C)$, system (1.1) reduces to the following system:

$$\begin{gathered}
\frac{d}{dt}\Phi_q(\dot{u}_1(t)) = \nabla_{u_1} F(u_1(t), u_2(t)), \quad \text{a.e. } t \in [0,T], \\
\frac{d}{dt}\Phi_p(\dot{u}_2(t)) = \nabla_{u_2} F(u_1(t), u_2(t)), \quad \text{a.e. } t \in [0,T], \\
u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\
u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0.
\end{gathered} \tag{1.7}$$

In [29, 30], Paşca and Tang obtained some existence results for system (1.7) by using the least action principle and saddle point theorem. Motivated by [17, 22–30], in this paper, we are concerned with system (1.1) and also use the least action principle and saddle point theorem to study the existence of periodic solution. Our results still improve those in [17] even if system (1.1) reduces to system (1.4).

A function $G : \mathbb{R}^N \to \mathbb{R}$ is called to be $(\lambda, \mu)$-quasiconcave if

$$G(\lambda(x+y)) \geq \mu(G(x) + G(y)), \tag{1.8}$$

for some $\lambda, \mu > 0$ and $x, y \in \mathbb{R}^N$.

Next, we state our main results.

**Theorem 1.1.** *Let $q'$ and $p'$ be such that $1/q + 1/q' = 1$ and $1/p + 1/p' = 1$. Suppose $F$ satisfies assumption (A) and the following conditions:*

(F1) *there exist*

$$0 < r_1 < \frac{(q'+1)^{q/q'}}{T^q \Theta(q, q')}, \qquad 0 < r_2 < \frac{(p'+1)^{p/p'}}{T^p \Theta(p, p')}, \tag{1.9}$$

*such that*

$$\begin{aligned}&\left(\nabla_{x_1}F(x_1,x_2)-\nabla_{y_1}F(y_1,y_2),x_1-y_1\right)\geq -r_1\left|x_1-y_1\right|^q,\quad \forall (x_1,x_2),(y_1,y_2)\in\mathbb{R}^N\times\mathbb{R}^N,\\&\left(\nabla_{x_2}F(x_1,x_2)-\nabla_{y_2}F(y_1,y_2),x_2-y_2\right)\geq -r_2\left|x_2-y_2\right|^p,\quad \forall (x_1,x_2),(y_1,y_2)\in\mathbb{R}^N\times\mathbb{R}^N,\end{aligned}\tag{1.10}$$

*where*

$$\begin{aligned}\Theta(q,q')&=\int_0^1\left[s^{q'+1}+\ (1-s)^{q'+1}\right]^{q/q'}ds,\\\Theta(p,p')&=\int_0^1\left[s^{p'+1}+(1-s)^{p'+1}\right]^{p/p'}ds,\end{aligned}\tag{1.11}$$

(F2) $F(x)\ \to\ +\infty$, *as* $|x|\ \to\ \infty$, *where* $x=(x_1,x_2)$,

(I1) *there exists* $\beta\in\mathbb{R}$ *such that*

$$\begin{aligned}I_j(x)&\geq\beta,\quad \forall x\in\mathbb{R}^N,\ j\in B,\\K_m(x)&\geq\beta,\quad \forall x\in\mathbb{R}^N,\ m\in C.\end{aligned}\tag{1.12}$$

*Then, system* (1.1) *has at least one solution in* $W_T^{1,q}\times W_T^{1,p}$, *where* $W_T^{1,s}=\{u:[0,T]\ \to\ \mathbb{R}^N\mid u$ *is absolutely continuous,* $u(0)=u(T)$ *and* $\dot{u}\in L^s\ (0,T;\mathbb{R}^N)\}$, $s\in\mathbb{R}$.

*Furthermore, if* $I_j\equiv 0\ (j\in B)$, $K_m\equiv 0\ (m\in C)$ *and the following condition holds:*

(F3) *there exist* $\delta>0$, $a\in[0,(q'+1)^{q/q'}/qT^q\Theta(q,q'))$ *and* $b\in[0,\ (p'+1)^{p/p'}/(pT^p\Theta(p,p')))$ *such that*

$$-a|x_1|^q-b|x_2|^p\leq F(x_1,x_2)\leq 0,\quad \forall|x_1|\leq\delta,\ \ |x_2|\leq\delta,\tag{1.13}$$

*then system* (1.7) *has at least two nonzero solutions in* $W_T^{1,q}\times W_T^{1,p}$.

When $p=q=2$, $F(x_1,x_2)=F_1(x_1)$, by Theorem 1.1, it is easy to get the following corollary.

**Corollary 1.2.** *Suppose* $F_1$ *satisfies the following conditions:*

(A)′ $F_1(z)$ *is continuously differentiable in* $z$ *and there exists* $a_1\in C(\mathbb{R}^+,\mathbb{R}^+)$ *such that*

$$\begin{aligned}|F_1(z)|\leq a_1(|z|),&\qquad |\nabla F_1(z)|\leq a_1(|z|),\\\left|I_j(z)\right|\leq a_1(|z|),&\qquad \left|\nabla I_j(z)\right|\leq a_1(|z|),\quad j\in B,\end{aligned}\tag{1.14}$$

*for all* $z\in\mathbb{R}^N$.

(F1)$'$ *there exists* $0 < r < 6/T^2$ *such that*

$$(\nabla_z F_1(z) - \nabla_w F_1(w), z - w) \geq -r|z - w|^2, \quad \forall z, w \in \mathbb{R}^N, \tag{1.15}$$

(F2)$'$ $F_1(z) \to +\infty$, *as* $|z| \to \infty$, $z \in \mathbb{R}^N$;

(I1)$'$ *there exists* $\beta \in \mathbb{R}$ *such that*

$$I_j(z) \geq \beta, \quad \forall z \in \mathbb{R}^N, \quad j \in B. \tag{1.16}$$

*Then, system* (1.6) *has at least one solution in* $W_T^{1,2}$. *Furthermore, if* $I_j \equiv 0$ $(j \in B)$ *and the following condition holds:*

(F3)$'$ *there exist* $\delta > 0$ *and* $a \in [0, (3/T^2))$ *such that*

$$-a|z|^2 \leq F_1(z) \leq 0, \quad \forall z \in \mathbb{R}^N, \ |z| \leq \delta, \tag{1.17}$$

*then system* (1.4) *has at least two nonzero solutions in* $W_T^{1,2}$.

For the Sobolev space $\widetilde{W}_T^{1,2}$, one has the following sharp estimates (see in [3, Proposition 1.2]):

$$\int_0^T |u(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2 dt \quad (\text{Wirtinger's inequality}), \tag{1.18}$$

$$\|u\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt \quad (\text{Sobolev's inequality}). \tag{1.19}$$

By the above two inequalities, we can obtain the following better results than by Corollary 1.2.

**Theorem 1.3.** *Suppose* $F_1$ *satisfies assumption (A)$'$, (F2)$'$, (I1)$'$ and*

(F1)$''$ *there exists* $0 < r < 4\pi^2/T^2$ *such that* (1.15) *holds.*

*Then, system* (1.6) *has at least one solution in* $W_T^{1,2}$. *Furthermore, if* $I_j \equiv 0$ $(j \in B)$ *and the following condition holds:*

(F3)$''$ *there exist* $\delta > 0$ *and* $a \in [0, (2\pi^2)/T^2)$ *such that*

$$-a|z|^2 \leq F_1(z) \leq 0, \quad \forall z \in \mathbb{R}^N, \ |z| \leq \delta, \tag{1.20}$$

*then system* (1.4) *has at least two nonzero solutions in* $W_T^{1,2}$.

Moreover, for system (1.6), we have the following additional result.

**Theorem 1.4.** *Suppose* $F_1$ *satisfies assumption (A)$'$, (F1)$''$ and the following conditions:*

(F4) $F_1(z)$ *is* $(\lambda, \mu)$*-quasiconcave on* $\mathbb{R}^N$, (F5) $F_1(z) \to -\infty$ *as* $|z| \to +\infty$, $z \in \mathbb{R}^N$,

(I2) *there exist* $d_j > 0$ $(j \in B)$ *such that*

$$\left|\nabla I_j(z)\right| \leq d_j, \quad \forall z \in \mathbb{R}^N, \; j \in B, \tag{1.21}$$

(I3) *there exist* $b_j > 0$, $c_j > 0$, $\gamma_j \in \mathbb{R}$, $\alpha_j \in [0, 2)(j \in B)$ *such that*

$$-b_j|z|^{\alpha_j} - c_j \leq I_j(z) \leq \gamma_j, \quad \forall z \in \mathbb{R}^N, \; j \in B. \tag{1.22}$$

*Then, system* (1.6) *has at least one solution in* $W_T^{1,2}$.

*Remark 1.5.* In [17], Yang considered the second-order Hamiltonian system with no impulsive effects, that is, system (1.4). When $I_j \equiv 0$ $(j \in B)$, our Theorems 1.3 and 1.4 still improve those results in [17]. To be precise, the restriction of $r$ is relaxed, and some unnecessary conditions in [17] are deleted. In [17], the restriction of $r$ is $0 < r < T/12$, which is not right. In fact, from his proof, it is easy to see that it should be $0 < r < 12/T^2$. Obviously, our restriction $0 < r < 4\pi^2/T^2$ is better. Moreover, in our Theorem 1.4, we delete such conditions (of in [17, Theorem 1]): $\nabla F_1(0) = 0$, and there exist positive constants $M, N$ such that

$$F_1(z) \geq -M\ |z|^2 - N, \quad z \in \mathbb{R}^N. \tag{1.23}$$

Finally, it is remarkable that Theorems 1.3 and 1.4 are also different from those results in [1–16]. We can find an example satisfying our Theorem 1.3 but not satisfying the results in [1–21]. For example, let

$$F_1(z) = \frac{\pi^2}{2T^2}\left(|z_1|^4 + |z_2|^4 + \cdots + |z_N|^4\right) - \frac{\pi^2}{4T^2}|z|^2, \tag{1.24}$$

where $z = (z_1, \ldots, z_N)^\tau$. We can also find an example satisfying our Theorem 1.4 but not satisfying the results in [1–21]. For example, let

$$F_1(z) = -\frac{r}{2}|z|^2, \tag{1.25}$$

where $12/T^2 < r < 4\pi^2/T^2$.

## 2. Variational Structure and Some Preliminaries

The norm in $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \left[\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt\right]^{1/p}. \tag{2.1}$$

Set

$$\|u\|_p = \left(\int_0^T |u(t)|^p dt\right)^{1/p}, \qquad \|u\|_\infty = \max_{t\in[0,T]}|u(t)|. \tag{2.2}$$

Let

$$\widetilde{W}_T^{1,p} = \left\{u \in W_T^{1,p} \mid \int_0^T u(t)dt = 0\right\}. \tag{2.3}$$

Obviously, $W_T^{1,p}$ is a reflexive Banach space. It is easy to know that $\widetilde{W}_T^{1,p}$ is a subset of $W_T^{1,p}$ and $W_T^{1,p} = \mathbb{R}^N \oplus \widetilde{W}_T^{1,p}$. In this paper, we will use the space $W$ defined by

$$W = W_T^{1,q} \times W_T^{1,p}, \quad u(t) = (u_1(t), u_2(t)), \tag{2.4}$$

with the norm $\|(u_1,u_2)\|_W = \|u_1\|_{W_T^{1,q}} + \|u_2\|_{W_T^{1,p}}$. It is clear that $W$ is a reflexive Banach space. Let $\widetilde{W} = \widetilde{W}_T^{1,q} \times \widetilde{W}_T^{1,p}$. Then, $W = (\widetilde{W}_T^{1,q} \times \widetilde{W}_T^{1,p}) \oplus (\mathbb{R}^N \times \mathbb{R}^N)$.

**Lemma 2.1** (see [31] or [32]). *Each $u \in W_T^{1,p}$ and each $v \in W_T^{1,q}$ can be written as $u(t) = \overline{u} + \tilde{u}(t)$ and $v(t) = \overline{v} + \tilde{v}(t)$ with*

$$\begin{aligned} \overline{u} &= \frac{1}{T}\int_0^T u(t)dt, \qquad \int_0^T \tilde{u}(t)dt = 0, \\ \overline{v} &= \frac{1}{T}\int_0^T v(t)dt, \qquad \int_0^T \tilde{v}(t)dt = 0. \end{aligned} \tag{2.5}$$

*Then,*

$$\|\tilde{u}\|_\infty \le \left(\frac{T}{p'+1}\right)^{1/p'}\left(\int_0^T |\dot{u}(s)|^p ds\right)^{1/p}, \qquad \|\tilde{v}\|_\infty \le \left(\frac{T}{q'+1}\right)^{1/q'}\left(\int_0^T |\dot{v}(s)|^q ds\right)^{1/q}, \tag{2.6}$$

$$\int_0^T |\tilde{u}(s)|^p ds \le \frac{T^p\Theta(p,p')}{(p'+1)^{p/p'}}\int_0^T |\dot{u}(s)|^p ds, \qquad \int_0^T |\tilde{v}(s)|^q ds \le \frac{T^q\Theta(q,q')}{(q'+1)^{q/q'}}\int_0^T |\dot{v}(s)|^q ds, \tag{2.7}$$

*where*

$$\Theta(p,p') = \int_0^1 \left[s^{p'+1} + (1-s)^{p'+1}\right]^{p/p'} ds, \qquad \Theta(q,q') = \int_0^1 \left[s^{q'+1} + (1-s)^{q'+1}\right]^{q/q'} ds. \tag{2.8}$$

Note that if $u \in W_T^{1,p}$, then $u$ is absolutely continuous. However, we cannot guarantee that $\dot{u}$ is also continuous. Hence, it is possible that $\Delta\Phi_p(\dot{u}(t)) = \Phi_p(\dot{u}(t^+)) - \Phi_p(\dot{u}(t^-)) \neq 0$, which results in impulsive effects.

Following the idea in [22], one takes $v_1 \in W_T^{1,q}$ and multiplies the two sides of

$$\frac{d}{dt}\left(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)\right) - \nabla_{x_1}F(u_1(t),u_2(t)) = 0, \tag{2.9}$$

by $v_1$ and integrate from 0 to $T$, one obtains

$$\int_0^T\left[\frac{d}{dt}\left(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)\right) - \nabla_{x_1}F(u_1(t),u_2(t))\right]v_1(t)dt = 0. \tag{2.10}$$

Note that $v_1(t)$ is continuous. So, $v_1(t_j^-) = v_1(t_j^+) = v_1(t_j)$. Combining $\dot{u}_1(0) - \dot{u}_1(T) = 0$, one has

$$\begin{aligned}
\int_0^T\left(\frac{d\Phi_q(\dot{u}_1(t))}{dt}, v_1(t)\right)dt &= \sum_{j=0}^{l}\int_{t_j}^{t_{j+1}}\left(\frac{d(\Phi_q(\dot{u}_1(t)))}{dt}, v_1(t)\right)dt \\
&= \sum_{j=0}^{l}\left[\left(\Phi_q\left(\dot{u}_1\left(t_{j+1}^-\right)\right), v_1\left(t_{j+1}^-\right)\right) - \left(\Phi_q\left(\dot{u}_1\left(t_j^+\right)\right), v_1\left(t_j^+\right)\right)\right]dt \\
&\quad - \sum_{j=0}^{l}\int_{t_j}^{t_{j+1}}\left(\Phi_q(\dot{u}_1(t)), \dot{v}_1(t)\right)dt \\
&= \left(\Phi_q(\dot{u}_1(T)), v_1(T)\right) - \left(\Phi_q(\dot{u}_1(0)), v_1(0)\right) \\
&\quad - \sum_{j=1}^{l}\left(\Delta\Phi_q(\dot{u}_1(t_j)), v_1(t_j)\right) - \int_0^T\left(\Phi_q(\dot{u}_1(t)), \dot{v}_1(t)\right)dt \\
&= -\sum_{j=1}^{l}\left(\nabla I_j(u_1(t_j)), v_1(t_j)\right) - \int_0^T\left(\Phi_q(\dot{u}_1(t)), \dot{v}_1(t)\right)dt.
\end{aligned} \tag{2.11}$$

Combining with (2.10), one has

$$\begin{aligned}
&\int_0^T\left(\Phi_q(\dot{u}_1(t)), \dot{v}_1(t)\right)\, dt + \sum_{j=1}^{l}\left(\nabla I_j(u_1(t_j)), v_1(t_j)\right) \\
&\quad + \int_0^T\left(\nabla_{x_1}F(u_1(t),u_2(t)), v_1(t)\right)dt = 0.
\end{aligned} \tag{2.12}$$

Similarly, one can get

$$\begin{aligned}
&\int_0^T \left(\Phi_p(\dot{u}_2(t)), \dot{v}_2(t)\right)\, dt + \sum_{m=1}^{k} (\nabla K_m(u_2(s_m)), v_2(s_m)) \\
&\quad + \int_0^T (\nabla_{x_2} F(u_1(t), u_2(t)), v_2(t)) dt = 0,
\end{aligned} \tag{2.13}$$

for all $v_2 \in W_T^{1,p}$. Considering the above equalities, one introduces the following concept of the weak solution for system (1.1).

*Definition 2.2. We say that a function* $u = (u_1, u_2) \in W_T^{1,q} \times W_T^{1,p}$ *is a weak solution of system* (1.1) *if*

$$\begin{aligned}
&\int_0^T \left(\Phi_q(\dot{u}_1(t)), \dot{v}_1(t)\right)\, dt + \sum_{j=1}^{l} (\nabla I_j(u_1(t_j)), v_1(t_j)) = -\int_0^T (\nabla_{x_1} F(u_1(t), u_2(t)), v_1(t)) dt, \\
&\int_0^T \left(\Phi_p(\dot{u}_2(t)), \dot{v}_2(t)\right)\, dt + \sum_{m=1}^{k} (\nabla K_m(u_2(s_m)), v_2(s_m)) = -\int_0^T (\nabla_{x_2} F(u_1(t), u_2(t)), v_2(t)) dt
\end{aligned} \tag{2.14}$$

holds for any $v = (v_1, v_2) \in W_T^{1,q} \times W_T^{1,p}$.

Define the functional $\varphi : W_T^{1,q} \times W_T^{1,p} \to \mathbb{R}$ by

$$\begin{aligned}
\varphi(u_1, u_2) &= \frac{1}{q}\int_0^T |\dot{u}_1(\mathrm{t})|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T F(u_1(t), u_2(t)) dt \\
&\quad + \sum_{j=1}^{l} I_j(u_1(t_j)) + \sum_{m=1}^{k} K_m(u_2(s_m)) \\
&= \phi(u_1, u_2) + \psi(u_1, u_2),
\end{aligned} \tag{2.15}$$

where $(u_1, u_2) \in W_T^{1,q} \times W_T^{1,p}$,

$$\begin{aligned}
\phi(u_1, u_2) &= \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T F(u_1(t), u_2(t)) dt, \\
\psi(u_1, u_2) &= \sum_{j=1}^{l} I_j(u_1(t_j)) + \sum_{m=1}^{k} K_m(u_2(s_m)).
\end{aligned} \tag{2.16}$$

By assumption (A) and [33], we know that $\phi \in C^1(W_T^{1,q} \times W_T^{1,p}, \mathbb{R})$. The continuity of $I_j (j \in B)$ and $K_m (m \in C)$ implies that $\varphi \in C^1(W_T^{1,p} \times W_T^{1,p}, \mathbb{R})$. So, $\varphi \in C^1(W_T^{1,p}, \mathbb{R})$, and for all $(v_1, v_2) \in W_T^{1,q} \times W_T^{1,p}$, we have

$$\begin{aligned} \langle \varphi'(u_1, u_2), (v_1, v_2) \rangle = & \int_0^T (\Phi_q(\dot{u}_1(t)), \dot{v}_1(t)) dt + \int_0^T (\Phi_p(\dot{u}_2(t)), \dot{v}_2(t)) dt \\ & + \int_0^T (\nabla_{x_1} F(u_1(t), u_2(t)), v_1(t)) dt + \int_0^T (\nabla_{x_2} F(u_1(t), u_2(t)), v_2(t)) dt \\ & + \sum_{j=1}^{l} (\nabla I_j(u_1(t_j)), v_1(t_j)) + \sum_{m=1}^{k} (\nabla K_m(u_2(s_m)), v_2(s_m)). \end{aligned} \tag{2.17}$$

Definition 2.2 shows that the critical points of $\varphi$ correspond to the weak solutions of system (1.1).

We will use the following lemma to seek the critical point of $\varphi$.

**Lemma 2.3** (see [3, Theorem 1.1]). *If $\varphi$ is weakly lower semicontinuous on a reflexive Banach space $X$ and has a bounded minimizing sequence, then $\varphi$ has a minimum on $X$.*

**Lemma 2.4** (see [34]). *Let $\varphi$ be a $C^1$ function on $X = X_1 \oplus X_2$ with $\varphi(0) = 0$, satisfying (PS) condition, and assume that for some $\rho > 0$,*

$$\begin{aligned} \varphi(u) \geq 0, \quad & \text{for } u \in X_1, \quad \|u\| \leq \rho, \\ \varphi(u) \leq 0, \quad & \text{for } u \in X_2, \quad \|u\| \leq \rho. \end{aligned} \tag{2.18}$$

*Assume also that $\varphi$ is bounded below and $\inf_X \varphi < 0$, then $\varphi$ has at least two nonzero critical points.*

**Lemma 2.5** (see [35, Theorem 4.6]). *Let $X = X_1 \oplus X_2$, where $X$ is a real Banach space and $X_1 \neq \{0\}$ and is finite dimensional. Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies (PS)-condition and*

($\varphi$1) *there is a constant $\alpha$ and a bounded neighborhood $D$ of 0 in $X_1$ such that $\varphi|_{\partial D} \leq \alpha$,*

($\varphi$2) *there is a constant $\beta > \alpha$ such that $\varphi|_{X_2} \geq \beta$.*

*Then, $\varphi$ possesses a critical value $c \geq \beta$. Moreover, $c$ can be characterized as*

$$c = \inf_{h \in \Gamma} \max_{u \in \overline{D}} \varphi(h(u)), \tag{2.19}$$

*where,*

$$\Gamma = \left\{ h \in C\left(\overline{D}, X\right) \mid h = id \ \text{ on } \ \partial D \right\}. \tag{2.20}$$

## 3. Proof of Theorems

**Lemma 3.1.** *Under assumption (A), $\varphi$ is weakly lower semicontinuous on $W_T^{1,q} \times W_T^{1,p}$.*

*Proof.* Let

$$\phi_1(u_1,u_2) = \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt,$$
$$\phi_2(u_1,u_2) = \int_0^T F(u_1(t),u_2(t))dt. \tag{3.1}$$

Since

$$\begin{aligned}
\phi_1\left(\frac{u_1+v_1}{2},\frac{u_2+v_2}{2}\right) &= \frac{1}{q}\int_0^T \left|\frac{\dot{u}_1(t)+\dot{v}_1(t)}{2}\right|^q dt + \frac{1}{p}\int_0^T \left|\frac{\dot{u}_2(t)+\dot{v}_2(t)}{2}\right|^p dt \\
&\le \frac{2^{q-1}}{q}\int_0^T \frac{1}{2^q}|\dot{u}_1(t)|^q\, dt + \frac{2^{q-1}}{q}\int_0^T \frac{1}{2^q}\, |\dot{v}_1(t)|^q dt \\
&\quad + \frac{2^{p-1}}{p}\int_0^T \frac{1}{2^p}\, |\dot{u}_2(t)|^p dt + \frac{2^{p-1}}{p}\int_0^T \frac{1}{2^p}|\dot{v}_2(t)|^p dt \\
&\le \frac{1}{2q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{2q}\int_0^T |\dot{v}_1(t)|^q dt \\
&\quad + \frac{1}{2p}\int_0^T |\dot{u}_2(t)|^p dt + \frac{1}{2p}\int_0^T |\dot{v}_2(t)|^p dt \\
&= \frac{\phi_1(u_1,u_2)+\phi_1(v_1,v_2)}{2},
\end{aligned} \tag{3.2}$$

then $\phi_1$ is convex. Moreover, by [33], we know that $\phi_1$ is continuous, and so, it is lower semicontinuous. Thus, it follows from [3, Theorem 1.2] that $\phi_1$ is weakly lower continuous. By assumption (A), it is easy to verify that $\phi_2(u_1,u_2)$ is weakly continuous. We omit the details. Let

$$\varphi_1(u_1) = \sum_{j=1}^{l} I_j\left(u_1\left(t_j\right)\right), \qquad \varphi_2(u_2) = \sum_{m=1}^{k} K_m(u_2(s_m)). \tag{3.3}$$

Next, we show that $\varphi_1$ and $\varphi_2$ are weakly continuous on $W_T^{1,q}$ and $W_T^{1,p}$, respectively. In fact, if

$$u_{1n} \rightharpoonup u_1 \text{ weakly in } W_T^{1,p}, \quad \text{as } n \longrightarrow \infty, \tag{3.4}$$

then by in [3, Proposition 1.2], we know that

$$u_{1n} \longrightarrow u_1 \text{ strongly in } C\left(0,T;\mathbb{R}^N\right), \quad \text{as } n \longrightarrow \infty. \tag{3.5}$$

So, there exists $M_1 > 0$ such that $\|u_1\|_\infty \le M_1$ and $\|u_{1n}\|_\infty \le M_1$, for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \left|\varphi_1(u_{1n}) - \varphi_1(u_1)\right| &= \left|\sum_{j=1}^{l} I_j(u_{1n}(t_j)) - \sum_{j=1}^{l} I_j(u_1(t_j))\right| \\ &\le \sum_{j=1}^{l}\left|I_j(u_{1n}(t_j)) - I_j(u_1(t_j))\right| \\ &= \sum_{j=1}^{l}\left|\int_0^1 (\nabla I_j(u_1(t_j) + s(u_{1n}(t_j) - u_1(t_j))), u_{1n}(t_j) - u_1(t_j))ds\right| \\ &\le \|u_{1n} - u_1\|_\infty \sum_{j=1}^{l} \max_{t\in[0,3M_1]} a_1(t) \longrightarrow 0. \end{aligned} \tag{3.6}$$

Hence, $\varphi_1$ is weakly continuous on $W_T^{1,q}$. Similarly, we can prove that $\varphi_2$ is also weakly continuous on $W_T^{1,p}$. Thus, we complete the proof. □

*Proof of Theorem 1.1.* It follows from (F1) and (2.7) that

$$\begin{aligned} &\int_0^T [F(u_1(t), u_2(t)) - F(u_1(t), \overline{u}_2)] \\ &\quad = \int_0^T \int_0^1 \frac{1}{s}(\nabla F_{x_2}(u_1(t), \overline{u}_2 + s\widetilde{u}_2(t)), s\widetilde{u}_2(t))ds\,dt \\ &\quad = \int_0^T \int_0^1 \frac{1}{s}(\nabla F_{x_2}(u_1(t), \overline{u}_2 + s\widetilde{u}_2(t)) - \nabla F_{x_2}(\overline{u}_1, \overline{u}_2), s\widetilde{u}_2(t))ds\,dt \\ &\quad \ge -\frac{r_2}{p}\int_0^T |\widetilde{u}_2(t)|^p dt \\ &\quad \ge -\frac{r_2 T^p \Theta(p,p')}{p(p'+1)^{p/p'}}\int_0^T |\dot{u}_2(t)|^p dt, \qquad \forall (u_1, u_2) \in W, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\int_0^T [F(u_1(t), \overline{u}_2) - F(\overline{u}_1, \overline{u}_2)]dt \\ &\quad = \int_0^T \int_0^1 \frac{1}{s}(\nabla_{x_1} F(\overline{u}_1 + s\widetilde{u}_1(t), \overline{u}_2), s\widetilde{u}_1(t))ds\,dt \\ &\quad = \int_0^T \int_0^1 \frac{1}{s}(\nabla_{x_1} F(\overline{u}_1 + s\widetilde{u}_1(t), \overline{u}_2) - \nabla_{x_1} F(\overline{u}_1, \overline{u}_2), s\widetilde{u}_1(t))ds\,dt \end{aligned}$$

$$
\begin{aligned}
&\geq -\frac{r_1}{q}\int_0^T |\tilde{u}_1(t)|^q dt \\
&\geq -\frac{r_1 T^q \Theta(q,q')}{q\ (q'+1)^{q/q'}}\int_0^T |\dot{u}_1(t)|^q dt, \quad \forall (u_1,u_2)\in W.
\end{aligned}
\tag{3.8}
$$

Hence, by (I1), (3.7), and (3.8), we have

$$
\begin{aligned}
\varphi(u_1,u_2) &= \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T [F(u_1(t),u_2(t)) - F(u_1(t),\overline{u}_2)]dt \\
&\quad + \int_0^T [F(u_1(t),\overline{u}_2) - F(\overline{u}_1,\overline{u}_2)]dt + TF(\overline{u}_1,\overline{u}_2) + \sum_{j=1}^{l} I_j(u_1(t_j)) + \sum_{m=1}^{k} K_m(u_2(s_m)) \\
&\geq \left(\frac{1}{p} - \frac{r_2 T^p \Theta(p,p')}{p\ (p'+1)^{p/p'}}\right)\int_0^T |\dot{u}_2(t)|^p dt + \left(\frac{1}{q} - \frac{r_1 T^q \Theta(q,q')}{q\ (q'+1)^{q/q'}}\right)\int_0^T |\dot{u}_1(t)|^q dt \\
&\quad + TF(\overline{u}_1,\overline{u}_2) - (l+k)\left|\beta\right|.
\end{aligned}
\tag{3.9}
$$

Note that for $u \in W_T^{1,p}$,

$$
\|u\|_{W_T^{1,p}} \longrightarrow \infty \Longleftrightarrow \left(|\overline{u}|^p + \int_0^T |\dot{u}(t)|^p dt\right)^{1/p} \longrightarrow \infty, \tag{3.10}
$$

and for $v \in W_T^{1,q}$,

$$
\|v\|_{W_T^{1,q}} \longrightarrow \infty \Longleftrightarrow \left(|\overline{v}|^q + \int_0^T |\dot{v}(t)|^q dt\right)^{1/q} \longrightarrow \infty. \tag{3.11}
$$

So, (F2) and (3.9) imply that

$$
\varphi(u_1,u_2) \longrightarrow +\infty, \text{ as } \|(u_1,u_2)\|_W \longrightarrow \infty. \tag{3.12}
$$

Thus, by Lemma 2.3, we know that $\varphi$ has at least one critical point which minimizes $\varphi$ on $W$.

Furthermore, if $I_j(u_1(t_j)) \equiv 0$ $(j \in B)$ and $K_m(u_2(s_m)) \equiv 0$ $(m \in C)$, then system (1.1) reduces to (1.7). When (F3) also holds, we will use Lemma 2.4 to obtain more critical points of $\varphi$. Let $X = W$, $X_2 = \mathbb{R}^N \times \mathbb{R}^N$ and $X_1 = \widetilde{W} = \widetilde{W}_T^{1,q} \times \widetilde{W}_T^{1,p}$.

By (3.9), we know that $\varphi(u_1,u_2) \to +\infty$ as $\|(u_1,u_2)\|_W \to \infty$. So, $\varphi$ satisfies (PS) condition and is bounded below. Take $\rho = \delta/c_1$, where $c_1$ is a positive constant such that

$\|u_1\|_\infty \le c_1\|u_1\|_{W_T^{1,q}} \le c_1\|u\|_W$ and $\|u_2\|_\infty \le c_1\|u_2\|_{W_T^{1,p}} \le c_1\|u\|_W$ for all $(u_1, u_2) \in W$. It follows from (F3) and Lemma 2.1 that

$$\begin{aligned}\varphi(u_1,u_2) &= \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T F(u_1(t),u_2(t))dt \\ &\ge \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt - a\int_0^T |u_1(t)|^q dt - b\int_0^T |u_2(t)|^p dt \\ &\ge \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt - a\frac{T^q\Theta(q,q')}{(q'+1)^{q/q'}}\int_0^T |\dot{u}_1(t)|^q dt \\ &\quad - b\frac{T^p\Theta(p,p')}{(p'+1)^{p/p'}}\int_0^T |\dot{u}_2(t)|^p dt, \qquad \forall (u_1,u_2) \in X_1.\end{aligned} \tag{3.13}$$

Since $a \le (q'+1)^{q/q'}/(qT^q\Theta(q,q'))$ and $b \le (p'+1)^{p/p'}/(pT^p\Theta(p,p'))$, (3.13) implies that $\varphi(u_1,u_2) \ge 0$ for all $(u_1,u_2) \in X_1$ with $\|u\|_W \le \rho$. By (F3), it is easy to obtain that $\varphi(u_1,u_2) \le 0$, for all $(u_1,u_2) \in X_2$ with $\|u\|_W \le \rho$.

If $\inf\{\varphi(u_1,u_2) : (u_1,u_2) \in W\} = 0$, then from above, we have $\varphi(u_1,u_2) = 0$ for all $(u_1,u_2) \in X_2$ with $\|(u_1,u_2)\|_W \le \rho$. Hence, all $(u_1,u_2) \in X_2$ with $\|(u_1,u_2)\|_W \le \rho$ are minimal points of $\varphi$, which implies that $\varphi$ has infinitely many critical points. If $\inf\{\varphi(u_1,u_2) : (u_1,u_2) \in W\} < 0$, then by Lemma 2.4, $\varphi$ has at least two nonzero critical points. Hence, system (1.7) has at least two nontrivial solutions in $W$. We complete our proof. □

*Proof of Theorem 1.3.* We only need to use (1.18) and (1.19) to replace (2.6) and (2.7) in the proof Theorem 1.1 with $p = q = 2$, $F(t,u_1,u_2) = F_1(u_1)$ and $K_m(u_2) \equiv 0$ $(m \in C)$. It is easy. So, we omit it. □

**Lemma 3.2.** *Under the assumptions of Theorem 1.4, the functional $\varphi_1$ defined by*

$$\varphi_1(u_1) = \frac{1}{2}\int_0^T |\dot{u}_1(t)|^2 dt + \int_0^T F_1(u_1(t))dt + \sum_{j=1}^{l} I_j(u_1(t_j))dt \tag{3.14}$$

*satisfies (PS) condition.*

*Proof.* Suppose that $\{u_{1n}\}$ is a (PS) sequence for $\varphi_1$; that is, there exists $D_1 > 0$ such that

$$\left|\varphi(u_{1n})\right| \le D_1, \quad \forall n \in \mathbb{N}, \quad \varphi'(u_{1n}) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

Hence, for $n$ large enough, we have $\|\varphi'(u_{1n})\| \le 1$. It follows from (F1)$''$, (I2), and (1.18) that

$$\begin{aligned}\|\tilde{u}_{1n}\|_{W_T^{1,2}} \ge \langle \varphi_1'(u_{1n}), \tilde{u}_{1n}\rangle &= \int_0^T |\dot{u}_{1n}(t)|^2 dt + \int_0^T (\nabla_{x_1} F_1(u_{1n}(t)), \tilde{u}_{1n}(t))dt \\ &\quad + \sum_{j=1}^{l} (\nabla I_j(u_{1n}(t_j)), \tilde{u}_{1n}(t_j))\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T |\dot{u}_{1n}(t)|^2 dt + \int_0^T (\nabla_{x_1} F_1(u_{1n}(t)) - \nabla_{x_1} F_1(\overline{u}_{1n}(t)), \widetilde{u}_{1n}(t)) dt \\
&\quad + \sum_{j=1}^{l} (\nabla I_j(u_{1n}(t_j)), \widetilde{u}_{1n}(t_j)) \\
&\geq \int_0^T |\dot{u}_{1n}(t)|^2 dt - r\frac{T^2}{4\pi^2} \int_0^T |\dot{u}_{1n}(t)|^2 dt - \|\widetilde{u}_{1n}\|_\infty \sum_{j=1}^{l} d_j \\
&\geq \left[1 - r\frac{T^2}{4\pi^2}\right] \int_0^T |\dot{u}_{1n}(t)|^2 dt - \left(\frac{T}{12}\right)^{1/2} \left(\int_0^T |\dot{u}_{1n}(t)|^2 dt\right)^{1/2} \sum_{j=1}^{l} d_j,
\end{aligned}
\tag{3.16}
$$

for $n$ large enough. By (1.18), we have

$$
\|\widetilde{u}_{1n}\|_{W_T^{1,2}} \leq \left[\frac{T^2}{4\pi^2} + 1\right]^{1/2} \left(\int_0^T |\dot{u}_{1n}(t)|^2 dt\right)^{1/2}, \tag{3.17}
$$

and (3.16), (3.17), and $r < 4\pi^2/T^2$ imply that there exists $D_2, D_3 > 0$ such that

$$
\int_0^T |\dot{u}_{1n}(t)|^2 dt \leq D_2, \quad \|\widetilde{u}_{1n}\|_{W_T^{1,2}} \leq D_3. \tag{3.18}
$$

It follows from (F4), (3.15), (I3), (1.18), and (3.18) that

$$
\begin{aligned}
-\mathrm{D}_1 \leq \varphi_1(u_{1n}) &= \frac{1}{2}\int_0^T |\dot{u}_{1n}(t)|^2 dt + \int_0^T F_1(u_{1n}(t)) dt + \sum_{j=1}^{l} I_j(u_1(t_j)) \\
&\leq \frac{1}{2}\int_0^T |\dot{u}_{1n}(t)|^2 \mathrm{d}t + \frac{1}{\mu}\int_0^T F_1(\lambda \overline{u}_{1n}) dt - \int_0^T F_1(-\widetilde{u}_{1n}(t)) dt + \sum_{j=1}^{l} \gamma_j \\
&= \frac{1}{2}\int_0^T |\dot{u}_{1n}(t)|^2 dt + \frac{T}{\mu} F_1(\lambda \overline{u}_{1n}) - T F_1(0) - \int_0^T [F_1(-\widetilde{u}_{1n}(t)) - F_1(0)] dt + \sum_{j=1}^{l} \gamma_j \\
&= \frac{1}{2}\int_0^T |\dot{u}_{1n}(t)|^2 dt + \frac{T}{\mu} F_1(\lambda \overline{u}_{1n}) - T F_1(0) + \sum_{j=1}^{l} \gamma_j
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^T\int_0^1 \frac{1}{s}(\nabla F_1(-s\widetilde{u}_{1n}(t)) - \nabla F_1(0), -s\widetilde{u}_{1n}(t))ds\,dt \\
&\le \frac{1}{2}\int_0^T |\dot{u}_{1n}(t)|^2 dt + \frac{T}{\mu}F_1(\lambda\overline{u}_{1n}) + r\int_0^T\int_0^1 s|\widetilde{u}_{1n}(t)|^2 ds\,dt - TF_1(0) + \sum_{j=1}^{l}\gamma_j \\
&\le \frac{1}{2}\int_0^T |\dot{u}_{1n}(t)|^2 dt + \frac{T}{\mu}F_1(\lambda\overline{u}_{1n}) + \frac{r}{2}\int_0^T |\widetilde{u}_{1n}(t)|^2 dt - TF_1(0) + \sum_{j=1}^{l}\gamma_j \\
&\le \frac{\max\{1,r\}}{2}\|\widetilde{u}_{1n}\|^2_{W_T^{1,2}} + \frac{T}{\mu}F_1(\lambda\overline{u}_{1n}) - TF_1(0) + \sum_{j=1}^{l}\gamma_j \\
&\le \frac{\max\{1,r\}}{2}D_3^q + \frac{T}{\mu}F_1(\lambda\overline{u}_{1n}) - TF_1(0) + \sum_{j=1}^{l}\gamma_j,
\end{aligned}
\tag{3.19}
$$

for all $n$ and (3.19) and (F5) imply that $\{\overline{u}_{1n}\}$ is bounded. Combining (3.18), we know that $\{u_{1n}\}$ is a bounded sequence. Similar to the argument in [25], it is easy to obtain that $\varphi$ satisfies (PS) condition. □

*Proof of Theorem 1.4.* From (I3) and (F5), it is easy to see that for $x_1 \in \mathbb{R}^N$,

$$\varphi_1(x_1) \longrightarrow -\infty, \quad \text{as } |x_1| \longrightarrow \infty. \tag{3.20}$$

For all $u_1 \in \widetilde{W}_T^{1,2}$, by (1.18), (F1)″ and (I3), we have

$$
\begin{aligned}
\varphi_1(u_1) &= \frac{1}{2}\int_0^T |\dot{u}_1(t)|^2 dt + \int_0^T F_1(u_1(t))dt + \sum_{j=1}^{l} I_j(u_1(t_j)) \\
&= \frac{1}{2}\int_0^T |\dot{u}_1(t)|^2 dt + \int_0^T [F_1(u_1(t)) - F_1(0)]dt + TF_1(0) + \sum_{j=1}^{l} I_j(u_1(t_j)) \\
&= \frac{1}{2}\int_0^T |\dot{u}_1(t)|^2 dt + \int_0^T\int_0^1 (\nabla F_{1x_1}(su_1(t)), u_1(t))ds\,dt + \sum_{j=1}^{l} I_j(u_1(t_j)) + TF_1(0) \\
&= \frac{1}{2}\int_0^T |\dot{u}_1(t)|^2 dt + \int_0^T\int_0^1 \frac{1}{s}(\nabla F_{1x_1}(su_1(t)) - \nabla F_{1x_1}(0), su_1(t))ds\,dt \\
&\quad + \sum_{j=1}^{l} I_j(u_1(t_j)) + TF_1(0)
\end{aligned}
$$

$$\begin{aligned}
&\geq \frac{1}{2}\int_0^T |\dot{u}_1(t)\ |^2 dt - \frac{r_1}{2}\int_0^T |u_1(t)|^2 dt + TF_1(0) - \sum_{j=1}^{l} b_j \left|u_1(t_j)\right|^{\alpha_j} - \sum_{j=1}^{l} c_j \\
&\geq \frac{1}{2}\int_0^T |\dot{u}_1(t)|^2 dt - \frac{r_1 T^2}{4\pi^2}\int_0^T |\dot{u}_1(\mathrm{t})|^2 dt + TF_1(0) - \sum_{j=1}^{l} b_j \|u_1\|_\infty^{\alpha_j} - \sum_{j=1}^{l} c_j \\
&\geq \left(\frac{1}{2} - \frac{r_1 T^2}{4\pi^2}\right)\int_0^T |\dot{u}_1(t)|^2 dt + TF_1(0) \\
&\quad - \left(\frac{T}{12}\right)^{\alpha_j/2} \sum_{j=1}^{l} b_j \left(\int_0^T |\dot{u}_1(t)|^2 dt\right)^{\alpha_j/2} - \sum_{j=1}^{l} c_j.
\end{aligned} \tag{3.21}$$

Note that for all $u_1 \in \widetilde{W}_T^{1,2}$, $\|u_1\|_{W_T^{1,2}}$ is equivalent to $\|\dot{u}_1\|_{L^2}$. Then, $r_1 < 4\pi^2/T^2$, $\alpha_j < 2(j \in B)$ and (3.21) imply that

$$\varphi_1(u_1) \longrightarrow +\infty, \quad \text{as } \|u_1\|_{W_T^{1,2}} \longrightarrow \infty, \quad u_1 \in \widetilde{W}_T^{1,2}. \tag{3.22}$$

It follows from (3.20) and (3.22) that $\varphi_1$ satisfies ($\varphi$1) and ($\varphi$2) in Lemma 2.5. Combining with Lemma 3.2, Lemma 2.5 shows that $\varphi_1$ has at least one critical point. Thus, we complete the proof. □

## 4. Examples

*Example 4.1.* Let $q = 4$, $p = 2$, $T = \pi$, $t_1 = 1$, and $s_1 = 2$. Consider the following system:

$$\begin{gathered}
\frac{d}{dt}\Phi_4(\dot{u}_1(t)) = \nabla_{u_1} F(u_1(t)\ , u_2(t)), \quad \text{a.e. } t \in [0,\pi], \\
\frac{d}{dt}\Phi_2(\dot{u}_2(t)) = \nabla_{u_2} F(u_1(t), u_2(t)), \quad \text{a.e. } t \in [0,\pi], \\
u_1(0) - u_1(\pi) = \dot{u}_1(0) - \dot{u}_1(\pi) = 0, \\
u_2(0) - u_2(\pi) = \dot{u}_2(0) - \dot{u}_2(\pi) = 0, \\
\Delta\Phi_4(\dot{u}_1(1)) = \Phi_q(\dot{u}_1(1^+)) - \Phi_q(\dot{u}_1(1^-)) = \nabla I_1(u_1(1)), \\
\Delta\Phi_2(\dot{u}_2(2)) = \Phi_p(\dot{u}_2(2^+)) - \Phi_p(\dot{u}_2(2^-)) = \nabla K_1(u_2(2)),
\end{gathered} \tag{4.1}$$

where $F(x_1, x_2) = x_{11}^4 + x_{12}^4 + \cdots + x_{1N}^4 + (1/\pi^2)(x_{21}^4 + x_{22}^2 + \cdots + x_{2N}^2) - (1/2\pi^2)|x_2|^2$, $x_1 = (x_{11}, x_{12}, \ldots, x_{1N})$, $x_2 = (x_{21}, x_{22}, \ldots, x_{2N})$, $I_1(x) = e^{|x|^2}$, $K_1(x) = e^{|x|^2}$, $x \in \mathbb{R}^N$. It is easy to verify that all conditions of Theorem 1.1 hold so that system (4.1) has at least one weak solution. Moreover, if $F(x_1, x_2) = (1/\pi^2)(x_{21}^4 + x_{22}^4 + \cdots + x_{2N}^4) - 1/2\pi^2|x_2|^2$, $x_2 = (x_{21}, x_{22}, \ldots, x_{2N})$, $I_1(x) = 0$ and $K_1(x) = 0$, $x \in \mathbb{R}^N$, then system (4.1) has at least two nonzero solutions.

*Example 4.2.* Let $T = 2$, $t_1 = 1$. Consider the following autonomous second-order Hamiltonian system with impulsive effects:

$$\begin{aligned} \ddot{u}(t) &= \nabla_u F(u(t)), \quad \text{a.e. } t \in [0,2], \\ u(0) - u(2) &= \dot{u}(0) - \dot{u}(2) = 0, \\ \dot{u}(1^+) - \dot{u}(1^-) &= \nabla I_1(u(1)), \end{aligned} \tag{4.2}$$

where $F(z) = z_1^4 + z_2^2 + \cdots + z_N^2 - 1/2|z|^2, I_1(z) = e^{|z|^2}$, $z = (z_1, \ldots, z_N)^\tau \in \mathbb{R}^N$. It is easy to verify that all conditions of Theorem 1.3 hold so that system (4.2) has at least one weak solution. Moreover, if $F(z) = z_1^4 + z_2^4 + \cdots + z_N^4 - 1/2|z|^2$ and $I_1(z) = 0$, $z \in \mathbb{R}^N$, then system (4.2) has at least two nonzero solutions.

*Example 4.3.* Let $T = \pi$, $t_1 = 2$. Consider the following autonomous second-order Hamiltonian system with impulsive effects:

$$\begin{aligned} \ddot{u}(t) &= \nabla_u F(u(t)), \quad \text{a.e. } t \in [0,\pi], \\ u(0) - u(\pi) &= \dot{u}(0) - \dot{u}(\pi) = 0, \\ \dot{u}(2^+) - \dot{u}(2^-) &= \nabla I_1(u(2)), \end{aligned} \tag{4.3}$$

where $F(z) = -|z|^2$, $I_1(z) = 2\sin z_1$, $z = (z_1, \ldots, z_N)^\tau \in \mathbb{R}^N$. It is easy to verify that all conditions of Theorem 1.4 hold so that system (4.3) has at least one weak solution.

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