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Research Article

Existence and Uniqueness of the Solution for a Time-Fractional Diffusion Equation with Robin Boundary Condition

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Existence and uniqueness of the solution for a time-fractional diffusion equation with Robin boundary condition on a bounded domain with Lyapunov boundary is proved in the space of continuous functions up to boundary. Since a Green matrix of the problem is known, we may seek the solution as the linear combination of the single-layer potential, the volume potential, and the Poisson integral. Then the original problem may be reduced to a Volterra integral equation of the second kind associated with a compact operator. Classical analysis may be employed to show that the corresponding integral equation has a unique solution if the boundary data is continuous, the initial data is continuously differentiable, and the source term is Hölder continuous in the spatial variable. This in turn proves that the original problem has a unique solution.

1. Introduction

In this paper, we study solvability of the time-fractional diffusion equation (TFDE)

$$\begin{gathered}
\partial_t^\alpha \Phi(x,t) - \Delta_x \Phi(x,t) = f(x,t), \quad \text{in } Q_T = \Omega \times (0,T], \\
\frac{\partial \Phi(x,t)}{\partial n(x)} + \beta(x,t)\Phi(x,t) = g(x,t), \quad \text{on } \Sigma_T = \Gamma \times (0,T], \\
\Phi(x,0) = \psi(x), \quad x \in \overline{\Omega},
\end{gathered} \tag{1.1}$$

where $f,\ g,\ \psi$ are any given functions, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with Lyapunov boundary $\Gamma \in C^{1+\lambda}$, $0 < \lambda < 1$, and

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\tau)^{-\alpha}u(\tau)\mathrm{d}\tau - t^{-\alpha}u(0)\right) \tag{1.2}$$

is the fractional Caputo time derivative of order $0 < \alpha < 1$. Physically fractional diffusion equations describe anomalous diffusion on complex systems like some amorphous semiconductors or strongly porous materials (see [1] and references therein).

As to the mathematical theory of fractional diffusion equations, only the first steps have been taken. In the literature, mainly the Cauchy problems for these equations have been considered until now (see [2–5] and references therein). Existence and uniqueness of a generalized solution for an initial-boundary-value problem for the generalized time-fractional diffusion equation is proved in [6]. However, uniqueness and existence of the classical solution is given only in a special 1-dimensional case.

Our model problem is much simpler than those treated for example, in [3, 5]. However, the boundary integral approach used in this paper can be used in more general situations as well. We decided to concentrate on a simple model instead of the more general ones to clarify the basic idea. Boundary integral approach also allows us to study (TFDE) or its generalizations in weaker spaces such as L^p-spaces or in the scale of anisotropic Sobolev spaces.

The paper is organized as follows. In Preliminaries, we recall the definitions of the potentials and the Poisson integral. We introduce their well-known properties from theory of PDEs of parabolic type, which are needed for proving the existence and uniqueness of the solution. That is, we recall the boundary behavior of the single-layer potential. We show that the volume potential solves the nonhomogeneous TFDE with the zero initial condition. Moreover, we prove that the Poisson integral solves the homogeneous TFDE with a given initial datum. The final section is dedicated to the proof of existence and uniqueness of the solution.

2. Preliminaries

Here we recall the potentials and the Poisson integral and their basic properties. In the sequel we shall assume that the functions appearing in the definitions are smooth enough such that the corresponding integrals exist.

The single-layer potential can be defined as

$$(S\varphi)(x,t) = \int_0^t\int_\Gamma \partial_{n(y)}G(x-y,t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau, \quad x \in \Omega, \tag{2.1}$$

where $n(y)$ denotes the outward unit normal at $y \in \Gamma$ and

$$G(x,t) = \begin{cases} \pi^{-n/2}t^{\alpha-1}|x|^{-n}H_{12}^{20}\left[\frac{1}{4}|x|^2t^{-\alpha}\ \Big|\ \begin{matrix}(\alpha,\alpha)\\ (n/2,1),(1,1)\end{matrix}\right], & x \in \mathbb{R}^n,\ t > 0, \\ 0, & x \in \mathbb{R}^n,\ t < 0, \end{cases} \tag{2.2}$$

is the fundamental solution of (TFDE) [3, 7–9]. Here H_{12}^{20} is the Fox H-function, which is defined via Mellin-Barnes integral representation

$$H_{12}^{20}(z) := H_{12}^{20}\left[z \mid \begin{array}{c} (\alpha,\alpha) \\ (n/2,1),(1,1) \end{array}\right] = \frac{1}{2\pi \mathrm{i}}\int_{\mathcal{C}} \frac{\Gamma(n/2+s)\Gamma(1+s)}{\Gamma(\alpha+\alpha s)} z^{-s}\, \mathrm{d}s, \tag{2.3}$$

where $\mathcal{C}$ is an infinite contour on the complex plane circulating the negative real axis counterclockwise.

The volume potential is defined by

$$(V\varphi)(x,t) = \int_0^t \int_{\Omega} G(x-y,t-\tau)\varphi(y,\tau)\mathrm{d}y\,\mathrm{d}\tau, \quad x \in \Omega \tag{2.4}$$

for φ such that supp $\varphi(\cdot,t) \subset \Omega$ for any $t \in (0,T]$.

The Poisson integral is defined as

$$(P\varphi)(x,t) = \int_0^t \int_{\Omega_0} E(x-y,t)\varphi(y)\mathrm{d}y, \quad x \in \overline{\Omega}, \tag{2.5}$$

where Ω_0 is some neighborhood of $\overline{\Omega}$ and

$$E(x,t) = \pi^{-n/2}|x|^{-n}\widetilde{H}_{12}^{20}\left(\frac{1}{4}|x|^2 t^{-\alpha}\right) \tag{2.6}$$

with

$$\widetilde{H}_{12}^{20}(z) = H_{12}^{20}\left[z \mid \begin{array}{c} (1,\alpha) \\ (n/2,1),(1,1) \end{array}\right] = \frac{1}{2\pi \mathrm{i}}\int_{\mathcal{C}} \frac{\Gamma(n/2+s)\Gamma(1+s)}{\Gamma(1+\alpha s)} z^{-s}\mathrm{d}s \tag{2.7}$$

and $\mathcal{C}$ as in the definition of H_{12}^{20}.

Note that in contrast to classical parabolic partial differential equations, we have a Green matrix $\{E(x,t),G(x,t)\}$ instead of one fundamental solution. We also emphasize that the Green's functions have singularities both in time and spatial variable unlike in the case of classical parabolic PDEs, where singularity occurs only in time.

Let us now state the basic properties of the aforementioned quantities. Since the proofs are strongly based on the detailed analysis of the Fox H-functions, we shall recall their basic properties. For further details of these functions, we refer to [3, 10, 11].

In order to simplify the notations, we introduce the following function defined for $z > 0$:

$$H_{(p)}(z) = \frac{1}{2\pi \mathrm{i}}\int_{\mathcal{C}} \frac{\Gamma(n/2+s)\Gamma(1+s)^p}{\Gamma(\alpha+\alpha s)\Gamma(s)^{p-1}} z^{-s}\mathrm{d}s, \quad p = 1,2. \tag{2.8}$$

Note that, in particular, $H_{(1)}(z) = H^{20}_{12}(z)$. The following properties of $H_{(p)}$ are needed.

Lemma 2.1. *For the functions $H_{(p)}$, there holds*

(i) *differentiation formula* $(\mathrm{d}/\mathrm{d}z)H_{(1)}(z) = -z^{-1}H_{(2)}(z)$,

(ii) *the asymptotic behaviour at infinity,*

$$\left|H_{(p)}(z)\right| \le Cz^{n/2}\exp\left(-\sigma z^{1/(2-\alpha)}\right), \quad \sigma := \alpha^{\alpha/2-\alpha}(2-\alpha), \tag{2.9}$$

for $p = 1, 2$ *and* $z \ge 1$,

(iii) *the asymptotic behaviour near zero*

$$\left|H_{(p)}(z)\right| \le C\begin{cases} z^{n/2} & \text{if } n = 2 \text{ or } n = 3, \\ z^2\left|\log z\right| & \text{if } n = 4, \\ z^2 & \text{if } n > 4, \end{cases} \tag{2.10}$$

for $p = 1, 2$ *and* $z \le 1$.

The constants in (*ii*) *and* (*iii*) *can depend on* n, p, *and* α.

Proof. The proofs follow from the Mellin-Barnes integral representation and the analyticity of the functions $H_{(p)}$ [3, 10, 12]. □

Remark 2.2. Above and in the sequel, C denotes a generic constant, which may depend on various quantities. The only thing that matters is that in our calculations C will be independent of x and t.

Let us now concentrate on the properties of the potentials. We start with the single-layer potential $S\varphi$. First of all, standard calculations show that $S\varphi$ solves the equation $(\partial_t^\alpha - \Delta_x)u = 0$. Moreover, we need to know the boundary behavior of the single-layer potential, which is given in the following result.

Theorem 2.3. *Let* $\varphi \in C(\overline{\Sigma_T})$. *The single-layer potential defined by* (2.1) *is continuous in* $\overline{\Omega_T}$ *with the zero initial value. Moreover, for* $x \in \Omega$ *and* $x_0 \in \Gamma$, $\nabla_x(S\varphi)(x,t)\cdot n(x_0)$ *has the following limiting value:*

$$\begin{aligned} \partial_{n(x_0)}S\varphi(x_0,t) &:= \lim_{x\to x_0}\nabla_x(S\varphi)(x,t)\cdot n(x_0) \\ &= \frac{1}{2}\varphi(x_0,t) + \int_0^t\int_\Gamma \partial_{n(x_0)}G(x_0-y,t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau \end{aligned} \tag{2.11}$$

as x tends to x_0 nontangentially.

Proof. The proof follows the same lines as in the case of the single-layer potential for the heat equation [13, Chapter 5.2] and is based on a detailed analysis of the kernel G. Since the proof is rather lengthy, we give only the reference [12, Theorems 1 and 2]. □

For the volume potential, we have the following result.

Theorem 2.4. *Let* $f \in C(\overline{\Sigma_T})$ *such that* $f(\cdot,t)$ *is Hölder continuous uniformly in* $t \in [0,T]$ *and* $\operatorname{supp} f(\cdot,t) \subset \Omega$, $t \in [0,T]$. *Then the volume potential* Vf *with* V *defined by* (2.4) *solves* $\partial_t^\alpha u - \Delta_x u = f$ *with the zero initial condition.*

Proof. The zero initial condition follows since G is locally integrable. Indeed, we split the integral Vf into two parts $I_1 + I_2$ depending on whether $z = (1/4)(t-\tau)^{-\alpha}|x-y|^2 \ge 1$ or $z \le 1$.

If $z \ge 1$, we use the fact that $z^\gamma \exp(-\sigma z^{1/(2-\alpha)})$ is uniformly bounded for any $\gamma, \sigma > 0$. Then Lemma 2.1 together with the definition of G yields

$$\left|G(x-y,t-\tau)\right| \le C(t-\tau)^{\alpha+\alpha\gamma-1-\alpha n/2}\left|x-y\right|^{-2\gamma}. \tag{2.12}$$

If we choose $n/2 - 1 < \gamma < n/2$, we see that $\lim_{t\to 0+} I_1(x,t) = 0$.

On the other hand, if $z \le 1$, then Lemma 2.1 yields

$$\left|G(x-y,t-\tau)\right| \le C \begin{cases} (t-\tau)^{(2-n)\alpha/2}, & n = 2,3, \\ (t-\tau)^{-\alpha-1}\left(\left|\log\left(\left|x-y\right|^2(t-\tau)^{-\alpha}\right)\right| + 1\right), & n = 4, \\ (t-\tau)^{-\alpha-1}\left|x-y\right|^{4-n}, & n > 4. \end{cases} \tag{2.13}$$

Then $\lim_{t\to 0+} I_2(x,t) = 0$ follows immediately for $n = 2,3$. If $n = 4$, we may use the fact that $z^\gamma|\log z|$ is bounded in $(0,1]$ for any $\gamma > 0$. If $n > 4$, we use $z^{-\gamma} \ge 1$ for any $\gamma > 0$. In the preceeding two cases, we obtain

$$\left|G(x-y,t-\tau)\right| \le C(t-\tau)^{\alpha\gamma-\alpha-1}\left|x-y\right|^{4-n-2\gamma}. \tag{2.14}$$

If we choose $1 < \gamma < 2$, we see that $\lim_{t\to 0+} I_2(x,t) = 0$.

For the proof of the first claim, we refer to [3, Sections 5.2 and 5.3], where the proof is given in a much more general case of a time-fractional diffusion equation. □

Finally, for the Poisson integral there holds the following theorem.

Theorem 2.5. *Let* ψ *be a continuous function in* Ω_0. *Then the Poisson integral* $P\psi$ *with* P *defined by* (2.5) *solves* $\partial_t^\alpha u - \Delta_x u = 0$ *with* $u(x,0) = \psi(x)$, $x \in \overline{\Omega}$.

Proof. The fact that $P\psi$ solves $\partial_t^\alpha u - \Delta_x u = 0$ follows from the calculations given in [8]. Note that differentiation inside the integral is allowed because there is no singularity in t. Therefore, it remains to prove the initial condition.

We proceed as in [13, Proof of Theorem 1.2.1] and consider first the case of constant ψ. The integral is divided into two parts $(P\psi)(x,t) = I_R + I_R^c$, where I_R is the integral over the ball $B(x,R)$ with R being so small that $B(x,R)$ is contained in Ω_0 and I_R^c denotes its complementary part. Since R is fixed and there is no singularity in the spatial variable, the asymptotic behavior of G shows that $\lim_{t\to 0+} I_R^c = 0$. We need to prove that $\lim_{t\to 0+} I_R = \psi$.

Introducing spherical coordinates, we get

$$\begin{aligned} I_R &= \pi^{-n/2}\omega_n\psi \int_0^R r^{-1}\widetilde{H}_{12}^{20}\left(\frac{1}{4}t^{-\alpha}r^2\right)\mathrm{d}r \\ &= \pi^{-n/2}\omega_n\psi \int_0^{R/2t^{\alpha/2}} r^{-1}\widetilde{H}_{12}^{20}\left(r^2\right)\mathrm{d}r \\ &\longrightarrow \pi^{-n/2}\omega_n\psi \int_0^{\infty} r^{-1}\widetilde{H}_{12}^{20}\left(r^2\right)\mathrm{d}r, \quad t \longrightarrow 0+, \end{aligned} \tag{2.15}$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$.

To evaluate the last integral denoted by I, we note that the asymptotic behavior of the integrand guarantees the absolute integrability. Through the change of variables $r^2 = t$, we see that the integral is nothing but half of the Mellin transform of $\widetilde{H}_{12}^{20}$,

$$I = \frac{1}{2}\mathcal{M}\left(\widetilde{H}_{12}^{20}\right)(s) = \frac{\Gamma(n/2+s)\Gamma(1+s)}{2\Gamma(1+\alpha s)}, \tag{2.16}$$

evaluated at the point $s = 0$.

Therefore, we may conclude that the claim in the case ψ is constant. In the case of general ψ we may proceed as in [13, Proof of Theorem 1.2.1.] □

3. Existence and Uniqueness of the Solution

As it was mentioned in Introduction, we seek the solution in a form of

$$u(x,t) = (S\varphi)(x,t) + (P\psi)(x,t) + (Vf)(x,t), \tag{3.1}$$

where φ is to be determined. The density φ is determined by reducing the original problem to a corresponding integral equation.

We assume that β is a continuous function on $\overline{\Sigma_T}$. We need to calculate the normal derivative of $S\varphi$, Vf, and $P\psi$. For $P\psi$, we observe that differentiation inside the integral defining P is allowed since there is no singularity in t. For Vf, the differentiation inside the integral is justified by the calculations given in [8]. Finally, Theorem 2.3 gives the boundary value for the normal derivative of $S\varphi$. Then the Robin boundary condition is equivalent with

$$\left(\frac{1}{2}I + W + \beta S\right)\varphi = F, \tag{3.2}$$

where

$$(W\varphi)(x,t) = \int_0^t \int_\Gamma \partial_{n(x)} G(x-y,t-\tau)\varphi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau \tag{3.3}$$

is the integral in Theorem 2.3 and

$$
\begin{aligned}
F(x,t) = g(x,t) &- \int_{\Omega_0} \partial_{n(x)} E(x-y,t)\varphi(y)\mathrm{d}y \\
&- \int_0^t \int_{\Omega} \partial_{n(x)} G(x-y,t-\tau) f(y,\tau)\mathrm{d}y\,\mathrm{d}\tau \\
&- \beta(x,t)(P\varphi)(x,t) - \beta(x,t)(Vf)(x,t).
\end{aligned}
\tag{3.4}
$$

We will prove that (3.2) admits a unique solution for any bounded function F. Therefore, it is needed to determine the conditions, which guarantee boundedness.

For the second integral on the right-hand side of (3.4), we use the following result.

Lemma 3.1. *Let $z = (1/4)|x-y|^2 t^{-\alpha}$ with $x \in \Gamma$ and $y \in \Omega$. The following estimates for the normal derivative of G hold:*

(1) *if $z \geq 1$, then*

$$
\left|\partial_{n(x)} G(x-y,t)\right| \leq C t^{-\alpha n/2-1} |x-y| \exp\left\{-\sigma t^{-\alpha/(2-\alpha)} |x-y|^{2/(2-\alpha)}\right\}; \tag{3.5}
$$

(2) *if $z \leq 1$, then*

$$
\left|\partial_{n(x)} G(x-y,t)\right| \leq C \begin{cases} t^{-\alpha-1}|x-y|\left|\log\left(|x-y|^2 t^{-\alpha}\right)\right| & \text{if } n = 2, \\ t^{-\alpha-1}|x-y|^{3-n} & \text{if } n \geq 3. \end{cases} \tag{3.6}
$$

Proof. Applying the differentiation formula in Lemma 2.1, we get

$$
\partial_{n(x)} G(x-y,t) = -\pi^{-n/2} \frac{\langle x-y, n(x)\rangle}{|x-y|^{n+2}} t^{\alpha-1} \left\{ n H_{(1)}(z) + 2 H_{(2)}(z) \right\}, \tag{3.7}
$$

where $z = (1/4)|x-y|^2 t^{-\alpha}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

Using the definition of $H_{(p)}$ and the property $\Gamma(z+1) = z\Gamma(z)$ of the Gamma function, it follows that the Mellin transform of $nH_{(1)} + 2H_{(2)}$ is

$$
\mathcal{M}\left(nH_{(1)} + 2H_{(2)}\right)(s) = \frac{1}{2} \frac{\Gamma((n+2)/2+s)\Gamma(1+s)}{\Gamma(\alpha+\alpha s)}, \tag{3.8}
$$

which is nothing but half of the Mellin transform of $H_{(1)}$ with n replaced by $n+2$.

Using the estimate (2.9) of Lemma 2.1 with $n+2$ instead of n, we obtain the first estimate for $z \geq 1$.

If $z \leq 1$, we use the estimate (2.10) of Lemma 2.1 with $n+2 = 4$ and $n+2 > 4$ to obtain the second estimate. □

Let us return to the estimation of the second integral on r.h.s. of (3.4). Once again we split the integral into two parts I_1 and I_2 depending on whether $z \geq 1$ or $z \leq 1$ with $z = (1/4)|x-y|^2(t-\tau)^{-\alpha}$. If f is a bounded function and $z \geq 1$, there holds

$$|I_1| \leq C\|f\|_{L^\infty(\Omega_T)} \int_0^t (t-\tau)^{\alpha/2-1} d\tau \int_4^\infty r^n \exp\left(-\sigma r^{2/(2-\alpha)}\right) dr \leq C\|f\|_{L^\infty(\Omega_T)}, \tag{3.9}$$

where we have used the spherical coordinates with $r = (t-\tau)^{-\alpha/2}|x-y|$.

If $z \leq 1$, we have to consider different cases of n's separately. As an example, let us consider case $n = 3$. We have

$$|I_2| \leq C\|f\|_{L^\infty(\Omega_T)} \int_{\Omega\times(0,t)\cap\{z\leq 1\}} (t-\tau)^{\alpha\gamma-\alpha-1} |x-y|^{-2\gamma} dy\, d\tau, \tag{3.10}$$

where the fact $z^{-\gamma} \geq 1$ for any $\gamma > 0$ is used. Choosing $1 < \gamma < 3/2$, we see that I_2 is bounded.

Using the estimates (2.12) and (2.13) in the proof of Theorem 2.4, we see that Vf is bounded as well.

For the first integral on the right-hand side, we use the following result [3, Proposition 1].

Lemma 3.2. *Let $z = (1/4)|x|^2 t^{-\alpha}$. For E there holds the following:*

(1) *if $z \geq 1$, then*

$$|\nabla_x E(x,t)| \leq Ct^{-\alpha(n+1)/2} \exp\left\{-\sigma t^{-\alpha/(2-\alpha)} |x|^{2/(2-\alpha)}\right\}; \tag{3.11}$$

(2) *if $z \leq 1$, then*

$$|\nabla_x E(x,t)| \leq Ct^{-\alpha}|x|^{-n+1}. \tag{3.12}$$

We split the first integral on right-hand side of (3.4) into two parts I_1 and I_2 depending whether $z = (1/4)|x-y|^2 t^{-\alpha} \geq 1$ or $z \leq 1$. If ψ is a bounded function, then using Lemma 3.2 we have

$$|I_1| \leq Ct^{-\alpha+\alpha\gamma}\|\psi\|_{L^\infty(\Omega_0)} \tag{3.13}$$

for any $\gamma > 0$, since $z \mapsto z^\gamma \exp(-\sigma z^\beta)$ is uniformly bounded on $[1,\infty)$ for any $\beta,\ \gamma,\ \sigma > 0$. Similarly, for I_2 there holds

$$|I_2| \leq C\|\psi\|_{L^\infty(\Omega_0)} \int_{\Omega_0\cap\{z\leq 1\}} t^{-\alpha+\alpha\gamma} |x-y|^{-n+1-2\gamma} dy \tag{3.14}$$

for any $\gamma > 0$, since $z^{-\gamma} \geq 1$ for any $\gamma > 0$. Choosing $\gamma < 1/2$, we have

$$|I_2| \leq Ct^{-\alpha+\alpha\gamma}\|\psi\|_{L^\infty(\Omega_0)}. \tag{3.15}$$

We see that I_2 blows up as $t \to 0+$. Therefore, we have to assume more smoothness on ψ to guarantee boundedness. Assume that ψ is a continuously differentiable function in $\overline{\Omega_0}$. Then integration by parts yields a better kernel E. Asymptotic behavior of E guarantees that the resulting integral is uniformly bounded on $x,\ t$ (see [3, Proposition 1]). We have

$$\left|\int_{\Omega_0} \partial_{n(x)} E(x-y,t)\psi(y)\,\mathrm{d}y\right| \leq C\left(\|\psi\|_{L^\infty(\overline{\Omega_0})} + \|\nabla\psi\|_{L^\infty(\overline{\Omega_0})}\right). \tag{3.16}$$

The same reason as above implies that $P\psi$ is bounded.

Now we are ready to prove that (3.2) has a unique solution.

Theorem 3.3. *Let $f \in L^\infty(\Omega_T)$, $g \in C(\overline{\Sigma_T})$, and $\psi \in C^1(\overline{\Omega_0})$. Then the boundary integral equation* (3.2) *admits a unique bounded, continuous solution φ.*

Proof. Using similar estimates as in Lemma 3.1 and in the proof of Theorem 2.4, we see that $W + \beta S$ is an integral operator with a weakly singular kernel. Note that the estimates for the normal derivative given in Lemma 3.1 can be multiplied by $|x - y|^\lambda$ in the estimates for W due to the Lyapunov smoothness of the boundary Γ. For details we refer to [12].

We conclude that $W+\beta S$ is a compact operator in $C(\overline{\Sigma_T})$ [14, Theorem 2.22]. Moreover, similarly as in [15] we can prove that there exists an integer k_0 such that

$$\|(2W + 2\beta S)^{k_0 l}\varphi\|_{L^\infty(\Sigma_T)} \leq \frac{(MT)^l}{l!}\|\varphi\|_{L^\infty(\Sigma_T)} \tag{3.17}$$

for some constant M and for all $l \in \mathbb{N}$. This, in particular, implies that the homogeneous equation $((1/2)I+W+\beta S)\varphi = 0$ has a unique solution. Moreover, $(2W+2\beta S)^{k_0 l}$ is a contraction for some $l \in \mathbb{N}$. Therefore, $(1/2)I + W + \beta S$ is invertible and the inverse is given by the Neumann series

$$((1/2)I + W + \beta S)^{-1} = 2\sum_{k=0}^{\infty}(-1)^k(2W + 2\beta S)^k. \tag{3.18}$$

Since the series is uniformly convergent, we have

$$\|\varphi\|_{L^\infty(\Sigma_T)} \leq C\|F\|_{L^\infty(\Sigma_T)}, \tag{3.19}$$

and continuity of φ follows from that of F. □

In conclusion, u defined by (3.1) solves (TFDE) provided φ solves (3.2). Combining our results with the results in [3, 8], we have proved our main result, which is stated as follows.

Theorem 3.4. *Let $g \in C(\overline{\Sigma_T})$, $\psi \in C^1(\overline{\Omega_0})$, and $f \in C(\overline{\Sigma_T})$ such that $f(\cdot,t)$ is Hölder continuous uniformly in $t \in [0,T]$ and* $\operatorname{supp} f(\cdot,t) \subset \Omega$, $t \in [0,T]$. *Then (TFDE) admits a unique classical solution and the solution depends continuously on the data in the following sense:*

$$\|u(x,t)\|_{C(\overline{\Omega_T})} \le C\Big(\|f\|_{C(\overline{\Omega_T})} + \|g\|_{C(\Sigma_T)} + \|\psi\|_{C^1(\overline{\Omega_0})}\Big). \tag{3.20}$$

If ψ has compact support in Ω, we may relax the smoothness assumption on ψ and proof of Theorem 3.4 implies the following.

Corollary 3.5. *Let g and f satisfy the assumptions in Theorem 3.4, and let $\psi \in C(\Omega)$ with compact support. Then (TFDE) admits a unique classical solution and the solution depends continuously on the data in the following sense:*

$$\|u(x,t)\|_{C(\overline{\Omega_T})} \le C\Big(\|f\|_{C(\overline{\Omega_T})} + \|g\|_{C(\Sigma_T)} + \|\psi\|_{C(\overline{\Omega})}\Big). \tag{3.21}$$

Proof. All the arguments are the same as in Theorem 3.4 except now we can choose $\gamma = 1$ in the estimates for I_1 and I_2 of

$$\int_\Omega \partial_{n(x)} E(x-y,t)\psi(y)\,\mathrm{d}y = I_1 + I_2 \tag{3.22}$$

in Theorem 3.4. Therefore, we obtain

$$\left|\int_\Omega \partial_{n(x)} E(x-y,t)\psi(y)\,\mathrm{d}y\right| \le C\Big(1 + \operatorname{dist}\left(\operatorname{supp}\,\psi, \Gamma\right)^{-n-1}\Big)\|\psi\|_{L^\infty(\Omega)} \tag{3.23}$$

and the claim follows. □

Remark 3.6. The estimates given for F reveal that if ψ is merely continuous, we have $|F(x,t)| \le Ct^{-\beta}$ for some $-1 < \beta < -\alpha/2$. Then the same technique as in [13, Section 5.3] may be employed to prove that (TFDE) with the initial condition replaced by $u(x,0) = \psi(x)$, $x \in \Omega$, has a unique solution, which may not be even continuous and can be unbounded near $\Gamma \times \{0\}$ and therefore is not a classical solution.

Remark 3.7. The same technique as above may be used for more general time-fractional diffusion equations, where $-\Delta$ is replaced by a uniformly elliptic second-order differential operator in nondivergence form with bounded continuous real-valued coefficients depending on x.

Remark 3.8. In [15], we have proved existence and uniqueness of the solution of TFDE with the zero initial condition and the zero source term with Dirichlet boundary condition. Using the same technique as above, we may also consider the case of nonzero initial condition and nontrivial source term. Indeed, use of the double-layer ansatz leads to a Volterra integral equation of the second kind as in this paper. Then, using the same arguments as above, we can prove uniqueness and existence of a classical solution without any restrictions on n or on boundary conditions such as in [6].

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