Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 265316, 19 pages
doi:10.1155/2011/265316

Research Article

Some Opial-Type Inequalities on Time Scales

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Received 30 January 2011; Accepted 31 March 2011

Academic Editor: W. A. Kirk

We will prove some dynamic inequalities of Opial type on time scales which not only extend some results in the literature but also improve some of them. Some discrete inequalities are derived from the main results as special cases.

1. Introduction

In 1960, Opial [1] proved the following inequality:

$$\int_a^b |x(t)||x'(t)|dt \leq \frac{(b-a)}{4}\int_a^b |x'(t)|^2 dt, \tag{1.1}$$

where x is absolutely continuous on $[a,b]$ and $x(a) = x(b) = 0$ with a best constant $1/4$. Since the discovery of Opial inequality, much work has been done, and many papers which deal with new proofs, various generalizations, and extensions have appeared in the literature. In further simplifying the proof of the Opial inequality which had already been simplified by Olech [2], Beescak [3], Levinson [4], Mallows [5], and Pederson [6], it is proved that if x is real absolutely continuous on $(0,b)$ and with $x(0) = 0$, then

$$\int_0^b |x(t)||x'(t)|dt \leq \frac{b}{2}\int_0^b |x'(t)|^2 dt. \tag{1.2}$$

These inequalities and their extensions and generalizations are the most important and fundamental inequalities in the analysis of qualitative properties of solutions of different types of differential equations.

In recent decades, the asymptotic behavior of difference equations and inequalities and their applications have been and still are receiving intensive attention. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. So, it is expected to see the discrete versions of the above inequalities. In fact, the discrete analogy of (1.1) which has been proved by Lasota [7] is given by

$$\sum_{i=1}^{h-1}|x_i\Delta x_i| \leq \frac{1}{2}\left[\frac{h+1}{2}\right]\sum_{i=0}^{h-1}|\Delta x_i|^2, \tag{1.3}$$

where $\{x_i\}_{0\leq i\leq h}$ is a sequence of real numbers with $x_0 = x_h = 0$ and $[\cdot]$ is the greatest integer function. The discrete analogy of (1.2) is proved in [8, Theorem 5.2.2] and given by

$$\sum_{i=1}^{h-1}|x_i\Delta x_i| \leq \frac{h-1}{2}\sum_{i=0}^{h-1}|\Delta x_i|^2, \tag{1.4}$$

where $\{x_i\}_{0\leq i\leq h}$ is a sequence of real numbers with $x_0 = 0$. These difference inequalities and their generalizations are also important and fundamental in the analysis of the qualitative properties of solutions of difference equations.

Since the continuous and discrete inequalities are important in the analysis of qualitative properties of solutions of differential and difference equations, we also believe that the unification of these inequalities on time scales, which leads to dyanmic inequalities on time scales, will play the same effective act in the analysis of qualitative properties of solutions of dynamic equations. The study of dynamic inequalities on time scales helps avoid proving results twice: once for differential inequality and once again for difference inequality. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is the so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$.

In this paper, we are concerned with a certain class of Opial-type dynamic inequalities on time scales and their extensions. If the time scale equals the reals (or to the integers), the results represent the classical results for differential (or difference) inequalities. A cover story article in New Scientist [9] discusses several possible applications. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [10]), that is, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$. For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [11, 12] which summarize and organize much of the time scale calculus.

In the following, we recall some of the related results that have been established for differential inequalities and dynamic inequalities on time scales that serve and motivate the contents of this paper. For a generalization of (1.1), Beescak [3] proved that if x is an absolutely continuous function on $[a, X]$ with $x(a) = 0$, then

$$\int_a^X |x(t)||x'(t)|dt \leq \frac{1}{2}\int_a^X \frac{1}{r(t)}dt \int_a^X r(t)|x'(t)|^2 dt, \tag{1.5}$$

where $r(t)$ is positive and continuous function with $\int_a^X dt/r(t) < \infty$, and if $x(b) = 0$, then

$$\int_X^b |x(t)||x'(t)|dt \le \frac{1}{2}\int_X^b \frac{1}{r(t)}dt \int_X^b r(t)|x'(t)|^2 dt. \tag{1.6}$$

Yang [13] simplified the Beesack proof and extended the inequality (1.5) and proved that if x is an absolutely continuous function on (a, b) with $x(a) = 0$, then

$$\int_a^b q(t)|x(t)||x'(t)|dt \le \frac{1}{2}\int_a^b \frac{1}{r(t)}dt \int_a^b r(t)q(t)|x'(t)|^2 dt, \tag{1.7}$$

where $r(t)$ is a positive and continuous function with $\int_a^X dt/r(t) < \infty$ and $q(t)$ is a positive, bounded, and nonincreasing function on $[a, b]$.

Hua [14] extended the inequality (1.2) and proved that if x is an absolutely continuous function with $x(a) = 0$, then

$$\int_a^b |x(t)|^p |x'(t)|dt \le \frac{(b-a)^p}{p+1}\int_a^b |x'(t)|^{p+1} dt, \tag{1.8}$$

where p is a positive integer. We mentioned here that the results in [14] fail to apply for general values of p.

Maroni [15] generalized (1.5) and proved that if x is an absolutely continuous function on $[a, b]$ with $x(a) = 0 = x(b)$, then

$$\int_a^b |x(t)||x'(t)|dt \le \frac{1}{2}\left(\int_a^b \left(\frac{1}{r(t)}\right)^{\alpha-1} dt\right)^{2/\alpha} \left(\int_a^X r(t)|x'(t)|^{\nu} dt\right)^{2/\nu}, \tag{1.9}$$

where $\int_a^b (1/r(t))^{\alpha-1} dt < \infty$, $\alpha \ge 1$ and $1/\alpha + 1/\nu = 1$.

Boyd and Wong [16] extended the inequality (1.8) for general values of $p > 0$ and proved that if x is an absolutely continuous function on $[a, b]$ with $x(0) = 0$, then

$$\int_0^a s(t)|x(t)|^p |x'(t)|dt \le \frac{1}{\lambda_0(p+1)}\int_0^a r(t)|x'(t)|^{p+1} dt, \tag{1.10}$$

where r and s are nonnegative functions in $C^1[0, a]$ and λ_0 is the smallest eigenvalue of the boundary value problem

$$\left(r(t)(u'(t))^p\right)' = \lambda s'(t)u^p(t), \tag{1.11}$$

with $u(0) = 0$ and $r(a)(u'(a))^p = \lambda s'(a)u^p(a)$ for which $u' > 0$ in $[0, a]$.

Yang [13] extended the inequality (1.8) and proved that if x is an absolutely continuous function on $[a,b]$ with $x(a)=0$, $p\geq 0$ and $q\geq 1$, then

$$\int_a^b |x(t)|^p \left|x'(t)\right|^q dt \leq \frac{q}{p+q}(b-a)^p \int_a^b \left|x'(t)\right|^{p+q} dt. \tag{1.12}$$

Yang [17] extended the inequality (1.12) and proved that if $r(t)$ is a positive, bounded function and x is an absolutely continuous on $[a,b]$ with $x(a)=0$, $p\geq 0$,$q\geq 1$, then

$$\int_a^b r(t)|x(t)|^p \left|x'(t)\right|^q dt \leq \frac{q}{p+q}(b-a)^p \int_a^b r(t)\left|x'(t)\right|^{p+q} dt. \tag{1.13}$$

However, as mentioned by Beesack and Das [18], the inequalities (1.12) and (1.13) are sharp when $q=1$ but are not sharp for $q>1$. Considering this problem, Beesack and Das [18] extended and improved the inequalities (1.12) and (1.13) when $x(a)=0$ or $x(b)=0$ or both. In fact, the extensions dealt with integral inequalities of the form

$$\int_a^b s(t)\left|y(t)\right|^p \left|y'(t)\right|^q dt \leq C(p,q) \int_a^b r(t)\left|y'(t)\right|^{p+q} dt, \tag{1.14}$$

where the functions r,s are nonnegative measurable functions on $I=[a,b]$, y is absolutely continuous on I, $pq>0$, and either $p+q\geq 1$ or $p+q<0$, with a sharp constant $C(p,q)$ depends on r,s,p, and q. For applications of the inequality (1.14) on zeros of differential equations, we refer the reader to the paper [19].

However, the study of dynamic inequalities of Opial types on time scales is initiated by Bohner and Kaymakçalan [20], and only recently received a lot of attention and few papers have been written, see [20–24] and the references cited therein. For contribution of different types of inequalities on time scales, we refer the reader to the papers [25–28] and the references cited therein.

Throughout the paper, we denote $f^\sigma := f\circ\sigma$, where the forward jump operator σ is defined by $\sigma(t):=\inf\{s\in\mathbb{T}: s>t\}$. By $x:\mathbb{T}\to\mathbb{R}$ is rd-continuous, we mean x is continuous at all right-dense points $t\in\mathbb{T}$ and at all left-dense points $t\in\mathbb{T}$ left hand limits exist (finite). The graininess function $\mu:\mathbb{T}\to\mathbb{R}^+$ is defined by $\mu(t):=\sigma(t)-t$. Also $\mathbb{T}^\kappa:=\mathbb{T}-\{m\}$ if $\mathbb{T}$ has a left-scattered maximum m, otherwise $\mathbb{T}^\kappa:=\mathbb{T}$. We will assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}}:=[a,b]\cap\mathbb{T}$.

In [20], the authors extended the inequality (1.1) on time scales and proved that if $x:[0,b]\cap\mathbb{T}\to\mathbb{R}$ is delta differentiable with $x(0)=0$, then

$$\int_0^h |x(t)+x^\sigma(t)|\left|x^\Delta(t)\right|\Delta t \leq h\int_0^h \left|x^\Delta(t)\right|^2 \Delta t. \tag{1.15}$$

Also in [20] the authors extended the inequality (1.7) of Yang and proved that if r and q are positive rd-continuous functions on $[0,b]$, $\int_0^b(\Delta t/r(t)) < \infty$, q nonincreasing and $x : [0,b]\cap\mathbb{T} \to \mathbb{R}$ is delta differentiable with $x(0) = 0$, then

$$\int_0^b q^\sigma(t)\left|(x(t)+x^\sigma(t))x^\Delta(t)\right|\Delta t \le \int_0^b \frac{\Delta t}{r(t)}\int_0^b r(t)q(t)\left|x^\Delta(t)\right|^2\Delta t. \tag{1.16}$$

Karpuz et al. [21] proved an inequality similar to the inequality (1.16) replaced $q^\sigma(t)$ by $q(t)$ of the form

$$\int_a^b q(t)\left|(x(t)+x^\sigma(t))x^\Delta(t)\right|\Delta t \le K_q(a,b)\int_a^b \left|x^\Delta(t)\right|^2\Delta t, \tag{1.17}$$

where q is a positive rd-continuous function on, $x : [0,b]\cap\mathbb{T} \to \mathbb{R}$ is delta differentiable with $x(a) = 0$, and

$$K_q(a,b) = \left(2\int_a^b q^2(u)(\sigma(u)-a)\Delta u\right)^{1/2}. \tag{1.18}$$

We note that when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $x^\Delta(t) = x'(t)$, and then the inequality (1.15) becomes the opial inequality (1.2). When $\mathbb{T} = \mathbb{N}$, we have $\sigma(t) = t+1$, $x^\Delta(t) = \Delta x(t)$ and the inequality (1.15) reduces to the inequality

$$\sum_{i=1}^{h-1}|(x_i + x_{i+1})\Delta x_i| \le (h-1)\sum_{i=0}^{h-1}|\Delta x_i|^2, \tag{1.19}$$

which is different from the inequality (1.4). This means that the extensions obtained by Bohner and Kaymakçalan [20] and Karpuz et al. [21] do not give a unification of differential and difference inequalities. So, the natural question now is: *if it possible to find new Opial dynamic inequalities which contains* (1.2) *and* (1.4) *as special cases?* One of our aims in this paper is to give an affirmative answer to this question.

Srivastava et al. [22] extended the Maroni inequality on a time scale and proved that if $x : [0,b]\cap\mathbb{T} \to \mathbb{R}$ is delta differentiable with $x(a) = 0$, then

$$\begin{aligned}\int_a^b s(t)|x(t)|^p\left|x^\Delta(t)\right|\Delta t \le \frac{1}{r+1}&\left(\int_a^b \frac{1}{r^{\alpha-1}(t)}\Delta t\right)^{(1+p)/\alpha}\\ &\times\left(\int_a^X r(t)s^{\nu/1+p}(t)\left|x^\Delta(t)\right|^\nu\Delta t\right)^{(1+p)/\nu},\end{aligned} \tag{1.20}$$

where q and r are positive rd-continuous functions on $[0,b]$, such that $s(t)$ is bounded and decreasing, $\int_a^b(1/r(t))^{\alpha-1}\Delta t < \infty$, $\alpha \ge 1$ and $1/\alpha + 1/\nu = 1$. For the interested reader, it would be interesting to extend the inequality (1.20) with the term $\int_a^b s(t)|x(t)|^p|x^\Delta(t)|\Delta t$ replaced by $\int_a^b s(t)|x(t)|^p|x^\Delta(t)|^q\Delta t$.

Wong et al. [23] and Srivastava et al. [24] extended the Yang inequality on a time scale and proved that if r is a positive rd-continuous function on $[0, b]$, we have

$$\int_a^b r(t)|x(t)|^p \left|x^\Delta(t)\right|^q \Delta t \leq \frac{q}{p+q}(b-a)^p \int_a^b r(t)\left|x^\Delta(t)\right|^{p+1} \Delta t, \tag{1.21}$$

where $x : [0, b] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable with $x(a) = 0$. But according to Beesack and Das [18], the inequality (1.21) is only sharp when $q = 1$. So, the natural question now is: *if it is possible to prove new inequality of type* (1.21) *with two different functions* r *and* s*, instead of* r *with a best constant?* One of our aims in this paper is to give an affirmative answer to this question. Also one of our motivations comes from the fact that the inequality of type (1.21) cannot be applied on the study of the distribution of the generalized zeros of the general dynamic equation

$$\left(r(t)x^\Delta(t)\right)^\Delta + q(t)x(t) = 0, \qquad t \in [\alpha, \beta]_{\mathbb{T}}, \tag{1.22}$$

since this study needs an inequality with two different functions instead of a single function.

The paper is organized as follows. First, we will extend the inequality (1.10) on a time scale which gives a connection between a dynamic Opial-type inequality and a boundary value problem on time scales. Second, we prove some new Opial-type inequalities on time scales with best constants of type (1.14). The main results give the affirmative answers of the above posed questions. Throughout the paper, some special cases on continuous and discrete spaces are derived and compared by previous results.

2. Main Results

In this section, we will prove the main results, and this will be done by making use of the Hölder inequality (see [11, Theorem 6.13])

$$\int_a^h |f(t)g(t)|\Delta t \leq \left[\int_a^h |f(t)|^\gamma \Delta t\right]^{1/\gamma} \left[\int_a^h |g(t)|^\nu \Delta t\right]^{1/\nu}, \tag{2.1}$$

where $a, h \in \mathbb{T}$ and $f;\ g \in C_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$, $\gamma > 1$ *and* $1/\nu + 1/\gamma = 1$, and the inequality (see [29, page 39])

$$A^{p+1} + (p+1)B^{p+1} - (p+1)AB^p \geq 0, \quad \forall A \neq B > 0,\ p > 0. \tag{2.2}$$

We also need the product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable functions f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \qquad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}, \tag{2.3}$$

and the formula

$$\left(x^{\gamma}(t)\right)^{\Delta}=\gamma\int_{0}^{1}\left[hx^{\sigma}+(1-h)x\right]^{\gamma-1}dh\,x^{\Delta}(t), \tag{2.4}$$

which is a simple consequence of Keller's chain rule [11, Theorem 1.90]. We will assume that the boundary value problem

$$\begin{gathered}\left(r(t)\left(u^{\Delta}(t)\right)^{p}\right)^{\Delta}=\beta s^{\Delta}(t)u^{p}(t),\\ u(0)=0,\qquad r(b)\left(u^{\Delta}(b)\right)^{p}=\beta s(b)u^{p}(b),\end{gathered} \tag{2.5}$$

has a solution $u(t)$ such that $u^{\Delta}(t)\geq 0$ on the interval $[0,b]_{\mathbb{T}}$, where r,s are nonnegative rd-continuous functions on $(0,b)_{\mathbb{T}}$.

Now, we are ready to state and prove one of our main results in this section. We begin with an inequality of Opial type which gives a connection with boundary value problems (2.5) on time scales.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale with $0,b\in\mathbb{T}$ and let r,s be nonnegative rd-continuous functions on $(0,b)_{\mathbb{T}}$ such that (2.5) has a solution for some $\beta>0$. If $x:[0,b]\cap\mathbb{T}\to\mathbb{R}$ is delta differentiable with $x(0)=0$, then for $p>0$,*

$$\begin{aligned}&\int_{0}^{b}s(t)|x(t)|^{p}\left|x^{\Delta}(t)\right|\Delta t\\ &\quad\leq\frac{1}{(p+1)\beta_{0}}\int_{0}^{b}r(t)\left|x^{\Delta}(t)\right|^{p+1}\Delta t\\ &\qquad+\frac{p}{(p+1)\beta_{0}}\int_{0}^{b}\left(r(t)w^{(p+1)/p}(t)-r^{\sigma}(t)w^{\sigma}(t)w^{1/p}(t)\right)|x^{\sigma}(t)|^{p+1}\Delta t.\end{aligned} \tag{2.6}$$

Proof. Let $u(t)$ be a solution of the boundary value problem (2.5) and denote

$$f(t)=\left|x^{\Delta}(t)\right|,\qquad F(t)=\int_{0}^{t}f(t)\Delta t,\qquad w(t)=\left(\frac{u^{\Delta}(t)}{u(t)}\right)^{p}. \tag{2.7}$$

Using the inequality (2.2) and substituting f for A and $w^{1/p}\,F^{\sigma}$ for B, we obtain

$$f^{p+1}+pw^{\lambda}(F^{\sigma})^{p+1}-(p+1)fw(F^{\sigma})^{p}\geq 0,\quad\text{where }\lambda=\frac{(p+1)}{p}. \tag{2.8}$$

Multiplying this inequality by $r(t)$, integrating from 0 to b, and using the fact that $F^{\Delta}(t) = f(t) > 0$, we have

$$\begin{aligned}
&\int_0^b r(t) f^{p+1}(t)\Delta t + p\int_0^b r(t) w^{\lambda}(t)(F^{\sigma}(t))^{p+1}\Delta t \\
&\quad\geq (p+1)\int_0^b r(t) w(t) f(t)(F^{\sigma}(t))^{p}\Delta t \\
&\quad= (p+1)\int_0^b r(t) w(t)(F^{\sigma}(t))^{p} F^{\Delta}(t)\Delta t.
\end{aligned} \tag{2.9}$$

By the chain rule (2.4) and the fact that $F^{\Delta}(t) > 0$, we obtain

$$\left(F^{p+1}(t)\right)^{\Delta} = (p+1)\int_0^1 [(1-h)F(t) + hF^{\sigma}(t)]^p dh F^{\Delta}(t). \tag{2.10}$$

Noting that if f is rd-continuous and $F^{\Delta} = f$, we see that

$$\int_t^{\sigma(t)} f(s)\Delta s = F(\sigma(t)) - F(t) = \mu(t)F^{\Delta}(t) = \mu(t) f(t) > 0. \tag{2.11}$$

From the definition of $F(t)$, we see that

$$F^{\sigma}(t) = \int_0^{\sigma(t)} f(t)\Delta t = \int_0^t f(t)\Delta t + \int_t^{\sigma(t)} f(t)\Delta t = F(t) + \mu(t) f(t) \geq F(t). \tag{2.12}$$

Substituting into (2.10), we see that

$$(p+1)[F(t)]^p F^{\Delta}(t) \leq \left(F^{p+1}(t)\right)^{\Delta} \leq (p+1)[F^{\sigma}(t)]^p F^{\Delta}(t). \tag{2.13}$$

Substituting (2.13) into (2.9), we have

$$\begin{aligned}
&\int_0^b r(t) f^{p+1}(t)\Delta t + p\int_0^b r(t) w^{\lambda}(t)(F^{\sigma}(t))^{p+1}\Delta t \\
&\quad\geq \int_0^b r(t) w(t)\left(F^{p+1}(t)\right)^{\Delta}\Delta t.
\end{aligned} \tag{2.14}$$

Integrating by parts and using the assumption $F(0) = 0$, we see that

$$\begin{aligned}\int_0^b r(t)w(t)\left(F^{p+1}(t)\right)^{\Delta}\Delta t &= r(t)w(t)F^{p+1}(t)\Big|_0^b - \int_0^b (r(t)w(t))^{\Delta}(F^{\sigma}(t))^{p+1}\Delta t \\ &= r(b)w(b)F^{p+1}(b) - \int_0^b (r(t)w(t))^{\Delta}(F^{\sigma}(t))^{p+1}\Delta t.\end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we see that

$$\begin{aligned}&\int_0^b r(t)f^{p+1}(t)\Delta t + p\int_0^b r(t)w^{\lambda}(t)(F^{\sigma}(t))^{p+1}\Delta t \\ &\quad\geq r(b)w(b)F^{p+1}(b) - \int_0^b (r(t)w(t))^{\Delta}(F^{\sigma}(t))^{p+1}\Delta t.\end{aligned} \tag{2.16}$$

From the definition of the function $w(t)$, we see that

$$r(t)w(t) = \frac{r(t)\left(u^{\Delta}(t)\right)^{p}}{u^{p}(t)}. \tag{2.17}$$

From this and (2.3), we obtain

$$(r(t)w(t))^{\Delta} = \frac{1}{u^{p}(t)}\left(r(t)\left(u^{\Delta}(t)\right)^{p}\right)^{\Delta} + \left(r\left(u^{\Delta}\right)^{p}\right)^{\sigma}\left[\frac{-(u^{p}(t))^{\Delta}}{u^{p}(t)u^{p}(\sigma(t))}\right]. \tag{2.18}$$

In view of (2.5) and (2.18), we get that

$$(r(t)w(t))^{\Delta} = \beta s^{\Delta}(t) - \frac{\left(r\left(u^{\Delta}\right)^{p}\right)^{\sigma}(u^{p}(t))^{\Delta}}{u^{p}(t)u^{p}(\sigma(t))}. \tag{2.19}$$

Using the fact that $u^{\Delta}(t) \geq 0$ and the chain rule (2.4), we see that

$$\begin{aligned}(u^{p}(t))^{\Delta} &= p\int_0^1 \left[hu^{\sigma} + (1-h)u\right]^{p-1}u^{\Delta}(t)dh \\ &\geq p\int_0^1 \left[hu + (1-h)u\right]^{p-1}u^{\Delta}(t)dh = p(u(t))^{p-1}u^{\Delta}(t).\end{aligned} \tag{2.20}$$

It follows from (2.19) and (2.20) that

$$\begin{aligned}
(r(t)w(t))^{\Delta} &\leq \beta s^{\Delta}(t) - \frac{\left(r(u^{\Delta})^{p}\right)^{\sigma} p(u(t))^{p-1}u^{\Delta}(t)}{u^{p}(t)u^{p}(\sigma(t))}\\
&= \beta s^{\Delta}(t) - \frac{p\left(r(u^{\Delta})^{p}\right)^{\sigma} u^{\Delta}(t)}{u(t)u^{p}(\sigma(t))}\\
&= \beta s^{\Delta}(t) - \frac{pr^{\sigma}(t)\left((u^{\Delta})^{p}\right)^{\sigma} u^{\Delta}(t)}{u(t)u^{p}(\sigma(t))}\\
&= \beta s^{\Delta}(t) - pr^{\sigma}(t)w^{\sigma}(t)w^{1/p}(t).
\end{aligned} \tag{2.21}$$

From (2.21) and (2.16), we have

$$\begin{aligned}
&\int_0^b r(t)f^{p+1}(t)\Delta t + p\int_0^b r(t)w^{\lambda}(t)(F^{\sigma}(t))^{p+1}\Delta t\\
&\quad\geq r(b)w(b)F^{p+1}(b) - \int_0^b \beta s^{\Delta}(t)(F^{\sigma}(t))^{p+1}\Delta t\\
&\qquad + p\int_0^b r^{\sigma}(t)w^{\sigma}(t)w^{1/p}(t)(F^{\sigma}(t))^{p+1}\Delta t.
\end{aligned} \tag{2.22}$$

This implies that

$$\begin{aligned}
&\int_0^b r(t)f^{p+1}(t)\Delta t + p\int_0^b \left[r(t)w^{\lambda}(t) - r^{\sigma}(t)w^{\sigma}(t)w^{1/p}(t)\right](F^{\sigma}(t))^{p+1}\Delta t\\
&\quad\geq r(b)w(b)F^{p+1}(b) - \int_0^b \beta s^{\Delta}(t)(F^{\sigma}(t))^{p+1}\Delta t.
\end{aligned} \tag{2.23}$$

Using the integration by parts again and using (2.13), we see that

$$\begin{aligned}
&-\beta\int_0^b s^{\Delta}(t)(F^{\sigma}(t))^{p+1}\Delta t\\
&\quad= -\beta s(t)(F(t))^{p+1}\Big|_0^b + \int_0^b s(t)\left(F^{p+1}(t)\right)^{\Delta}\Delta t\\
&\quad= -\beta s(b)(F(b))^{p+1} + \int_0^b s(t)\left(F^{p+1}(t)\right)^{\Delta}\Delta t\\
&\quad\geq -\beta s(b)(F(b))^{p+1} + \beta(p+1)\int_0^b s(t)[F(t)]^{p}F^{\Delta}(t)\Delta t.
\end{aligned} \tag{2.24}$$

Substituting (2.24) into (2.23), we have

$$\begin{aligned}
&\int_0^b r(t)f^{p+1}(t)\Delta t + p\int_0^b \left[r(t)w^{\lambda}(t) - r^{\sigma}(t)w^{\sigma}(t)w^{1/p}(t)\right]\left(F^{\sigma}(t)\right)^{p+1}\Delta t \\
&\quad \geq \left[r(b)w(b - \beta s(b)\ \right]F^{p+1}(b) + (p+1)\beta\int_0^b s(t)[F(t)]^p F^{\Delta}(t)\Delta t.
\end{aligned} \tag{2.25}$$

From this, we obtain

$$\begin{aligned}
&\int_0^b r(t)f^{p+1}(t)\Delta t + p\int_0^b \left[r(t)w^{\lambda}(t) - r^{\sigma}(t)w^{\sigma}(t)w^{1/p}(t)\right](F^{\sigma}(t))^{p+1}\Delta t \\
&\quad \geq (p+1)\beta\int_0^b s(t)F^p(t)F^{\Delta}(t)\Delta t.
\end{aligned} \tag{2.26}$$

This implies that

$$\begin{aligned}
&\int_0^b r(t)f^{p+1}(t)\Delta t + p\int_0^b \left[r(t)w^{\lambda}(t) - r^{\sigma}(t)w^{\sigma}(t)w^{1/p}(t)\right](F^{\sigma}(t))^{p+1}\Delta t \\
&\quad \geq (p+1)\beta_0\int_0^b s(t)F^p(t)F^{\Delta}(t)\Delta t,
\end{aligned} \tag{2.27}$$

which is the desired inequality (2.6) after replacing f by $x^{\Delta}(t)$ and F by $x(t)$. The proof is complete. □

Remark 2.2. Note that when $\mathbb{T} = \mathbb{R}$, we have $r(t) = r^{\sigma}(t)$ and $w(t) = w^{\sigma}(t)$, so that $r(t)w^{\lambda}(t) - r^{\sigma}(t)w^{\sigma}(t)w^{1/p}(t) = 0$ and then (2.6) becomes the inequality (1.10) that has been proved by Boyed and Wong [16]. Note also that the inequality (2.6) can be applied for different values of r and s, and this will left to the interested reader.

When $\mathbb{T} = \mathbb{N}$, then (2.6) reduces to the following discrete inequality.

Corollary 2.3. *Assume that* $\{r_i\}_{0\leq i\leq N}$ *and* $\{s_i\}_{0\leq i\leq N}$ *be nonnegative sequences such that the boundary value problem*

$$\begin{aligned}
\left(r(n)\left(\Delta u^{\Delta}(n)\right)^p\right) &= \beta\Delta s(n)u^p(n), \\
u(0) = 0, \qquad r(b)(\Delta u(b))^p &= \beta s(b)u^p(b),
\end{aligned} \tag{2.28}$$

has a solution $u(n)$ such that $\Delta u(n) \geq 0$ on the interval $[0, N]$ for some $\beta > 0$. If $\{x_i\}_{0\leq i\leq N}$ is a sequence of real numbers with $x(0) = 0$, then for $p > 0$,

$$\sum_{n=1}^{N-1} s(n)|x(n)|^p|\Delta x(n)| \leq \frac{1}{(p+1)\beta_0}\sum_{n=1}^{N-1} r(n)|\Delta x(n)|^{p+1} + \frac{p}{(p+1)\beta_0}\sum_{n=0}^{N-1}\left[r(n)w^{p+1/p}(n) - r(n+1)w(n+1)w^{1/p}(n)\right]|x(n+1)|^{p+1}. \tag{2.29}$$

Next, in the following we will prove some Opial-type inequalities on time scales which can be considered as the extension of the inequality (1.14) obtained by Beesack and Das [18].

Theorem 2.4. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and p, q positive real numbers such that $p + q > 1$, and let r, s be nonnegative rd-continuous functions on $(a, X)_{\mathbb{T}}$ such that $\int_a^X r^{-1/(p+q-1)}(t)\Delta t < \infty$. If $y : [a, X] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$ (and y^Δ does not change sign in (a, X)), then*

$$\int_a^X s(x)|y(x)|^p\left|y^\Delta(x)\right|^q \Delta x \leq K_1(a, X, p, q)\int_a^X r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{2.30}$$

where

$$K_1(a, X, p, q) = \left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_a^X (s(x))^{(p+q)/p}(r(x))^{-(q/p)}\left(\int_a^x r^{-1/(p+q-1)}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{q/(p+q)}. \tag{2.31}$$

Proof. Let

$$|y(x)| = \int_a^x \left|y^\Delta(t)\right|\Delta t = \int_a^x \frac{1}{(r(t))^{1/(p+q)}}(r(t))^{1/(p+q)}\left|y^\Delta(t)\right|\Delta t. \tag{2.32}$$

Now, since r is nonnegative on (a, X), it follows from the Hölder inequality (2.1) (assuming that the integrals exist) with

$$f(t) = \frac{1}{(r(t))^{1/(p+q)}}, \qquad g(t) = (r(t))^{1/(p+q)}\left|y^\Delta(t)\right|, \qquad \gamma = \frac{p+q}{p+q-1}, \qquad \nu = p+q, \tag{2.33}$$

that

$$\int_a^x \left|y^\Delta(t)\right|\Delta t \leq \left(\int_a^x \frac{1}{(r(t))^{1/(p+q-1)}}\Delta t\right)^{(p+q-1)/(p+q)}\left(\int_a^x r(t)\left|y^\Delta(t)\right|^{p+q}\Delta t\right)^{1/(p+q)}. \tag{2.34}$$

Then, for $a \le x \le X$, we get that

$$\left|y(x)\right|^p \le \left(\int_a^x \frac{1}{(r(t))^{1/(p+q-1)}}\Delta t\right)^{p(p+q-1/p+q)} \left(\int_a^x r(t)\left|y^\Delta(t)\right|^{p+q}\Delta t\right)^{p/(p+q)}. \tag{2.35}$$

Setting

$$z(x) := \int_a^x r(t)\left|y^\Delta(t)\right|^{p+q}\Delta t. \tag{2.36}$$

We see that $z(a) = 0$, and

$$z^\Delta(x) = r(x)\left|y^\Delta(x)\right|^{p+q} > 0. \tag{2.37}$$

This gives us

$$\left|y^\Delta(x)\right|^q = \left(\frac{z^\Delta(x)}{r(x)}\right)^{q/(p+q)}. \tag{2.38}$$

Thus, if s is a nonnegative on (a, X), we have from (2.35) and (2.38) that

$$\begin{aligned} s(x)\left|y(x)\right|^p\left|y^\Delta(x)\right|^q &\le s(x)\left(\frac{1}{r(x)}\right)^{q/(p+q)} \\ &\quad\times\left(\int_a^x \frac{1}{r^{1/(p+q-1)}(t)}\Delta t\right)^{p(p+q-1/p+q)}(z(x))^{p/(p+q)}\left(z^\Delta(x)\right)^{q/(p+q)}. \end{aligned} \tag{2.39}$$

This implies that

$$\begin{aligned} &\int_a^X s(x)\left|y(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x \\ &\quad\le \int_a^X s(x)\left(\frac{1}{r(x)}\right)^{q/(p+q)}\left(\int_a^x \frac{1}{r^{1/(p+q-1)}(t)}\Delta t\right)^{p(p+q-1/p+q)}(z(x))^{p/(p+q)}\left(z^\Delta(x)\right)^{q/(p+q)}\Delta x. \end{aligned} \tag{2.40}$$

Supposing that the integrals in (2.40) exist and again applying the Hölder inequality (2.1) with indices $p+q/p$ and $p+q/q$, we have

$$\begin{aligned}
&\int_a^X s(x)\left|y(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\\
&\quad\le\left(\int_a^X s^{(p+q)/p}(x)\left(\frac{1}{r(x)}\right)^{q/p}\left(\int_a^x\frac{1}{r^{1/(p+q-1)}(t)}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}\\
&\quad\quad\times\left(\int_a^X z^{p/q}(x)z^\Delta(x)\Delta x\right)^{q/(p+q)}.
\end{aligned}\tag{2.41}$$

From (2.37), the chain rule (2.4), and the fact that $z^\Delta(t)>0$, we obtain

$$z^{p/q}(x)z^\Delta(x)\le\frac{q}{p+q}\left(z^{(p+q)/q}(x)\right)^\Delta.\tag{2.42}$$

Substituting (2.42) into (2.41) and using the fact that $z(a)=0$, we have

$$\begin{aligned}
&\int_a^X s(x)\left|y(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\\
&\quad\le\left(\int_a^X s^{(p+q)/p}(x)\left(\frac{1}{r(x)}\right)^{q/p}\left(\int_a^x\frac{1}{r^{1/p+q-1}(t)}\Delta t\right)^{(p+q-1)}dx\right)^{p/(p+q)}\\
&\quad\quad\times\left(\frac{p}{p+q}\right)^{q/(p+q)}\left(\int_a^X\left(z^{(p+q)/q}(t)\right)^\Delta\Delta t\right)^{q/(p+q)}\\
&\quad=\left(\int_a^X s^{(p+q)/p}(x)\left(\frac{1}{r(x)}\right)^{q/p}\left(\int_a^x\frac{1}{r^{1/(p+q-1)}(t)}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}\\
&\quad\quad\times\left(\frac{q}{p+q}\right)^{q/(p+q)}z(X).
\end{aligned}\tag{2.43}$$

Using (2.36), we have from the last inequality that

$$\int_a^X s(x)\left|y(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\le K_1(a,b,p,q)\int_a^X r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x,\tag{2.44}$$

which is the desired inequality (2.30). The proof is complete. □

Here, we only state the following theorem, since its proof is the same as that of Theorem 2.4, with $[a,X]$ replaced by $[b,X]$.

Theorem 2.5. *Let $\mathbb{T}$ be a time scale with $a,b\in\mathbb{T}$ and p,q positive real numbers such that $p+q>1$, and let r,s be nonnegative rd-continuous functions on $(b,X)_{\mathbb{T}}$ such that $\int_X^b r^{-1/(p+q-1)}(t)\Delta t<\infty$.*

If $y : [X,b] \cap \mathbb{T} \to \mathbb{R}$ *is delta differentiable with* $y(b) = 0$, *(and* y^{Δ} *does not change sign in* (X,b)*), then one has*

$$\int_X^b s(x)|y(x)|^p \left|y^{\Delta}(x)\right|^q \Delta x \leq K_2(X,b,p,q) \int_X^b r(x) \left|y^{\Delta}(x)\right|^{p+q} \Delta x, \tag{2.45}$$

where

$$K_2(X,b,p,q) = \left(\frac{q}{p+q}\right)^{q/(p+q)} \left(\int_X^b (s(x))^{(p+q)/p} (r(x))^{-(q/p)} \left(\int_x^b r^{-1/(p+q-1)}(t)\Delta t\right)^{(p+q-1)} \Delta x\right)^{p/(p+q)}. \tag{2.46}$$

In the following, we assume that there exists $h \in (a,b)$ which is the unique solution of the equation

$$K(p,q) = K_1(a,h,p,q) = K_2(h,b,p,q) < \infty, \tag{2.47}$$

where $K_1(a,h,p,q)$ and $K_2(h,b,p,q)$ are defined as in Theorems 2.4 and 2.5. Note that since

$$\int_a^b s(x)|y(x)|^p \left|y^{\Delta}(x)\right|^q \Delta x = \int_a^X s(x)|y(x)|^p \left|y^{\Delta}(x)\right|^q \Delta x + \int_X^b s(x)|y(x)|^p \left|y^{\Delta}(x)\right|^q \Delta x, \tag{2.48}$$

then the proof of the following theorem will be a combination of Theorems 2.4 and 2.5 and due to the limited space we omit the details.

Theorem 2.6. *Let* $\mathbb{T}$ *be a time scale with* $a,b \in \mathbb{T}$ *and* p,q *positive real numbers such that* $pq > 0$ *and* $p+q > 1$, *and let* r,s *be nonnegative rd-continuous functions on* $(a,b)_{\mathbb{T}}$ *such that* $\int_a^b r^{-1/(p+q-1)}(t)\Delta t < \infty$. *If* $y : [a,b] \cap \mathbb{T} \to \mathbb{R}$ *is delta differentiable with* $y(a) = 0 = y(b)$, *(and* y^{Δ} *does not change sign in* (a,b), *then one has*

$$\int_a^b s(x)|y(x)|^p \left|y^{\Delta}(x)\right|^q \Delta x \leq K(p,q) \int_a^b r(x) \left|y^{\Delta}(x)\right|^{p+q} \Delta x. \tag{2.49}$$

For $r = s$ in (2.30), we obtain the following special case from Theorem 2.4, which improves the inequality (1.21) obtained by Wong et al. [23].

Corollary 2.7. *Let* $\mathbb{T}$ *be a time scale with* $a,b \in \mathbb{T}$ *and* p,q *positive real numbers such that* $p+q > 1$, *and let* r *be a nonnegative rd-continuous function on* $(a,X)_{\mathbb{T}}$ *such that* $\int_a^X r^{-1/(p+q-1)}(t)\Delta t < \infty$.

If $y : [a, X] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$, (and y^Δ does not change sign in (a, X)), then one has

$$\int_a^X r(x)\left|y(x)\right|^p \left|y^\Delta(x)\right|^q \Delta x \leq K_1^*(a, X, p, q) \int_a^X r(x)\left|y^\Delta(x)\right|^{p+q} \Delta x, \tag{2.50}$$

where

$$K_1^*(a, X, p, q) = \left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_a^X r(x)\left(\int_a^x r^{-1/(p+q-1)}(t)\Delta t\right)^{(p+q-1)} \Delta x\right)^{p/(p+q)}. \tag{2.51}$$

From Theorems 2.5 and 2.6 one can derive inequalities similar to the inequality in (2.50) by setting $r = s$. The details are left to the reader.

On a time scale $\mathbb{T}$, we note as a consequence from the chain rule (2.4) that

$$\begin{aligned} \left((t-a)^{p+q}\right)^\Delta &= (p+q)\int_0^1 [h(\sigma(t)-a) + (1-h)(t-a)]^{p+q-1} dh \\ &\geq (p+q)\int_0^1 [h(t-a) + (1-h)(t-a)]^{p+q-1} dh \\ &= (p+q)(t-a)^{p+q-1}. \end{aligned} \tag{2.52}$$

This implies that

$$\int_a^X (x-a)^{(p+q-1)} \Delta x \leq \int_a^X \frac{1}{(p+q)}\left((x-a)^{p+q}\right)^\Delta \Delta x = \frac{(X-a)^{p+q}}{(p+q)}. \tag{2.53}$$

From this and (2.51) with $r(t) = 1$, one gets that

$$\begin{aligned} K_1^*(a, X, p, q) &= \left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_a^X (x-a)^{(p+q-1)} \Delta x\right)^{p/(p+q)} \\ &\leq \left(\frac{q}{p+q}\right)^{q/(p+q)} \left(\frac{(X-a)^{p+q}}{(p+q)}\right)^{p/(p+q)} = \frac{q^{q/(p+q)}}{p+q}(X-a)^p. \end{aligned} \tag{2.54}$$

So setting $r = 1$ in (2.50) and using (2.54), one has the following inequality.

Corollary 2.8. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and p, q be positive real numbers such that $p+q > 1$. If $y : [a, X] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$, (and y^Δ does not change sign in (a, X)), then, one has*

$$\int_a^X \left|y(x)\right|^p \left|y^\Delta(x)\right|^q \Delta x \leq \frac{q^{q/(p+q)}}{p+q} \times (X-a)^p \int_a^X \left|y^\Delta(x)\right|^{p+q} \Delta x. \tag{2.55}$$

Remark 2.9. Note that when $\mathbb{T} = \mathbb{R}$, the inequality (2.55) becomes

$$\int_a^X |y(x)|^p |y'(x)|^q dx \le \frac{q^{q/(p+q)}}{p+q} \times (X-a)^p \int_a^X |y'(x)|^{p+q} dx, \tag{2.56}$$

which gives an improvement of the inequality (1.13).

When $\mathbb{T} = \mathbb{N}$, we have form (2.55) the following discrete Opial-type inequality.

Corollary 2.10. *Assume that p, q be positive real numbers such that $p + q > 1$ and $\{r_i\}_{0\le i\le N}$ a nonnegative real sequence. If $\{y_i\}_{0\le i\le N}$ is a sequence of real numbers with $y(0) = 0$, then*

$$\sum_{n=1}^{N-1} r(n)|y(n)|^p |\Delta y(n)|^q \le \frac{q^{q/(p+q)}}{p+q} \times (N-a)^p \sum_{n=0}^{N-1} r(n) |\Delta y(n)|^{p+q}. \tag{2.57}$$

The inequality (2.55) has immediate application to the case where $y(a) = y(b) = 0$. Choose $X = (a+b)/2$ and apply (2.51) to $[a, c]$ and $[c, b]$ and then add we obtain the following inequality.

Corollary 2.11. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and p, q positive real numbers such that $p+q > 1$. If $y : [a, X] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable with $y(a) = 0 = y(b)$, then one has*

$$\int_a^b |y(x)|^p \left|y^\Delta(x)\right|^q \Delta x \le \frac{q^{q/(p+q)}}{p+q} \times \left(\frac{b-a}{2}\right)^p \int_a^b \left|y^\Delta(x)\right|^{p+q} \Delta x. \tag{2.58}$$

From this inequality, we have the following discrete Opial-type inequality.

Corollary 2.12. *Assume that p, q be positive real numbers such that $p + q > 1$. If $\{y_i\}_{0\le i\le N}$ is a sequence of real numbers with $y(0) = 0 = y(N)$, then*

$$\sum_{n=1}^{N-1} r(n)|y(n)|^p |\Delta y(n)|^q \le \frac{q^{q/(p+q)}}{p+q} \times \left(\frac{N-a}{2}\right)^p \sum_{n=0}^{N-1} r(n) |\Delta y(n)|^{p+q}. \tag{2.59}$$

By setting $p = q = 1$ in (2.58), we have the following Opial type inequality on a time scale.

Corollary 2.13. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$. If $y : [a, X] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable with $y(a) = 0 = y(b)$, then one has*

$$\int_a^b |y(x)| \left|y^\Delta(x)\right| \Delta x \le \frac{(b-a)}{4} \int_a^b \left|y^\Delta(x)\right|^2 \Delta x. \tag{2.60}$$

When $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, we have from (2.60) the following inequalities.

Corollary 2.14. *If $y : [a, X] \cap \mathbb{R} \to \mathbb{R}$ is differentiable with $y(a) = 0 = y(b)$, then one has*

$$\int_a^b |y(x)| \left|y'(x)\right| dx \leq \frac{(b-a)}{4} \int_a^b \left|y'(x)\right|^2 dx. \tag{2.61}$$

Corollary 2.15. *If $\{y_i\}_{0\leq i\leq N}$ is a sequence of real numbers with $y(0) = 0 = y(N)$, then*

$$\sum_{n=1}^{N-1} |y(n)||\Delta y(n)| \leq \frac{N}{4} \sum_{n=0}^{N-1} |\Delta y(n)|^2. \tag{2.62}$$

Acknowledgments

This project was supported by King Saud University, Dean-ship of Scientific Research, College of Science Research Centre.

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