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Research Article

New-Type Solutions of the Modified Fischer-Kolmogorov Equation

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We prove the existence of new-type solutions of the modified Fischer-Kolmogorov equation with slow/fast diffusion and with possibly nonsmooth double-well potential. We show that a certain relation between the rate of the diffusion and the smoothness of the potential may originate new type solutions which do not occur in the classical Fischer-Kolmogorov equation. The main focus of this paper is to show the sensitivity of the mathematical modelling with respect to the chosen form of the diffusion term and the shape of the double-well potential.

1. Introduction

The Fischer-Kolmogorov (FK) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3 \tag{1.1}$$

was originally proposed in 1937 to model the interaction of dispersal and fitness in biological populations; see [1]. In connection with the modelling of an active pulse transmission line simulating a nerve axon, it is called the Newell-Whitehead equation or the Nagumo equation; see [2]. The stationary solutions of FK equation are solutions of the second-order equation

$$u'' + u - u^3 = 0. \tag{1.2}$$

This equation has a first integral, and we can define the energy functional

$$\mathcal{E}(u) \stackrel{\text{def}}{=} -\frac{1}{2}(u')^2 + \frac{1}{4}\left(1-u^2\right)^2, \tag{1.3}$$

which is constant along solutions of (1.2).

Since (1.2) is equivalent to the first-order system

$$\begin{aligned} u' &= v, \\ v' &= -u + u^3, \end{aligned} \tag{1.4}$$

solutions of (1.2) can be depicted as orbits of the system (1.4) in the (u, u')-plane; see [1, page 11]. Complete classification of solutions of (1.2) is given in [1, pages 10–12].

Let us focus on the solutions $u(x)$ of (1.2) with zero energy

$$\mathcal{E}(u(x)) = 0, \quad x \in \mathbb{R}. \tag{1.5}$$

It is easy to see that the only solutions of (1.2) satisfying (1.5) are

(1) *constant solutions:* $u(x) \equiv -1$ and $u(x) \equiv 1$,

(2) *kinks* (called also *domain walls*): heteroclinic orbits that connect the saddle points $(u, u') = (\pm 1, 0)$. These solutions describe transition layers between the two stable uniform states -1 and $+1$ and have explicit form

$$u(x) = \pm \tanh\left(\frac{x - x_0}{\sqrt{2}}\right), \tag{1.6}$$

where $x_0 \in \mathbb{R}$ is arbitrary but fixed number. We may assume, without loss of generality, that $x_0 = 0$ due to the fact that (1.2) is autonomous.

In contrast with some fourth-order problems, the dynamical system (1.4) associated with the second-order FK equation does not admit any homoclinic orbits, and so (1.2) does not possess any solutions which are called *pulses* (see [1, page 12]). In this paper, we want to show that this fact is a consequence of the combined effect of the linear diffusion term and the C^2-regularity of the double-well potential

$$W(u) \stackrel{\text{def}}{=} \frac{1}{4}\left(1-u^2\right)^2. \tag{1.7}$$

It may well happen in practice that the diffusion is slower than the linear one. Then besides kinks, also many *new types of nonconstant stationary solutions on zero energy* level may occur. These solutions might be of the shape similar to the kinks mentioned above, but the shape of solutions might be of pulse type as well. Moreover, one can also get periodic solutions or any random C^1-transition between the stable states -1 and $+1$; see the next section. This may happen even in the case when the double-well potential is a smooth,

C^∞-function. Similar phenomenon occurs also when the diffusion is expressed by the linear Laplace operator and the double-well potential lacks its C^2-regularity at the global minimizers.

On the other hand, if the slow diffusion is compensated by higher regularity of the double-well potential, then the kinks are the only nonconstant stationary solutions on zero energy level. In general, we show that a certain relation between the rate of the diffusion and the smoothness of the potential may originate new type solutions which do not occur in the classical Fischer-Kolmogorov equation or if such a relation does not hold true.

The *main focus of this paper* is to show *the sensitivity of the mathematical modelling* with respect to the chosen form of the diffusion term and the shape of the double-well potential as well.

2. Modified FK Equation

We consider modified FK equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + u\left(1-u^2\right)^{\alpha-1}, \tag{2.1}$$

where $\alpha > 1$ and $p > 1$ are real parameters. Obviously, for $\alpha = p = 2$, (2.1) reduces to FK equation. The one-dimensional p-Laplacian $(\partial/\partial x)(|\partial u/\partial x|^{p-2}(\partial u/\partial x))$ corresponds to the fast or slow diffusion if $p \in (1,2)$ or $p \in (2,\infty)$, respectively. The double-well potential

$$W_\alpha(u) \stackrel{\text{def}}{=} \frac{1}{2\alpha}\left(1-u^2\right)^{\alpha} \tag{2.2}$$

keeps the original shape of $W = W_2$, but it exhibits the lack of C^2-smoothness at the global minimizers ± 1 if $\alpha \in (1,2)$.

A counterpart of (1.2) is the quasilinear elliptic second-order equation

$$\left(|u'|^{p-2}u'\right)' + u\left(1-u^2\right)^{\alpha-1} = 0, \tag{2.3}$$

and the energy functional is given by

$$\mathcal{E}_{\alpha,p}(u) \stackrel{\text{def}}{=} -\frac{1}{p}|u'|^p + \frac{1}{2\alpha}\left(1-u^2\right)^{\alpha}. \tag{2.4}$$

It is clear that $u(x) \equiv \pm 1$ are constant solutions of (2.3) which correspond to the zero energy $\mathcal{E}_{\alpha,p}(u(x)) = 0$, $x \in \mathbb{R}$. We observe that for any solution corresponding to the zero energy we have $u'(x) = 0$ if and only if either $u(x) = -1$ or else $u(x) = +1$. It then follows that a solution corresponding to the zero energy is either monotone or constant (equal to either -1 or $+1$) on suitable subintervals of $\mathbb{R}$. In particular, we also observe that a bounded monotone solution of (2.3) corresponding to the zero energy must be increasing or decreasing transition between two stable uniform states -1 and $+1$.

Let $u(x)$ be such a transition. Since (2.3) is autonomous, we may assume that $u(0) = 0$. Without loss of generality, we may also assume that $u'(0) = (p/2\alpha)^{1/p}$ due to the uniqueness theorem for the initial value problem, see, for example, [3]. Then there is x_0, $0 < x_0 \leq \infty$ such that $1 > u(x) > 0$, $u'(x) > 0$, $x \in (0, x_0)$. Hence, there exists a positive inverse function $x(u)$ of $u(x)$ on $(0, x_0)$. We then introduce new variables $t = u$ and $z(t) = (u'(x(t)))^p$, compare with [1, page 53]. Upon the substitution into the first integral

$$\mathcal{E}_{\alpha,p}(u) = -\frac{1}{p}|u'|^p + \frac{1}{2\alpha}\left(1-u^2\right)^\alpha = 0, \tag{2.5}$$

we obtain

$$-\frac{1}{p}z(t) + \frac{1}{2\alpha}\left(1-t^2\right)^\alpha = 0, \tag{2.6}$$

that is,

$$z(t) = \frac{p}{2\alpha}\left(1-t^2\right)^\alpha. \tag{2.7}$$

Hence,

$$x(= x(u)) = \int_0^{u(x)} \frac{dt}{(z(t))^{1/p}} = \left(\frac{2\alpha}{p}\right)^{1/p} \int_0^{u(x)} \frac{dt}{(1-t^2)^{\alpha/p}}, \quad x \in [0, x_0). \tag{2.8}$$

In particular, for $x \nearrow x_0$, $u(x) \nearrow 1$, and so we get

$$x_0 = \left(\frac{2\alpha}{p}\right)^{1/p} \int_0^1 \frac{dt}{(1-t^2)^{\alpha/p}}. \tag{2.9}$$

It then follows from (2.9) that

(1) $x_0 = \infty$ if $\alpha \geq p$,

(2) $0 < x_0 < \infty$ if $\alpha < p$.

In the first case, we thus have strictly increasing solution $u(x)$ on $[0, \infty)$ such that $u(x) \nearrow 1$ as $x \nearrow \infty$. We extend u to $(-\infty, 0)$ as $u(x) = -u(-x)$, $x \in (-\infty, 0)$, and by the uniqueness theorem, we get a global solution of (2.3) which is defined on the entire $\mathbb{R}$. It is actually an *increasing odd kink* of the shape similar to that given by $u(x) = \tanh(x/\sqrt{2})$ (which occurs in the case $\alpha = p = 2$). Similarly, we obtain a *decreasing odd kink*, and all other bounded nonconstant solutions differ from these two kinks just modulo translations; see Figure 1.

We want to emphasize that the situation is completely different in the second case. A solution $u(x)$ of (2.3) on $[0, x_0)$ is given implicitly by (2.8). We have $u(0) = 0$, $u'(0) = (p/2\alpha)^{1/p}$, and we can define $u(x_0) = 1$, $u'(x_0) = 0$. We then extend $u(x)$ as follows: $u(x) = -u(-x)$, $x \in [-x_0, 0)$, $u(x) = -1$, $x \in (-\infty, -x_0)$, $u(x) = 1$, $x \in (x_0, \infty)$. We obtain a global solution of (2.3) which is of *kink type*; see Figure 2. It differs from the previous increasing odd kink by the property that the transition from -1 to $+1$ occurs on the finite interval $[-x_0, x_0]$.

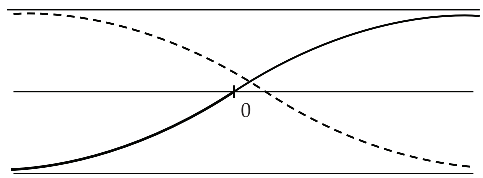

Figure 1

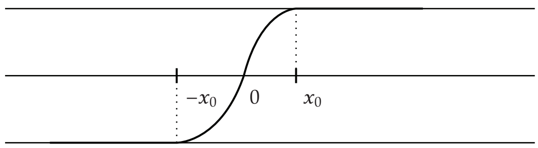

Figure 2

In the language of the dynamical system

$$u' = |v|^{p'-2}v,$$
$$v' = u\left(1-u^2\right)^{\alpha-1}, \tag{2.10}$$

$p' = p/(p-1)$, associated with (2.3), this solution is a heteroclinic orbit that connects the saddle points $(u,u') = (\pm 1, 0)$. In contrast with the case $\alpha \geq p$, this orbit connects the equilibria in a "finite time."

Notice that we have a plenty of other possibilities how to extend $u(x)$ given by (2.8) from $[0, x_0]$ to get a global solution of (2.3). In particular, we can extend $u(x)$ as follows: $u(x) = -u(-x)$, $x \in [-x_0, 0)$, $u(x) = u(2x_0 - x)$, $x \in (x_0, 3x_0]$, $u(x) = -1$, $x \in (-\infty, -x_0) \cup (3x_0, \infty)$; see Figure 3. We obtain a global solution of (2.3) which is of *pulse type*. It differs from the pulses defined in [1] by the property that the solution leaves one stable uniform state (-1), it touches the other one $(+1)$, and it returns back to the original stable state (-1). Both transitions occur on finite intervals of the same length. Such a solution is a homoclinic orbit of (2.10) which emanates from $(-1, 0)$, passes through $(+1, 0)$, and returns back to $(-1, 0)$ in a "finite time."

Similarly, we can construct infinitely many (even continua) of stationary solutions of (2.3) on zero energy level. Some of them might be *periodic*; the other ones are just *random transitions* between uniform steady states (-1) and $(+1)$; see Figures 4 and 5.

3. Discussions

We would like to point out that the same results hold true if the double-well potential W_α is replaced by more general nonnegative even function $F(u)$ with a positive local maximum at 0 and two global minima at ± 1, $F(\pm 1) = 0$, F strictly concave in $(-\xi, \xi)$, strictly convex in $(-\infty, -\xi) \cup (\xi, \infty)$ for some $\xi \in (0,1)$, $F \in C^2$ with possible exception at ± 1, where we might

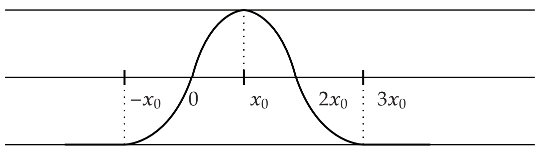

Figure 3

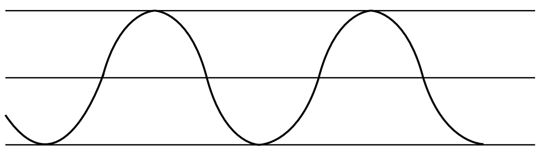

Figure 4

have $F \in C^1$ only. Then the analogues of (2.8) and (2.9) read as

$$x = \int_0^{u(x)} \frac{dt}{(F(t))^{1/p}}, \quad x \in [0, x_0), \qquad x_0 = \int_0^1 \frac{dt}{(F(t))^{1/p}}, \tag{3.1}$$

respectively, and we have either

(1) $x_0 = \int_0^1 (dt/(F(t))^{1/p}) = \infty$, or else,

(2) $0 < x_0 = \int_0^1 (dt/(F(t))^{1/p}) < \infty$.

The only bounded nonconstant solutions in the first case are kinks. The existence of new-type solutions in the second case is then related to the nonuniqueness of the solution of the initial value problem

$$\begin{gathered} \left(|u'|^{p-2}u'\right)' - F'(u) = 0, \\ u(x_0) = \pm 1, \qquad u'(x_0) = 0. \end{gathered} \tag{3.2}$$

It is clear from Section 2 that the existence of new-type solutions depends on the relation between the rate of the diffusion and the regularity of the double-well potential at its global minimizers. In particular, equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\left(1 - u^2\right)^{\alpha - 1} \tag{3.3}$$

with $1 < \alpha < 2$ and

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right) + u - u^3 \tag{3.4}$$

with $p > 2$ exhibit the existence of new-type stationary solutions.

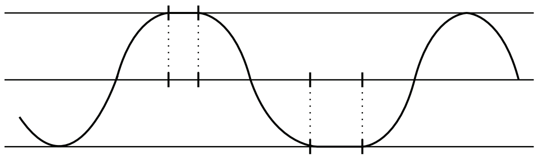

Figure 5

The generalization of these results to reaction-diffusion type equations in higher space dimensions is far from being obvious. For example, in the radial symmetric case, the counterpart of (2.3) is not an autonomous equation. Hence, we cannot use similar calculations with a first integral, and continua of new solutions cannot be constructed simply by graphing transitions between steady stable states as in one-dimensional case. Nevertheless, the author conjectures that even in higher space dimensions new-type solutions occur under suitable conditions on p and/or α. To prove this fact and to classify these solutions might be an interesting open problem.

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