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Research Article

Some Topological and Geometrical Properties of a New Difference Sequence Space

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We introduce the new difference sequence space $a_p^r(\Delta)$. Further, it is proved that the space $a_p^r(\Delta)$ is the BK-space including the space bv_p, which is the space of sequences of pbounded variation. We also show that the spaces $a_p^r(\Delta)$, and ℓ_p are linearly isomorphic for $1 \leq p < \infty$. Furthermore, the basis and the α-, β- and γ-duals of the space $a_p^r(\Delta)$ are determined. We devote the final section of the paper to examine some geometric properties of the space $a_p^r(\Delta)$.

1. Preliminaries, Background, and Notation

By ω, we will denote the space of all real valued sequences. Any vector subspace of ω is called as a *sequence space*. We will write ℓ_∞, c, and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also, by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely, and p-absolutely convergent series, respectively, where $1 < p < \infty$.

A sequence space λ with a linear topology is called a K-*space*, provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A K-space λ is called an *FK-space*, provided λ is a complete linear metric space. An FK-space whose topology is normable is called a *BK-space* (see [1, pages 272-273]).

Let λ, μ be two sequence spaces and $A = (a_{nk})$ an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$; if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$ we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

If a normed sequence space λ contains a sequence (b_n) with the property that for every $x \in \lambda$, there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0, \tag{1.2}$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$.

For a sequence space λ, the *matrix domain* λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}, \tag{1.3}$$

which is a sequence space. The new sequence space λ_A generated by the limitation matrix A from the space λ either includes the space λ or is included by the space λ, in general; that is, the space λ_A is the expansion or the contraction of the original space λ.

We will define $B_r = (b^r_{nk})$ by

$$b^r_{nk} = \begin{cases} \dfrac{1-r}{n+1} r^k, & 0 \le k \le n-1, \\ \dfrac{r^n+1}{n+1}, & k = n, \\ 0, & k > n, \end{cases} \qquad (0 < r < 1), \tag{1.4}$$

for all $n, k \in \mathbb{N}$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. We will also use the convention that any term with negative subscript is equal to naught.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been recently employed by Wang [2], Ng and Lee [3], Malkowsky [4], and Altay et al. [5]. They introduced the sequence spaces $(\ell_p)_{N_q}$ in [2], $(\ell_p)_{C_1} = X_p$ in [3], $(\ell_\infty)_{R^t} = r^t_\infty$, $c_{R^t} = r^t_c$ and $(c_0)_{R^t} = r^t_0$ in [4] and $(\ell_p)_{E^r} = e^r_p$ in [5]; where N_q, C_1, R^t and E^r denote the Nörlund, arithmetic, Riesz and Euler means, respectively, and $1 \le p \le \infty$.

Recently, there has been a lot of interest in investigating geometric properties of sequence spaces besides topological and some other usual properties. In literature, there are many papers concerning the geometric properties of different sequence spaces. For example, in [6], Mursaleen et al. studied some geometric properties of normed Euler sequence space. Şimşek and Karakaya [7] investigated the geometric properties of sequence space $\ell_\rho(u, v, p)$ equipped with Luxemburg norm. Further information on geometric properties of sequence space can be found in [8, 9].

The main purpose of the present paper is to introduce the difference sequence space $a^r_p(\Delta)$ together with matrix domain and is to derive some inclusion relations concerning with $a^r_p(\Delta)$. Also, we investigate some topological properties of this new space and furthermore characterize geometric properties concerning Banach-Saks type p.

2. $a_p^r(\Delta)$ Difference Sequence Space

In the present section, we introduce the difference sequence space $a_p^r(\Delta)$ and emphasize its some properties. Although the difference sequence space $\lambda(\Delta)$ corresponding to the space λ was defined by Kızmaz [10] as follows:

$$\lambda(\Delta) = \{x = (x_k) \in \omega : (x_k - x_{k+1}) \in \lambda\}, \tag{2.1}$$

the difference sequence space corresponding to the space ℓ_p was not examined, where λ denotes the anyone of the spaces c_0, c or ℓ_∞. So, Başar and Altay have recently studied the sequence space bv_p, the space of p-bounded variation, in [11] defined by

$$bv_p = \left\{x = (x_k) \in \omega : (x_k - x_{k-1}) \in \ell_p\right\}, \quad 1 \leq p < \infty, \tag{2.2}$$

which fills up the gap in the existing literature. Recently, Aydín and Başar [12] studied the sequence spaces a_0^r and a_c^r, defined by

$$\begin{aligned} a_0^r &= \left\{x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n}\left(1+r^k\right)x_k = 0\right\}, \\ a_c^r &= \left\{x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n}\left(1+r^k\right)x_k \text{ exists}\right\}. \end{aligned} \tag{2.3}$$

Aydín and Başar [13] introduced the difference sequence spaces $a_0^r(\Delta)$ and $a_c^r(\Delta)$, defined by

$$\begin{aligned} a_0^r(\Delta) &= \left\{x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n}\left(1+r^k\right)(x_k - x_{k-1}) = 0\right\}, \\ a_c^r(\Delta) &= \left\{x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n}\left(1+r^k\right)(x_k - x_{k-1}) \text{ exists}\right\}. \end{aligned} \tag{2.4}$$

Aydín [14] introduced a_p^r sequence space, defined by

$$a_p^r = \left\{x = (x_k) \in \omega : \sum_n \left|\frac{1}{n+1}\sum_{k=0}^{n}\left(1+r^k\right)x_k\right|^p < \infty\right\}; \quad 1 \leq p < \infty. \tag{2.5}$$

Define the matrix $\Delta = (\delta_{nk})$ by

$$\delta_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \leq k \leq n, \\ 0, & 0 \leq k < n-1 \quad \text{or} \quad k > n. \end{cases} \tag{2.6}$$

As was made by Başar and Altay in [11], we treat slightly more different than Kızmaz and the other authors following him and employ the technique obtaining a new sequence space by the matrix domain of a triangle limitation method. We will introduce the sequence space $a_p^r(\Delta)$ which is a natural continuation of Aydín and Başar [13], as follows:

$$a_p^r(\Delta) = \left\{ x = (x_k) \in \omega : \sum_n \left| \frac{1}{n+1} \sum_{k=0}^{n} \left(1 + r^k\right)(x_k - x_{k-1}) \right|^p < \infty \right\}; \quad 1 \leq p < \infty. \tag{2.7}$$

With the notation of (1.3), we may redefine the space $a_p^r(\Delta)$ by

$$a_p^r(\Delta) = \left(a_p^r\right)_\Delta. \tag{2.8}$$

Define the sequence $y = \{y_n(r)\}$ which will be frequently used as the B_r-transform of a sequence $x = (x_k)$, that is,

$$y_n(r) = \sum_{k=0}^{n-1} \frac{(1-r)r^k}{1+n} x_k + \frac{1+r^n}{1+n} x_n, \quad n \in \mathbb{N}. \tag{2.9}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *The set $a_p^r(\Delta)$ becomes the linear space with the coordinatewise addition and scalar multiplication which is the BK-space with the norm*

$$\|x\|_{a_p^r(\Delta)} = \|y\|_{\ell_p}, \tag{2.10}$$

where $1 \leq p < \infty$.

Proof. Since the proof is routine, we omit the details of the proof. □

Theorem 2.2. *The space $a_p^r(\Delta)$ is linearly isomorphic to the space ℓ_p; that is, $a_p^r(\Delta) \cong \ell_p$, where $1 \leq p < \infty$.*

Proof. It is enough to show the existence of a linear bijection between the spaces $a_p^r(\Delta)$ and ℓ_p for $1 \leq p < \infty$. Consider the transformation T defined, with the notation of (2.9), from $a_p^r(\Delta)$ to ℓ_p by $x \mapsto y = Tx$. The linearity of T is clear. Furthermore, it is trivial that $x = \theta$ whenever $Tx = \theta$, and hence, T is injective.

We assume that $y \in \ell_p$ for $1 \leq p < \infty$ and define the sequence $x = (x_k)$ by

$$x_n(r) = \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1+j}{1+r^k} y_j; \quad k \in \mathbb{N}. \tag{2.11}$$

Then, since

$$(\Delta x)_n = \sum_{j=n-1}^{n} (-1)^{n-j} \frac{1+j}{1+r^n} y_j; \quad n \in \mathbb{N}, \tag{2.12}$$

we get that

$$\sum_n \left| \frac{1}{1+n} \sum_{k=0}^{n} \left(1+r^k\right) \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1+j}{1+r^k} y_j \right|^p = \sum_n |y_n|^p < \infty. \tag{2.13}$$

Thus, we have that $x \in a_p^r(\Delta)$. In addition, one can derive that

$$\begin{aligned} \|x\|_{a_p^r(\Delta)} &= \left(\sum_n \left| \frac{1}{1+n} \sum_{k=0}^{n} \left(1+r^k\right)(x_k - x_{k-1}) \right|^p \right)^{1/p} \\ &= \left(\sum_n \left| \frac{1}{1+n} \sum_{k=0}^{n} \left(1+r^k\right) \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1+j}{1+r^k} y_j \right|^p \right)^{1/p} \\ &= \|y\|_{\ell_p} \end{aligned} \tag{2.14}$$

which means that T is surjective and is norm preserving. Hence, T is a linear bijection. □

We wish to exhibit some inclusion relations concerning with the space $a_p^r(\Delta)$.

Theorem 2.3. *The inclusion $bv_p \subset a_p^r(\Delta)$ strictly holds for $1 < p < \infty$.*

Proof. To prove the validity of the inclusion $bv_p \subset a_p^r(\Delta)$ for $1 < p < \infty$, it suffices to show the existence of a number $K > 0$ such that $\|x\|_{a_p^r(\Delta)} \leq K\|x\|_{bv_p}$ for every $x \in bv_p$.

Let $x \in bv_p$ and $1 < p < \infty$. Then, we obtain

$$\begin{aligned} \sum_n \left| \frac{1}{1+n} \sum_{k=0}^{n} \left(1+r^k\right)(x_k - x_{k-1}) \right|^p &\leq \sum_n \left[2 \sum_{k=0}^{n} \frac{|x_k - x_{k-1}|}{1+n} \right]^p \\ &< 2^p \left(\frac{p}{p-1} \right)^p \sum_n |x_n - x_{n-1}|^p, \end{aligned} \tag{2.15}$$

as expected,

$$\|x\|_{a_p^r(\Delta)} \leq \left(\frac{2p}{p-1} \right) \|x\|_{bv_p}, \tag{2.16}$$

for $1 < p < \infty$.

Furthermore, let us consider the sequence $x = \{x_n(r)\}$ defined by

$$x_n(r) = \sum_{k=0}^{n} \frac{(-1)^n}{1+r^k}, \quad n \in \mathbb{N}. \tag{2.17}$$

Then, the sequence x is in $a_p^r(\Delta) - bv_p$, as asserted. □

Lemma 2.4 (see [11, Theorem 2.4]). *The inclusion $\ell_p \subset bv_p$ strictly holds for $1 < p < \infty$.*

Combining Lemma 2.4 and Theorem 2.3, we get the following corollary.

Corollary 2.5. *The inclusion $\ell_p \subset a_p^r(\Delta)$ strictly holds for $1 < p < \infty$.*

3. The Basis for the Space $a_p^r(\Delta)$

In the present section, we will give a sequence of the points of the space $a_p^r(\Delta)$ which forms a basis for the space $a_p^r(\Delta)$, where $1 \leq p < \infty$.

Theorem 3.1. *Define the matrix $B_r = \{b_n^{(k)}(r)\}_{n\in\mathbb{N}}$ of elements of the space $a_p^r(\Delta)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(r) = \begin{cases} \dfrac{1+k}{1+r^k} - \dfrac{1+k}{1+r^{k+1}}, & 0 \leq k \leq n-1, \\ \dfrac{1+n}{1+r^n}, & k = n, \\ 0, & k > n, \end{cases} \tag{3.1}$$

for every fixed $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)}(r)\}_{k\in\mathbb{N}}$ is a basis for the space $a_p^r(\Delta)$, and any $x \in a_p^r(\Delta)$ has a unique representation of the form

$$x = \sum_k \lambda_k(r) b^{(k)}(r), \tag{3.2}$$

where $\lambda_k(r) = (B_r x)_k$ for all $k \in \mathbb{N}$ and $1 \leq p < \infty$.

Proof. It is clear that $\{b^{(k)}(r)\} \subset a_p^r(\Delta)$, since

$$B_r b^{(k)}(r) = e^{(k)} \in \ell_p, \quad k = 0, 1, 2, \ldots, \tag{3.3}$$

for $1 \leq p < \infty$; here, $e^{(k)}$ is the sequence whose only nonzero term is 1 in the kth place for each $k \in \mathbb{N}$. Let $x \in a_p^r(\Delta)$ be given. For every nonnegative integer m, we set

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) b^{(k)}(r). \tag{3.4}$$

Then, by applying B_r to (3.4), we obtain with (3.3) that

$$B_r x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) B_r b^{(k)}(r) = \sum_{k=0}^{m} (B_r x)_k e^{(k)},$$
$$\left\{B_r\left(x - x^{[m]}\right)\right\}_i = \begin{cases} 0, & 0 \le i \le m, \\ (B_r x)_i, & i > m, \end{cases} \tag{3.5}$$

where $i, m \in \mathbb{N}$. For a given $\varepsilon > 0$, there is an integer m_0 such that

$$\left[\sum_{i=m}^{\infty} |(B_r x)_i|^p\right]^{1/p} < \frac{\varepsilon}{2}, \tag{3.6}$$

for all $m \ge m_0$. Hence,

$$\left\|x - x^{[m]}\right\|_{a_p^r(\Delta)} = \left[\sum_{i=m}^{\infty} |(B_r x)_i|^p\right]^{1/p} \le \left[\sum_{i=m_0}^{\infty} |(B_r x)_i|^p\right]^{1/p} < \frac{\varepsilon}{2} < \varepsilon, \tag{3.7}$$

for all $m \ge m_0$, which proves that $x \in a_p^r(\Delta)$ is represented as in (3.2).

Let us show the uniqueness of representation for $x \in a_p^r(\Delta)$ given by (3.2). Assume, on the contrary, that there exists a representation $x = \sum_k \mu_k(r) b^{(k)}(r)$. Since the linear transformation T, from $a_p^r(\Delta)$ to ℓ_p, used in Theorem 2.2 is continuous, at this stage, we have

$$(B_r x)_n = \sum_k \mu_k(r) \left\{B_r b^{(k)}(r)\right\}_n = \sum_k \mu_k(r) e_n^{(k)} = \mu_n(r); \quad n \in \mathbb{N}, \tag{3.8}$$

which contradicts the fact that $(B_r x)_n = \lambda_n(r)$ for all $n \in \mathbb{N}$. Hence, the representation (3.2) of $x \in a_p^r(\Delta)$ is unique. This step concludes the proof. □

4. The α-, β-, and γ-Duals of the Space $a_p^r(\Delta)$

In this section, we state and prove theorems determining the α-, β-, and γ-duals of the space $a_p^r(\Delta)$. Since the case $p = 1$ can be proved by the same analogy and can be found in the literature, we omit the proof of that case and consider only the case $1 < p < \infty$ in the proof of Theorems 4.4–4.6.

For the sequence spaces λ and μ, define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \left\{z = (z_k) \in \omega : xz = (x_k z_k) \in \mu,\ \forall x \in \lambda\right\}. \tag{4.1}$$

With the notation of (4.1), α-, β- and γ-duals of a sequence space λ, which are, respectively, denoted by $\lambda^{\alpha}, \lambda^{\beta}$ and λ^{γ}, are defined by

$$\lambda^{\alpha} = S(\lambda, \ell_1), \qquad \lambda^{\beta} = S(\lambda, cs), \qquad \lambda^{\gamma} = S(\lambda, bs). \tag{4.2}$$

It is well-known for the sequence spaces λ and μ that $\lambda^{\alpha} \subseteq \lambda^{\beta} \subseteq \lambda^{\gamma}$ and $\lambda^{\eta} \supset \mu^{\eta}$ whenever $\lambda \subset \mu$, where $\eta \in \{\alpha, \beta, \gamma\}$.

We begin with to quoting the lemmas due to Stieglitz and Tietz [15], which are needed in the proof of the following theorems.

Lemma 4.1. *$A \in (\ell_p : \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_{nk} \right|^q < \infty \quad (1 < p \leq \infty). \tag{4.3}$$

Lemma 4.2. *$A \in (\ell_p : c)$ if and only if*

$$\lim_{n \longrightarrow \infty} a_{nk} \quad \textit{exists for each } k \in \mathbb{N}, \tag{4.4}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}|^q < \infty \quad 1 < p < \infty. \tag{4.5}$$

Lemma 4.3. *$A \in (\ell_p : \ell_\infty)$ if and only if* (4.5) *holds.*

Theorem 4.4. *Define the set a_q^r by*

$$a_q^r = \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} c_{nk}^r \right|^q < \infty \right\}, \tag{4.6}$$

where $C^r = (c_{nk}^r)$ is defined via the sequence $a = (a_n)$ by

$$c_{nk}^r = \begin{cases} \left(\dfrac{1+k}{1+r^k} - \dfrac{1+k}{1+r^{k+1}} \right) a_n, & 0 \leq k \leq n-1, \\ \dfrac{1+n}{1+r^n} a_n, & k = n, \\ 0, & k > n. \end{cases} \tag{4.7}$$

for all $n, k \in \mathbb{N}$. Then, $\{a_p^r(\Delta)\}^{\alpha} = a_q^r$, where $1 < p < \infty$.

Proof. Bearing in mind the relation (2.9), we immediately derive that

$$a_n x_n = \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1+j}{1+r^k} a_n y_j = (C^r y)_n, \quad n \in \mathbb{N}. \tag{4.8}$$

It follows from (4.8) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in a_p^r(\Delta)$ if and only if $C^r y \in \ell_1$ whenever $y \in \ell_p$. This means that $a = (a_n) \in \{a_p^r(\Delta)\}^\alpha$ if and only if $C^r \in (\ell_p : \ell_1)$. Then, we derive by Lemma 4.1 with C^r instead of A that

$$\sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} c_{nk}^r \right|^q < \infty. \tag{4.9}$$

This yields the desired consequence that $\{a_p^r(\Delta)\}^\alpha = a_q^r$. □

Theorem 4.5. *Define the sets a_1^r, a_2^r, and a_3^r by*

$$\begin{aligned}
a_1^r &= \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_k \left| e_{nk}^r \right|^q < \infty \right\}, \\
a_2^r &= \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} a_j \text{ exists for each fixed } k \in \mathbb{N} \right\}, \\
a_3^r &= \left\{ a = (a_k) \in \omega : \left\{ \frac{1+n}{1+r^n} a_n \right\} \in cs \right\},
\end{aligned} \tag{4.10}$$

where $E = (e_{nk}^r)$ is defined by

$$e_{nk}^r = \begin{cases} (k+1)\left[\dfrac{a_k}{1+r^k} + \left(\dfrac{1}{1+r^k} - \dfrac{1}{1+r^{k+1}} \right) \displaystyle\sum_{j=k+1}^{n} a_j \right], & 0 \le k \le n-1, \\ \dfrac{1+n}{1+r^n} a_n, & k = n, \\ 0, & k > n, \end{cases} \tag{4.11}$$

for all $n, k \in \mathbb{N}$. Then, $\{a_p^r(\Delta)\}^\beta = a_1^r \cap a_2^r \cap a_3^r$, where $1 < p < \infty$.

Proof. Consider the equation

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \left[\sum_{i=j-1}^{j} (-1)^{i-j} \frac{1+i}{1+r^j} y_j \right] \right\} a_k \\
&= \sum_{k=0}^{n-1} (k+1) \left[\frac{a_k}{1+r^k} + \left(\frac{1}{1+r^k} - \frac{1}{1+r^{k+1}} \right) \sum_{j=k+1}^{n} a_j \right] y_k \\
&\quad + \frac{1+n}{1+r^n} a_n y_n = (Ey)_n \quad n \in \mathbb{N}.
\end{aligned} \tag{4.12}$$

Thus, we deduce from Lemma 4.2 with (4.12) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in a_p^r(\Delta)$ if and only if $Ey \in c$ whenever $y = (y_k) \in \ell_p$. That is to say that $a = (a_k) \in \{a_p^r(\Delta)\}^\beta$ if and only if $E \in (\ell_p : c)$. Therefore, we derive from (4.4) and (4.5) that $\{a_p^r(\Delta)\}^\beta = a_1^r \cap a_2^r \cap a_3^r$. □

Theorem 4.6. *$\{a_p^r(\Delta)\}^\gamma = a_1^r$, where $1 < p < \infty$.*

Proof. It is natural that the present theorem may be proved by the same technique used in the proof of Theorems 4.4 and 4.5, above. But, we prefer here the following classical way.

Let $a = (a_k) \in a_1^r$ and $x = (x_k) \in a_p^r(\Delta)$. Then, we obtain by applying Hölder's inequality that

$$\begin{aligned} \left|\sum_{k=0}^{n} a_k x_k\right| &= \left|\sum_{k=0}^{n}\left\{\sum_{j=0}^{k}\left[\sum_{i=j-1}^{j}(-1)^{i-j}\frac{1+i}{1+r^j}y_j\right]\right\}a_k\right| = \left|\sum_{k=0}^{n} e_{nk}^r y_k\right| \\ &\leq \left(\sum_{k=0}^{n}\left|e_{nk}^r\right|^q\right)^{1/q}\left(\sum_{k=0}^{n}\left|y_k\right|^p\right)^{1/p}, \end{aligned} \tag{4.13}$$

which gives us by taking supremum over $n \in \mathbb{N}$ that

$$\begin{aligned} \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n} a_k x_k\right| &\leq \sup_{n\in\mathbb{N}}\left[\left(\sum_{k=0}^{n}\left|e_{nk}^r\right|^q\right)^{1/q}\left(\sum_{k=0}^{n}\left|y_k\right|^p\right)^{1/p}\right] \\ &\leq \|y\|_{\ell_p}\cdot\left(\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|e_{nk}^r\right|^q\right)^{1/q} < \infty. \end{aligned} \tag{4.14}$$

This means that $a = (a_k) \in \{a_p^r(\Delta)\}^\gamma$. Hence,

$$a_1^r \subset \left\{a_p^r(\Delta)\right\}^\gamma. \tag{4.15}$$

Conversely, let $a = (a_k) \in \{a_p^r(\Delta)\}^\gamma$ and $x = (x_k) \in a_p^r(\Delta)$. Then, one can easily see that $(\sum_{k=0}^{n} e_{nk}^r y_k)_{n\in\mathbb{N}} \in \ell_\infty$ whenever $(a_k x_k) \in bs$. This shows that the triangle matrix $E = (e_{nk}^r)$, defined by (4.11), is in the class $(\ell_p : \ell_\infty)$. Hence, the condition (4.5) holds with e_{nk}^r instead of a_{nk} which yields that $a = (a_k) \in a_1^r$. That is to say that

$$\left\{a_p^r(\Delta)\right\}^\gamma \subset a_1^r. \tag{4.16}$$

Therefore, by combining the inclusions (4.15) and (4.16), we deduce that the γ-dual of the space $a_p^r(\Delta)$ is the set a_1^r, and this step completes the proof. □

5. Some Geometric Properties of the Space $a_p^r(\Delta)$

In this section, we study some geometric properties of the space $a_p^r(\Delta)$.

A Banach space X is said to have the *Banach-Saks* property if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $\{t_k(z)\}$ is convergent in the norm in X [16], where

$$t_k(z) = \frac{1}{k+1}(z_0 + z_1 + \cdots + z_k) \quad (k \in \mathbb{N}). \tag{5.1}$$

A Banach space X is said to have the *weak Banach-Saks property* whenever given any weakly null sequence $(x_n) \subset X$ and there exists a subsequence (z_n) of (x_n) such that the sequence $\{t_k(z)\}$ strongly convergent to zero.

In [17], García-Falset introduce the following coefficient:

$$R(X) = \sup\left\{\liminf_{n\longrightarrow\infty} \|x_n - x\| : (x_n) \subset B(X), \quad x_n \xrightarrow{\omega} 0, \; x \in B(X)\right\}, \tag{5.2}$$

where $B(X)$ denotes the unit ball of X.

Remark 5.1. A Banach space X with $R(X) < 2$ has the weak fixed point property, [18].

Let $1 < p < \infty$. A Banach space is said to have the *Banach-Saks type p* or property $(\mathrm{BS})_p$, if every weakly null sequence (x_k) has a subsequence (x_{k_l}) such that for some $C > 0$,

$$\left\|\sum_{l=0}^{n} x_{k_l}\right\| < C(n+1)^{1/p}, \tag{5.3}$$

for all $n \in \mathbb{N}$ (see [19]).

Now, we may give the following results related to the some geometric properties, mentioned above, of the space $a_p^r(\Delta)$.

Theorem 5.2. *The space $a_p^r(\Delta)$ has the Banach-Saks type p.*

Proof. Let (ε_n) be a sequence of positive numbers for which $\sum \varepsilon_n \leq 1/2$, and also let (x_n) be a weakly null sequence in $B(a_p^r(\Delta))$. Set $b_0 = x_0 = 0$ and $b_1 = x_{n_1} = x_1$. Then, there exists $m_1 \in \mathbb{N}$ such that

$$\left\|\sum_{i=m_1+1}^{\infty} b_1(i) e^{(i)}\right\|_{a_p^r(\Delta)} < \varepsilon_1. \tag{5.4}$$

Since (x_n) is a weakly null sequence implies $x_n \rightarrow 0$ coordinatewise, there is an $n_2 \in \mathbb{N}$ such that

$$\left\|\sum_{i=0}^{m_1} x_n(i) e^{(i)}\right\|_{a_p^r(\Delta)} < \varepsilon_1, \tag{5.5}$$

where $n \geq n_2$. Set $b_2 = x_{n_2}$. Then, there exists an $m_2 > m_1$ such that

$$\left\| \sum_{i=m_2+1}^{\infty} b_2(i) e^{(i)} \right\|_{a_p^r(\Delta)} < \varepsilon_2. \tag{5.6}$$

By using the fact that $x_n \to 0$ coordinatewise, there exists an $n_3 > n_2$ such that

$$\left\| \sum_{i=0}^{m_2} x_n(i) e^{(i)} \right\|_{a_p^r(\Delta)} < \varepsilon_2, \tag{5.7}$$

where $n \geq n_3$.

If we continue this process, we can find two increasing subsequences (m_i) and (n_i) such that

$$\left\| \sum_{i=0}^{m_j} x_n(i) e^{(i)} \right\|_{a_p^r(\Delta)} < \varepsilon_j, \tag{5.8}$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=m_j+1}^{\infty} b_j(i) e^{(i)} \right\|_{a_p^r(\Delta)} < \varepsilon_j, \tag{5.9}$$

where $b_j = x_{n_j}$. Hence,

$$\begin{aligned}
\left\| \sum_{j=0}^{n} b_j \right\|_{a_p^r(\Delta)} &= \left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{m_{j-1}} b_j(i) e^{(i)} + \sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} + \sum_{i=m_j+1}^{\infty} b_j(i) e^{(i)} \right) \right\|_{a_p^r(\Delta)} \\
&\leq \left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{m_{j-1}} b_j(i) e^{(i)} \right) \right\|_{a_p^r(\Delta)} + \left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{a_p^r(\Delta)} \\
&\quad + \left\| \sum_{j=0}^{n} \left(\sum_{i=m_j+1}^{\infty} b_j(i) e^{(i)} \right) \right\|_{a_p^r(\Delta)} \\
&\leq \left\| \sum_{j=0}^{n} \left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{a_p^r(\Delta)} + 2 \sum_{j=0}^{n} \varepsilon_j.
\end{aligned} \tag{5.10}$$

On the other hand, it can be seen that $\|x_n\|_{a_p^r(\Delta)} < 1$. Therefore, $\|x_n\|_{a_p^r(\Delta)}^p < 1$. We have

$$\begin{aligned}\left\|\sum_{j=0}^{n}\left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i)e^{(i)}\right)\right\|_{a_p^r(\Delta)}^p &= \sum_{j=0}^{n}\sum_{i=m_{j-1}+1}^{m_j}\left|\sum_{k=0}^{i-1}\frac{(1-r)r^k}{1+i}x_j(k)+\frac{1+r^i}{1+i}x_i(k)\right|^p \\ &\leq \sum_{j=0}^{n}\sum_{i=0}^{\infty}\left|\sum_{k=0}^{i-1}\frac{(1-r)r^k}{1+i}x_j(k)+\frac{1+r^i}{1+i}x_i(k)\right|^p \\ &\leq (n+1).\end{aligned} \tag{5.11}$$

Hence, we obtain

$$\left\|\sum_{j=0}^{n}\left(\sum_{i=m_{j-1}+1}^{m_j} b_j(i)e^{(i)}\right)\right\|_{a_p^r(\Delta)} \leq (n+1)^{1/p}. \tag{5.12}$$

By using the fact $1 \leq (n+1)^{1/p}$ for all $n \in \mathbb{N}$, we have

$$\left\|\sum_{j=0}^{n} b_j\right\|_{a_p^r(\Delta)} \leq (n+1)^{1/p} + 1 \leq 2(n+1)^{1/p}. \tag{5.13}$$

Hence, $a_p^r(\Delta)$ has the Banach-Saks type p. This completes the proof of the theorem. □

Remark 5.3. Note that $R(a_p^r(\Delta)) = R(\ell_p) = 2^{1/p}$, since $a_p^r(\Delta)$ is linearly isomorphic to ℓ_p.

Hence, by the Remarks 5.1 and 5.3, we have the following.

Theorem 5.4. *The space $a_p^r(\Delta)$ has the weak fixed point property, where $1 < p < \infty$.*

References

[1] B. Choudhary and S. Nanda, *Functional Analysis with Applications*, John Wiley & Sons, New Delhi, India, 1989.

[2] C. S. Wang, "On Nörlund sequence spaces," *Tamkang Journal of Mathematics*, vol. 9, no. 2, pp. 269–274, 1978.

[3] P. N. Ng and P. Y. Lee, "Cesàro sequence spaces of non-absolute type," *Commentationes Mathematicae. Prace Matematyczne*, vol. 20, no. 2, pp. 429–433, 1977/78.

[4] E. Malkowsky, "Recent results in the theory of matrix transformations in sequence spaces," *Matematichki Vesnik*, vol. 49, no. 3-4, pp. 187–196, 1997.

[5] B. Altay, F. Başar, and M. Mursaleen, "On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I," *Information Sciences*, vol. 176, no. 10, pp. 1450–1462, 2006.

[6] M. Mursaleen, F. Başar, and B. Altay, "On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ II," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 65, no. 3, pp. 707–717, 2006.

[7] N. Şimşek and V. Karakaya, "On some geometrical properties of generalized modular spaces of Cesáro type defined by weighted means," *Journal of Inequalities and Applications*, vol. 2009, Article ID 932734, 13 pages, 2009.

[8] M. Mursaleen, R. Çolak, and M. Et, "Some geometric inequalities in a new Banach sequence space," *Journal of Inequalities and Applications*, vol. 2007, Article ID 86757, 6 pages, 2007.

[9] Y. Cui, C. Meng, and R. Płuciennik, "Banach-Saks property and property (β) in Cesàro sequence spaces," *Southeast Asian Bulletin of Mathematics*, vol. 24, no. 2, pp. 201–210, 2000.

[10] H. Kızmaz, "On certain sequence spaces," *Canadian Mathematical Bulletin*, vol. 24, no. 2, pp. 169–176, 1981.

[11] F. Başar and B. Altay, "On the space of sequences of p−bounded variation and related matrix mappings," *Ukrainian Mathematical Journal*, vol. 55, no. 1, pp. 108–118, 2003.

[12] C. AydÍn and F. Başar, "On the new sequence spaces which include the spaces c_0 and c," *Hokkaido Mathematical Journal*, vol. 33, no. 2, pp. 383–398, 2004.

[13] C. Aydın and F. Başar, "Some new difference sequence spaces," *Applied Mathematics and Computation*, vol. 157, no. 3, pp. 677–693, 2004.

[14] C. Aydın, *Isomorphic sequence spaces and infinite matrices*, Ph.D. thesis, Inönü Üniversitesi Fen Bilimleri Enstitüsü, Malatya, Turkey, 2002.

[15] M. Stieglitz and H. Tietz, "Matrix transformationen von folgenräumen eine ergebnisübersicht," *Mathematische Zeitschrift*, vol. 154, no. 1, pp. 1–16, 1977.

[16] J. Diestel, *Sequences and Series in Banach Spaces*, vol. 92 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1984.

[17] J. García-Falset, "Stability and fixed points for nonexpansive mappings," *Houston Journal of Mathematics*, vol. 20, no. 3, pp. 495–506, 1994.

[18] J. García-Falset, "The fixed point property in Banach spaces with the NUS-property," *Journal of Mathematical Analysis and Applications*, vol. 215, no. 2, pp. 532–542, 1997.

[19] H. Knaust, "Orlicz sequence spaces of Banach-Saks type," *Archiv der Mathematik*, vol. 59, no. 6, pp. 562–565, 1992.