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Research Article

On Carlson's Type Removability Test for the Degenerate Quasilinear Elliptic Equations

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Carlson's type theorem on removable sets for α-Holder continuous solutions is investigated for the quasilinear elliptic equations div $A(x,u,\nabla u)=0$, having degeneration ω in the Muckenhoupt class. In partial, when α is sufficiently small and the operator is weighted p-Laplacian, we show that the compact set E is removable if and only if the Hausdorff measure $\Lambda_\omega^{-p+(p-1)\alpha}(E)=0$.

1. Introduction

In this paper, we will consider questions of a removable singularity for the class of quasilinear elliptic equations of the form

$$\operatorname{div}(A(x,u,\nabla u))=0, \tag{1.1}$$

where $A(x,\xi,\zeta)=\{A_1(x,\xi,\zeta),A_2(x,\xi,\zeta),\dots,A_n(x,\xi,\zeta)\}:D\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous with respect to ξ, continuously differentiable with respect to ζ functions. For $\xi\in\mathbb{R}^1$, $\zeta\in\mathbb{R}^n$, it is assumed that $A_j(x,\xi,\zeta)$; $j=1,2,\dots n$ are measurable functions with respect to a variable x in the open domain $D\subset\mathbb{R}^n$. Let the following growth conditions be satisfied:

$$\begin{gathered}\sum_{i,k=1}^{n}\frac{\partial A_i}{\partial\zeta_k}(x,\xi,\zeta)\eta_i\eta_k\ge\lambda\omega(x)|\zeta|^{p-2}|\eta|^2,\\ \left|\frac{\partial A_i}{\partial\zeta_k}(x,\xi,\zeta)\right|\le\lambda^{-1}\omega(x)|\zeta|^{p-2},\quad k,i=1,2,\dots,n,\end{gathered}\tag{1.2}$$

where $1 < p < \infty$, $\lambda \in (0,1)$. Throughout the paper, $\omega : \mathbb{R}^n \to [0,\infty]$ is a measurable function satisfying the doubling condition: for any ball $B = B(x,r)$ with centre at a point x and of radius $r > 0$, the inequality

$$\int_{2B} \omega\, dx \leq C \int_{B} \omega\, dx \tag{1.3}$$

is satisfied, where the constant C is positive and does not depend on the ball $B \subset \mathbb{R}^n$. For the system of functions $A_j(x,\xi,\zeta)$; $j = 1,2,\ldots n$, we can write the following expression:

$$\begin{aligned} A_i(x,u(x),\nabla u(x)) &= \int_0^1 \sum_{k=1}^{n} \frac{\partial A_i}{\partial \zeta_k}(x,u(x),t\nabla u(x))u_{x_k} dt \\ &= \sum_{k=1}^{n} b_{ik}(x)\omega(x)|\nabla u(x)|^{p-2}u_{x_k}, \quad i = 1,2,\ldots,n, \end{aligned} \tag{1.4}$$

where $b_{ik}(x) = \omega(x)^{-1}|\nabla u(x)|^{2-p} \int_0^1 \sum_{k=1}^{n} \left(\dfrac{\partial A_i}{\partial \zeta_k}\right)(x,u(x),t\nabla u(x))dt$; $i,k = 1,2,\ldots,n$. Therefore, (1.1) can be written in the form

$$\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(\omega|\nabla u|^{p-2} b_{ik}(x) \frac{\partial u}{\partial x_k}\right) = 0. \tag{1.5}$$

By virtue of (1.2) and (1.4), the system of functions $\{b_{ik}(x)\}_{i,k=1,2,\ldots,n}$ is bounded and measurable. Moreover, the condition of uniform ellipticity is satisfied: for a.e. $x \in D$, $\eta \in \mathbb{R}^n$ there exist positive constants C_1, C_2 such that

$$C_1|\eta|^2 \leq \sum_{i,k=1}^{n} b_{ik}(x)\eta_i\eta_k = \sum_{i,k=1}^{n} \omega(x)^{-1}|\nabla u(x)|^{2-p} \int_0^1 \frac{\partial A_i}{\partial \zeta_k}(x,u(x),t\nabla u(x))\eta_i\eta_k dt \leq C_2|\eta|^2. \tag{1.6}$$

Denote by $C^\alpha(D)$, $0 < \alpha \leq 1$, the class of continuous in D functions $f : D \to \mathbb{R}$ satisfying the condition

$$|f(x) - f(y)| \leq K|x - y|^\alpha \tag{1.7}$$

with some $K > 0$ not depending on the points $x, y \in D$. Denote by $W^1_{p\omega}(D)$ the space of measurable functions in D, which have the finite norm

$$\|u\| = \|u\|_{L_{p\omega}(D)} + \sum_{j=1}^{n} \left\|u_{x_j}\right\|_{L_{p\omega}(D)}, \tag{1.8}$$

where u_{x_j} are the derivatives of a function $u \in L_{p\omega}(D)$ in the sense of the distribution theory, which belong to the space $L_{p\omega}(D)$. The norm of the space $L_{p\omega}(D)$ is given in the form $\|f\|_{L_{p\omega}(D)} = (\int_D \omega |f|^p dx)^{1/p}$ for $p \geq 1$; $\|f\|_{L_{p\omega}(D)} = \operatorname{ess\,sup}_{x\in D}|f(x)|$ for $p = \infty$. Denote by $\dot{W}^1_{p\omega}(D)$ a subspace of the space $W^1_{p\omega}(D)$, where the class of functions $C_0^\infty(D)$ is an everywhere dense set. Denote by $\widetilde{W}^1_{p\omega}(D)$ the closure of the set of functions $C^\infty(D)$ with respect to the norm $W^1_{p\omega}(D)$. The spaces $W^1_{p\omega}(D)$ and $\widetilde{W}^1_{p\omega}(D)$ coincide and are completely reflexive [1, 2] if the conditions $1 < p < \infty$ and A_p-Muckenhoupt are fulfilled for ω:

$$\left(\int_B \omega(x)dx\right)\left(\int_B \omega^{-1/(p-1)}(x)dx\right)^{p-1} \leq C_p|B|^p, \tag{1.9}$$

where $|B|$ denotes the Lebesgue measure of an arbitrary ball $B \subset \mathbb{R}^n$. In the sequel, we will also use the A_1-condition:

$$\int_{B(x,\rho)} \omega(x)dx \leq C\rho^n \inf_{x\in B} \omega. \tag{1.9$'$}$$

Definition 1.1. A function $u \in W^1_{p\omega}(D)$ is called a solution of (1.1) if the integral identity

$$\int_D A(x,u,\nabla u)\cdot \nabla\varphi\, dx = 0 \tag{1.10}$$

is fulfilled for any test function $\varphi \in \dot{W}^1_{p\omega}(D)$.

Definition 1.2. Let $E \subset\subset D$ be a compact subset of the bounded domain $D \subset \mathbb{R}^n$. One will say that the set E is removable for the class of $C^\alpha(D)$ of solutions of (1.1) if any solution of (1.1) in $D \setminus E$ from the space $W^1_{p\omega,\mathrm{loc}}(D\setminus E)$ belongs to the space $W^1_{p\omega}(D)$ throughout the domain D and is extendable inside the compactum E as solution.

Definition 1.3. Let $E \subset \mathbb{R}^n$ be a bounded closed subset, $h : \mathbb{R} \to (0,\infty)$ a continuous function, and $h(0) = 0$, μ some Radon measure. A finite system of balls $\{B_\nu = B(x_\nu, r_\nu)\}_{\nu=1,2,\dots,N}$, the radii of which do not exceed $\delta > 0$, covers the set E, that is, $E \subset \bigcup_\nu B_\nu$. Assume that $\Lambda_\mu^{h,\delta}(E) = \inf\{\sum_\nu h(r_\nu)\mu(B_\nu)\}$, where the lower bound is taken with respect to all the mentioned balls.

Assume that

$$\Lambda_\mu^h(E) = \lim_{\delta\to\infty} \Lambda_\mu^{h,\delta}(E). \tag{1.11}$$

In the case $\mu = dx$, $h(t) = t^{-p+(p-1)\alpha}$, the number $\Lambda_\mu^h(E)$ is a Hausdorff measure of order $n - p + (p-1)\alpha$ of the set E. We will sometimes denote it by $\mathrm{mes}_{n-p+(p-1)\alpha}(E)$. Denote by $\Lambda_\omega^{-p+(p-1)\alpha}(E)$ the term $\Lambda_\mu^{h(r)}(E)$ for $h(t) = t^{-p+(p-1)\alpha}$, $d\mu = \omega\, dx$.

By Carlson's theorem [3], a necessary and sufficient condition for the compact set E to be removable in the class of harmonic outside E functions belonging to the class $C^\alpha(D)$ is expressed in terms of a Hausdorff measure of order $n - 2 + \alpha$ having the form

$$\mathrm{mes}_{n-2+\alpha}E = 0, \quad 0 < \alpha < 1. \tag{1.12}$$

(For the case $\alpha = 1$, the same result was proved in [4, 5].) In [6], the corresponding result was proved for a general linear elliptic equation with variable coefficients (see also [7, 8]). In [9], for the case $p \geq 2$, a sufficient condition was proved for a solution of the p-Laplace equation ($A = |\nabla u|^{p-2}\nabla u$) to be removable in the class $C^\alpha(D)$ $(0 < \alpha \leq 1)$ in the form

$$\operatorname{mes}_{n-p+(p-1)\alpha} E = 0, \quad 0 < \alpha \leq 1. \tag{1.13}$$

Furthermore, a complete analogue of Carlson's result was proved in [10], where the authors not only proved the necessity of condition (1.13), but also gave another proof of sufficiency that includes also range of exponent $1 < p < 2$. Their approach was applied to the case of a metric measure space in [11].

It should be said that in [9] a somewhat general result was in fact obtained for the compact set E to be removable for the class $C^\alpha(D)$ of solutions of the equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\omega|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right) = 0 \tag{1.14}$$

in the form of a sufficient condition

$$\Lambda_\omega^{-p+(p-1)\alpha}(E) = 0. \tag{1.15}$$

Note, in [9], the case $p \geq 2$ was considered and it was required that the function ω to satisfy the doubling condition.

The present paper continues the development of the approach of [9]. We show that condition (1.15) is also the necessary one for the compact set E to be removable. Moreover, imposing some restrictions on the degeneration function, we manage to make the proof embrace a range of the exponent $1 < p < 2$.

We will use the following auxiliary statements.

Lemma 1.4 (see [12]). *Assume that a function $u \in L^1(D)$ satisfies the inequality*

$$\int_{B(x,r)} \left|u - (u)_{x,r}\right| dx \leq M r^{n+\alpha} \tag{1.16}$$

for any ball $B(x,r) \subset D$, where $\alpha \in (0,1)$. Then, $u \in C^\alpha(D)$ and for any $D' \subset\subset D$ the estimate

$$\sup_{D'}|u| + \sup_{x,y\in D'} \frac{\left|u(x) - u(y)\right|}{\left|x - y\right|^\alpha} \leq C\left(M + \|u\|_{L^1(D)}\right), \tag{1.17}$$

where $C = C(n, \alpha, D', D)$, is satisfied.

We also need the following analogue of the well-known Giaquinta's lemma [13].

Lemma 1.5. *Let $\phi(t)$, $\omega(t)$ be nonnegative nondecreasing functions on $[0,R]$. Assume that $s>0$ is such that*

$$\frac{\omega(\lambda r)}{\omega(r)} \geq \lambda^s \tag{1.18}$$

for all $r>0$ and $0<\lambda<1$. Suppose that

$$\phi(\rho) \leq A\left[\frac{\omega(\rho)}{\omega(r)}\left(\frac{\rho}{r}\right)^{\alpha}+\varepsilon\right]\phi(r)+B\omega(r)r^{\beta} \tag{1.19}$$

for any $0<\rho<r<R$, with A,B,α,β nonnegative constants and $\beta<\alpha$. Then, for any $\gamma\in(\beta,\alpha)$, there exists a constant $\varepsilon_0=\varepsilon_0(A,\alpha,\beta,\gamma,s)$ such that if $\varepsilon<\varepsilon_0$, then one has, far all $0<\rho<r<R$,

$$\phi(\rho) \leq c\left[\frac{\omega(\rho)}{\omega(r)}\left(\frac{\rho}{r}\right)^{\gamma}\phi(r)+B\omega(\rho)\rho^{\beta}\right], \tag{1.20}$$

where $c=c(\beta,\alpha,A,s,\gamma)>0$.

Proof. For $\tau\in(0,1)$ and $r<R$, we have

$$\phi(\tau r) \leq A\tau^{\alpha}\frac{\omega(\tau r)}{\omega(r)}\left[1+\varepsilon\tau^{-\alpha-s}\right]\phi(r)+B\tau^{-s}\omega(\tau r)r^{\beta}. \tag{1.21}$$

Choose $\tau<1$ in such a way that $2A\tau^{\alpha}=\tau^{\gamma}$ and assume $\varepsilon_0\tau^{-\alpha-s}\leq 1$. Then, we get, for every $r<R$,

$$\phi(\tau r) \leq \tau^{\gamma}\frac{\omega(\tau r)}{\omega(r)}\phi(r)+B\tau^{-s}\omega(\tau r)r^{\beta} \tag{1.22}$$

and therefore, for all integers $k>0$,

$$\begin{aligned}
\phi\left(\tau^{k+1}r\right) &\leq \tau^{\gamma}\frac{\omega(\tau^{k+1}r)}{\omega(\tau^{k}r)}\phi\left(\tau^{k}r\right)+B\tau^{-s}\omega\left(\tau^{k+1}r\right)\tau^{k\beta}r^{\beta} \\
&\leq \tau^{(k+1)\gamma}\frac{\omega(\tau^{k+1}r)}{\omega(r)}\phi(r)+B\tau^{-s}\omega\left(\tau^{k+1}r\right)\tau^{k\beta}r^{\beta}\sum_{j=0}^{k}\tau^{j(\gamma-\beta)} \\
&\leq \tau^{(k+1)\gamma}\frac{\omega(\tau^{k+1}r)}{\omega(r)}\phi(r)+\frac{B\tau^{-s}\tau^{k\beta}}{1-\tau^{\gamma-\beta}}r^{\beta}\omega\left(\tau^{k+1}r\right).
\end{aligned} \tag{1.23}$$

Choosing k such that $\tau^{k+2}r<\rho\leq\tau^{k+1}r$, the last inequality gives

$$\phi(\rho) \leq \frac{1}{\tau^{\gamma}}\left(\frac{\rho}{r}\right)^{\gamma}\frac{\omega(\rho)}{\omega(r)}\phi(r)+\frac{B\tau^{-s}}{(1-\tau^{\gamma-\beta})\tau^{s+2\beta}}\rho^{\beta}\omega(\rho). \tag{1.24}$$

This proves Lemma 1.5. □

We did not find the proof of the next inequality in the literature and therefore give here our proof.

Lemma 1.6. *Let $1 \le p \le 2$. Let $x, y \in \mathbb{R}^n$ be arbitrary points. Then, the estimate*

$$\left||x|^{p-2}x - |y|^{p-2}y\right| \le 2^{2-p}|x-y|^{p-1} \tag{1.25}$$

is valid.

Proof. Let us introduce the vector function

$$\varphi(\theta) = |\theta x + (1-\theta)y|^{p-2}(\theta x + (1-\theta)y), \quad 0 \le \theta \le 1 \tag{1.26}$$

acting from $[0,1]$ into $\mathbb{R}^n$. Applying the methods of differential calculus for the vector function, we obtain

$$\begin{aligned}\left||x|^{p-2}x - |y|^{p-2}y\right| &= |\varphi(1)-\varphi(0)| = \left|\int_0^1 \frac{d\varphi}{d\theta}d\theta\right| = (p-1)\left|\int_0^1 |\theta x + (1-\theta)y|^{p-2}(x-y)d\theta\right| \\ &\le (p-1)|x-y|\int_0^1 |\theta x + (1-\theta)y|^{p-2}d\theta.\end{aligned} \tag{1.27}$$

The set of points $\{l(\theta) \in \mathbb{R}^n : l(\theta) = \theta x + (1-\theta)y;\ 0 \le \theta \le 1\}$ in $\mathbb{R}^n$ forms a segment of the straight line that connects the point x with the point y. We denote this segment by $[x,y]$. Let $|dl|$ be a length element of this segment. It is obvious that $|dl| = |x-y|d\theta$. Therefore, for the above integral expression, we have the estimate

$$\le (p-1)\int_{[x,y]} |l(\theta)|^{p-2}|dl(\theta)| = (p-1)\int_0^{|x-y|} \frac{|dl(\theta)|}{\operatorname{dist}(l(\theta),0)^{2-p}}. \tag{1.28}$$

To proceed with the estimation of this expression, we introduce into consideration the triangle, the base of which is the segment $[x,y]$ and the vertex lies at the point 0. Now, the integration in the preceding estimate will be carried out with respect to the base of the triangle. It is not difficult to verify that the above integral expression takes a maximal value when the point 0 lies in the middle of the segment $[x,y]$, which means that for it we have the estimate

$$\le (p-1)\int_0^{|x-y|/2} \frac{ds}{s^{2-p}} = 2^{2-p}|x-y|^{p-1}. \tag{1.29}$$

To show that this is true, let us choose a new coordinate system, where the x_n-axis is directed along the segment $[x,y]$. Let $(u_1,u_2,\dots,u_n)$ be the coordinates of the the point 0 in the new coordinate system. Then, the preceding integral expression is equal to

$$\begin{aligned}&(p-1)\int_{-|x-y|/2}^{|x-y|/2}\left(\sqrt{u_1^2+u_2^2+\cdots+u_{n-1}^2+(u_n-x_n)^2}\right)^{p-2}dx_n\\&\quad=(p-1)\int_{-|x-y|/2}^{|x-y|/2}|u_n-s|^{p-2}ds\le 2(p-1)\int_0^{|x-y|/2}s^{p-2}ds=2^{2-p}|x-y|^{p-1}.\end{aligned}\tag{1.30}$$

□

The main result of this paper is contained in the following statements.

Theorem 1.7. *Let $D\subset\mathbb{R}^n$ be a bounded domain, $E\subset\subset D$ be a compact subset. Let $2<p<\infty$ and ω be a positive, locally integrable function satisfying condition* (1.3) *or $1<p<2$ and let any of the following conditions be fulfilled for the function ω:*

(1) *the function ω is integrable along any finite smooth $n-1$-dimensional surface and condition* (1.9′) *is fulfilled for it;*

(2) *for any $x\in D$ and sufficiently small $\rho>0$, the condition $\int_{B(x,\rho)}\omega\,dy\le C\rho^s$ is fulfilled for some $s>n-p+1$, where the constant $C>0$ does not depend on x.*

Then, for a compact set E to be removable in the class $C^\alpha(D)$, $0<\alpha\le 1$ of solutions of (1.1) *in $D\setminus E$, $u\in W^1_{p\omega,\mathrm{loc}}(D\setminus E)$, it is sufficient that condition* (1.15) *be fulfilled.*

Here, we will use also the fact that a solution of generating equations of the form (1.1) is Hölderian. According to [14], when a weight ω belongs to the Muckenhoupt A_p-class, a solution of (1.1) belongs to the class $C^\kappa(D_\rho)$ in any subdomain $D_\rho=\{x\in D:\operatorname{dist}(x,\partial D)>\rho\}$ of the domain D. For solutions, we have the estimate

$$\operatorname*{osc}_{B(x,\rho)} u\le C\left(\frac{\rho}{r}\right)^{\kappa}\operatorname*{osc}_{B(x,r)} u,\quad 0<\rho<r,\tag{1.31}$$

where $\kappa=\kappa(n,p,C_p,\lambda)\in(0,1]$ and $C=C(n,p,C_p,\lambda)$. Let κ denote a maximal number $\kappa=\kappa(n,p,C_p,\lambda)$, for which the estimate (1.31) holds for solutions of (1.1). The following statement is valid.

Theorem 1.8. *Let $\omega\in A_p$, $E\subset\subset D$ be a compact subset of the domain D. Let $0<\alpha<\kappa$ be some number. In that case, if $\Lambda_\omega^{-p+(p-1)\alpha}(E)>0$, then the set E is not removable in the class of $u\in W^1_{p\omega,\mathrm{loc}}(D\setminus E)$ solutions of* (1.1) *which belong to $C^\alpha(D)$.*

The foregoing statements give rise to the following corollaries.

Corollary 1.9. *Let $0<\alpha<\kappa$, $2\le p<\infty$, $\omega\in A_p$, or $1<p<2$ and any of the following conditions be fulfilled:*

(1) *the function ω satisfies condition* (1.9′) *and is integrable along any finite smooth $n-1$-dimensional surface;*

(2) *for any $x\in D$ and sufficiently small $\rho>0$, the condition $\int_{B(x,\rho)}\omega dy\le C\rho^s$ is fulfilled by some $s>n-p+1$, where the constant $C>0$ does not depend on x.*

Then, for the compact set E to be removable for the class of $W^1_{p\omega}(D \setminus E)$ *solutions of* (1.1) *belonging to* $C^\alpha(D)$, *it is necessary and sufficient that condition* (1.15) *be fulfilled.*

Corollary 1.10. *Let* $0 < \alpha \le 1$, $1 < p < \infty$. *Then, for the compact set E to be removable in the class* $W^1_{p,\mathrm{loc}}(D \setminus E)$ *of solutions of the equation*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right) = 0 \tag{1.32}$$

belonging to the class $C^\alpha(D)$ *throughout the domain D, it is necessary and sufficient that condition* (1.13) *be fulfilled.*

2. Proof of the Main Results

In [9], the method of proving Theorem 1.7 was based on the application of an analogue of Landis-Gerver's mean value theorem [15]. The restrictive condition $p \ge 2$ used in [9] was necessitated by the proof of Lemma 2.1 below (see also [15, Lemma 1]). Below, we prove a such type lemma for the case $1 < p < 2$, ignoring some smoothness of the function f and making some additional assumptions for the function ω.

Lemma 2.1. *Let D be a bounded domain. Let* $2 \le p < \infty$ *and the function* $\omega : \mathbb{R}^n \to [0,\infty]$ *satisfy condition* (1.3) *or* $1 < p < 2$, *and let any of the following conditions be fulfilled for the function* ω:

(1) *condition* (1.9′) *is fulfilled and* ω *is integrable along any finite smooth* $n-1$*-dimensional surface;*

(2) *for any* $x \in D$ *and sufficiently small* $\rho > 0$, *the condition* $\int_{B(x,\rho)} \omega\, dy \le C\rho^s$, *where the constant* $C > 0$ *does not depend on* x, *is fulfilled for some* $s > n - p + 1$.

Assume that $f : D \to \mathbb{R}$ *is a sufficiently smooth function (one can also assume the condition* $f(x) \in C^\beta(D)$, *where* $\beta \ge \min\{p', 1\}$*). Then, for any* $\varepsilon > 0$, *there exist a finite number of balls* $\{B_\nu\}$, $\nu = 1, 2, \ldots, N$, *such that*

$$\sum_{\nu=1}^{N} \int_{\partial B_\nu} \omega \left|\nabla f\right|^{p-1} ds < \varepsilon. \tag{2.1}$$

Proof. We will follow the same reasoning as that used in proving Lemma 1 in [9] (see also [3], Lemma 2.1). The set $O_f = \{x \in D : \nabla f(x) = 0\}$ is divided into two parts $O_f = O'_f \cup O''_f$; here, O'_f is the set of points where $\nabla^2 f(x) \ne 0$, and O''_f is the set of points where $\nabla^2 f(x) = 0$. Let $1 < p < 2$. Then, for the set O'_f, our reasoning is as follows. By virtue of the implicit function theorem, the set O'_f lies on a countable quantity of smooth $n-1$-dimensional surfaces $\{S_j\}$; $j = 1, 2, \ldots$. Let $x \in S_j$ be a fixed point on the j-th surface. For sufficiently small $r > 0$, we have

$$\int_{S(x,r)} \omega \left|\nabla f\right|^{p-1} ds \le 2C_1 r^{p-1} \omega(S(x,r)), \tag{2.2}$$

where $C_1 = \sup_D |\nabla^2 f|$ and $\omega(S(x,r))$ is integral omega over the $n-1$ dimensional surface $S(x,r)$. By virtue of Fubini's formula,

$$\int_{r_x}^{2r_x} \left(\int_{|y-x|=t} \omega\, ds_y\right) dt \le \int_{|y-x|<2r_x} \omega(y)\, dy. \tag{2.3}$$

Let E be the set of points $t \in (r_x, 2r_x)$, for which the following condition is fulfilled:

$$\int_{|y-x|=t} \omega\, ds > \frac{2}{r_x}\left(\int_{|y-x|<2r_x} \omega(x)dx\right). \tag{2.4}$$

Then

$$\begin{aligned}\frac{2}{r_x}\left(\int_{|y-x|<2r_x} \omega(y)dy\right)\mu_1(E) &\leq \int_E\left(\int_{|y-x|=t} \omega\, ds\right)dt \leq \int_{r_x}^{2r_x}\left(\int_{|y-x|=t} \omega\, ds\right)dt \\ &\leq \int_{|y-x|<2r_x} \omega(y)dy,\end{aligned} \tag{2.5}$$

whence for the one-dimensional Lebesgue measure of the set E we obtain the estimate $\mu_1(E) \leq r_x/2$. Hence, by virtue of the doubling condition, there exists a point $t_x \in (r_x, 2r_x)$, for which

$$\int_{|y-x|=t_x} \omega\, ds < \frac{2}{r_x}\left(\int_{|y-x|<2r_x} \omega(y)dy\right) \leq \frac{2C}{r_x}\left(\int_{|y-x|<r_x} \omega(y)dy\right) \leq \frac{4C}{t_x}\left(\int_{|y-x|<t_x} \omega(x)dx\right). \tag{2.6}$$

Then, for sufficiently small $t_j > 0$, for any $x \in S_j$, it can be assumed that there exists a number $\rho_x \in (t_j, 2t_j)$, for which

$$\omega(S(x,\rho_x)) \leq \frac{4C}{\rho_x}\omega(B(x,\rho_x)). \tag{2.7}$$

Therefore,

$$\int_{S(x,\rho_x)} \omega|\nabla f|^{p-1}ds \leq 4C_1\rho_x^{p-1}\omega(S(x,\rho_x)) \leq 16CC_1\rho_x^{p-2}\omega(B(x,\rho_x)). \tag{2.8}$$

For the surface S_j, from the system of balls $\{B(x,\rho_x); x \in S_j\}$, we can extract, by virtue of Besicovtich theorem [16], a subcovering $\{B(x_\nu,\rho_\nu);\ x_\nu \in S_j, \nu \in \mathbb{N}\}$ with finite intersections:

$$\sum_\nu \chi_{B(x_\nu,\rho_\nu)}(x) \leq C_n, \qquad \bigcup_\nu B(x_\nu,\rho_\nu) \supset S_j. \tag{2.9}$$

Therefore and by construction, for $x_\nu \in S_j,\ \nu = 1,2,\ldots$, we have $\rho_\nu \in (t_j, 2t_j)$. Thus,

$$\begin{aligned}\sum_\nu \int_{S(x_\nu,\rho_\nu)} \omega|\nabla f|^{p-1}ds &\leq \sum_\nu 16CC_1\rho_\nu^{p-2}\omega(B(x_\nu,\rho_\nu)) \leq \sum_\nu 16CC_1C_2\rho_\nu^{n+p-2}\inf_{x\in B(x_\nu,\rho_\nu)}\omega \\ &\leq \sum_\nu 32CC_1C_2\rho_\nu^{p-1}\omega(B(x_\nu,\rho_\nu)\cap S_j) \leq 32\cdot 2^{p-1}Ct_j^{p-1}\omega(S_j), \\ &\qquad j = 1,2,\ldots.\end{aligned} \tag{2.10}$$

Here, we have used the sufficient smallness of t_j, condition (1.9′), and the inequality $\rho_\nu^{n-1}\inf_{x\in B(x_\nu,\rho_\nu)}\omega \le 2\omega(B(x_\nu,\rho_\nu)\cap S_j)$. Choosing now t_j, $32\cdot 2^{p-1}CC_1C_2t_j^{p-1}\omega(S_j) \le \varepsilon/2^j$, where $\varepsilon > 0$ is an arbitrary number, we obtain $\sum_\nu \int_{S(x_\nu,\rho_\nu)} \omega|\nabla f|^{p-1}ds < \varepsilon/2^j$, whence, after summing the inequalities over all surfaces S_j, $j \in \mathbb{N}$, we find

$$\sum_j \sum_{x_\nu\in S_j} \int_{S(x_\nu,\rho_\nu)} \omega\left|\nabla f\right|^{p-1}ds < \varepsilon. \tag{2.11}$$

In the case of the second condition $1 < p < 2$, using $\int_{B(x,\rho)} \omega\, dy \le C\rho^s$, we immediately pass from the inequality (2.8) to (2.10) as

$$\omega(S(x,\rho_x)) \le \frac{4C}{\rho_x}\omega(B(x,\rho_x)) \le 4CC_4\rho_x^{n-p} = 8CC_4\rho_x^{2-p}\operatorname{mes}_{n-1}(B(x,\rho_x)\cap S_j). \tag{2.12}$$

Due to the latter inequality, an estimate analogous to (2.10) will have the form

$$\begin{gathered}\sum_\nu \int_{S(x_\nu,\rho_\nu)} \omega\left|\nabla f\right|^{p-1}ds \le \sum_\nu 2C_1\rho_\nu^{p-2}\omega(B(x_\nu,\rho_\nu)) \le \sum_\nu 4C_1\rho_\nu^{p-2+s}, \\ \sum_\nu 8C_1\rho_\nu^{s-n+p-1}\operatorname{mes}_{n-1}(B(x_\nu,\rho_\nu)\cap S_j) \le 16\cdot 2^{p-1}Ct_j^{s-n+p-1}\omega(S_j), \quad j = 1,2,\ldots.\end{gathered} \tag{2.13}$$

After choosing t_j sufficiently small and taking the condition $s > n-p+1$ into account, we can make the right-hand part smaller than $\varepsilon/2^j$.

In the case $p \ge 2$, the whole reasoning of [9] is applicable. Note that only instead of the inequality (2.10) we will have

$$\sum_\nu \int_{S(x_\nu,\rho_\nu)} \omega\left|\nabla f\right|^{p-1}ds \le C\sum_\nu \rho_\nu^{p-1}\omega(B(x_\nu,\rho_\nu)) \le C_1t_j^{p-2}\omega\left(S_{t_j}^j\right), \tag{2.14}$$

where $S_{t_j}^j$ is the t_j neighborhood of the surface S^j. After choosing a sufficiently small t_j, we can make the right-hand part of this inequality smaller than $\varepsilon/2^j$. This is possible because the n-dimensional Lebesgue measure of the surface S^j is equal to zero.

Now, it remains to obtain the covering for the set of points O_f''. Let $1 < p < 2$. Let us decompose $O_f'' = O_f''' \cup O_f''''$, where O_f''' is the set of points O_f'', for which $\nabla^3 f \ne 0$. Here, we repeat the reasoning for O_f'. As above, the set O'''' is divided into two parts. In one part, we have $\nabla^4 f(x) \ne 0$, to which we apply the same reasoning as for O_f'. The second part of O'''', where $\nabla^4 f(x) = 0$, is again divided into two parts. At the k-th step, when $k(p-1) \ge 1$ and $t > 0$ is sufficiently small, this process yields the estimate

$$\begin{aligned}\sum_\nu \int_{S(x_\nu,\rho_\nu)} \omega\left|\nabla f\right|^{p-1}ds &\le \eta\sum_\nu C{\rho_\nu}^{k(p-1)}\omega(S(x_\nu,\rho_\nu)) \\ &\le \eta\sum_\nu 2C{\rho_\nu}^{k(p-1)-1}\omega(B(x_\nu,\rho_\nu)) \le 2\eta CC_3\omega(D),\end{aligned} \tag{2.15}$$

where $\eta > 0$ is arbitrary.

Note that, in the case $p \geq 2$, the foregoing estimate gives the desired results immediately at the first step ($k = 1$). □

Remark 2.2. It is not difficult to verify that under the assumptions of Lemma 2.1, instead of the condition of sufficient smoothness it sufficed to assume that $f(x) \in C^{\beta}(D)$, where $\beta \geq \min\{p', 1\}$.

Applying approaches similar to those in [15, Theorem 2.2, page 128] and [9], we prove the following analogue of Landis-Gerver's lemma.

Lemma 2.3. *Let $2 \leq p < \infty$ and the function $\omega : \mathbb{R}^n \to [0, \infty]$ satisfy condition (1.3) or $1 < p < 2$ and let any of the following conditions be fulfilled for the function ω:*

(1) *condition (1.9′) is fulfilled and ω is integrable along any finite smooth $n-1$-dimensional surface;*

(2) *for any $x \in D$ and sufficiently small $\rho > 0$, the condition $\int_{B(x,\rho)} \omega\, dy \leq C\rho^s$ is fulfilled by some $s > n - p + 1$, where the constant $C > 0$ does not depend on x.*

Let D be some domain lying in the spherical layer $B(x_0, 2r) \setminus \overline{B}(x_0, r)$ and having limit points on the surfaces of the spheres $S(x_0, 2r)$ and $S(x_0, r)$. Let $\sum_{i,k=1}^{n} a_{ik}(x)\eta_i\eta_k$ be the quadratic form, the coefficients of which are well defined and continuously differentiable in the domain D and for which the inequalities

$$\lambda|\eta|^2 \leq \sum_{i,k=1}^{n} a_{ik}(x)\eta_i\eta_k \leq \lambda^{-1}|\eta|^2 \tag{2.16}$$

are fulfilled for any $x \in D$, $\eta \in \mathbb{R}^n$ for some $\lambda \in (0,1)$. Assume that $f : D \to \mathbb{R}$ is a sufficiently smooth function.

Then, there exists a piecewise-smooth surface Σ, separating, in the domain D, the surfaces of the spheres $S(x_0, r)$ and $S(x_0, 2r)$ and being such that

$$\int_{\Sigma} \omega|\nabla f|^{p-2}\left|\frac{\partial f}{\partial \nu}\right| ds \leq K\frac{(\operatorname{osc}_D f)^{p-1}\omega(D)}{r^p}, \tag{2.17}$$

where $\partial f/\partial \nu = \sum_{j=1}^{n} a_{ij}(x) f_{x_i} n_j$ is conormal derivative on Σ, $n = (n_1, n_2, \dots, n_n)$ is unit orthogonal vector to the surface Σ, and the constant K depends on p, λ, and the dimension n.

Proof. It suffices to consider the case $r = 1$. Indeed, after the change of variables $x = ry$, the function $f : D \to R$ transforms to the function $\tilde{f} : \tilde{D} \to R$, where $\tilde{f}(y) = f(ry)$. Also, $|\nabla_y \tilde{f}| = |\nabla_x f| r$, $\partial\tilde{f}/\partial v_y = (\partial f/\partial v_x) r$, $\omega(\tilde{D}) = \omega(D) r^{-n}$. $\tilde{D}$ lies in the spherical layer $B(0,2) \setminus B(0,1)$. It suffices to show that $\partial\tilde{f}/\partial v_y = (\partial f/\partial v_x) r$. Indeed, let a sufficiently small element of the surface Σ satisfy the equation $\varphi(x) = 0$ in coordinates x. Then, after the change of variables, this equation takes the form $\tilde{\varphi}(y) = 0$, where $\tilde{\varphi}(y) = \varphi(ry)$. In other words, the normals of the surfaces Σ and $\tilde{\Sigma}$ are related by $n_x = \nabla_x\varphi/|\nabla_x\varphi| = \nabla_y\tilde{\varphi}/|\nabla_y\tilde{\varphi}| = n_y$. Therefore,

$$\frac{\partial\tilde{f}}{\partial v_y} = \sum_{i,k=1}^{n} \tilde{a}_{ik}(y)\frac{\partial\tilde{f}}{\partial y_i}\tilde{n}_k = r\sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial f}{\partial x_i} n_k = r\frac{\partial f}{\partial v_x}, \quad \tilde{a}_{ik}(y) = a_{ik}(ry),\ i,k = 1,2,\dots,n. \tag{2.18}$$

Applying these equalities, from the estimate

$$\int_{\Sigma}\left|\nabla_y\tilde{f}\right|^{p-2}\left|\frac{\partial\tilde{f}}{\partial v_y}\right|ds_y\le K\left(\operatorname*{osc}_{\tilde{D}}\tilde{f}\right)^{p-1}\omega\left(\tilde{D}\right), \tag{2.19}$$

we obtain (2.17).

Let us now prove (2.19). Following the notation and reasoning of [15] (see also [9]), we assume that $\varepsilon=\omega(D)(\operatorname{osc} f)^{p-1}$. For this ε, we find the corresponding balls $Q_1,Q_2,\ldots,Q_N$ of Lemma 2.1 and remove them from the domain D. Assume that $D^*=D\setminus\bigcup_{m=1}^{N}Q_m$ and intersect D^* with the closed layer $(1+1/4)\le|x|\le(1+3/4)$. Denote this intersection by D'. On the closed set D', we have $\nabla f\neq 0$. Let us choose some δ-neighborhood D'_δ with $\delta<1/4$ so small that in D'_δ we would have $|\nabla f|>\alpha>0$. We consider on D'_δ the system of equations

$$\frac{dx_i}{dt}=\sum_{k=1}^{n}a_{ik}(x)\frac{\partial f}{\partial x_k},\quad i=1,2,\ldots,n. \tag{2.20}$$

In D'_δ, there are no stationary points of the system (2.20), and at every point $x\in D'_\delta$ the direction of the field forms with the direction of the gradient an angle different from the straight angle. Let $l(x)$ be the vector of the field at the point x. Then, using $\cos(l(x),\nabla f)=(\sum_{k=1}^{n}a_{ik}(x)(\partial f/\partial x_k),\nabla f)/|\sum_{k=1}^{n}a_{ik}(x)(\partial f/\partial x_k)||\nabla f|>\lambda|\nabla f|^2/\lambda^{-1}|\nabla f|^2=\lambda^2$, we obtain

$$\left|\frac{\partial f}{\partial l}\right|>\lambda^2\left|\nabla f\right|>\gamma\alpha>0,\quad \gamma=\lambda^2. \tag{2.21}$$

From this inequality, it follows that in D'_δ there are no closed trajectories and all the trajectories have the uniformly bounded length.

Let some surface S be tangential, at each of its points, to the field direction. Then,

$$\int_S\omega\left|\nabla f\right|^{p-2}\left|\frac{\partial f}{\partial v}\right|ds=0, \tag{2.22}$$

since the integrand is identically zero. We will use this fact in constructing the needed surface Σ. The base of Σ consists of ruled surfaces, while the generatrices are the trajectories of the system (2.20). Note that they will add nothing to the integral in which we are interested. These surfaces will have the form of fine tubes which will cover the entire D'. Let us insert partitions into some of the tubes. The integral over these partitions will not any longer be equal to zero, but we can make it infinitesimal. The construction of tubes practically repeats that given in [15, pages 129–132].

Denote by E the intersection of D' with the sphere $S^0_{(1+3/4)}$. Let N be the set of points $x\in E$, where the direction of the field of the system (2.20) is tangential to the sphere $S^0_{(1+3/4)}$. At the points $x\in N$, we have $\partial f/\partial v=0$, where $\partial/\partial v$ is the derivative with respect to the conormal to the sphere $S^0_{(1+3/4)}$. Cover N by a set G, open on the sphere $S^0_{(1+3/4)}$ and being such that

$$\int_G\omega\left|\nabla f\right|^{p-2}\left|\frac{\partial f}{\partial v}\right|ds\le\omega(D)(\operatorname{osc} f)^{p-1}. \tag{2.23}$$

Put $E' = E \setminus G$. At the points $x \in E'$, the direction of the field is transversal to the sphere. Cover E' on the sphere $S^0_{(1+3/4)}$ by a finite number of uncovered domains with piecewise-smooth boundaries. We will call them *cells*. We will choose their diameters so small that at the points of the cells the field would be transversal to the sphere and the bundle of trajectories passing through each of the cells would diverge by $\delta/2n$ at most. The surface with trajectories lying inside the ball $|x| < (1+3/4)$ and passing through the cell boundary will be called *a tube*. Thus, we obtain a finite number of tubes. We will call a tube a *through* tube if, without intersecting this tube, we can connect by a broken line a point of its corresponding cell with a point of the sphere $S^0_{(1+1/4)-(\delta/2)}$ within the limits of the intersection of D' with a spherical layer $1+1/4-\delta < |x| < 1+3/4$. Such through tubes are denoted by $T_1, T_2, \dots, T_s$. If every through tube is partitioned, then the spheres S^0_1 and S^0_2 are separated in D by the set-theoretic sum of nonthrough tubes, partitions $T_1, T_2, \dots, T_s$, the spheres $S_1, S_2, \dots, S_N$, and the set G on the sphere $S^0_{(1+3/4)}$.

Let us now take care to choose partitions in such a way that the integral $\int \omega |\nabla f|^{p-2} |\partial f/\partial v| ds$ over them would have the value which we need. Denote by U_i the domain bounded by T_i. Choose any trajectory on this tube. Denote it by L_i. The length $\mu_1 L_i$ of the curve L_i satisfies the inequality $\mu_1 L_i > 1/2$. Introduce, on L_i, the parameter l which is the length of the arc counted from $S^0_{(1+1/4)}$. Denote by $\sigma_i(l)$ the section U_i with a hypersurface which is orthogonal, at the point l, to the trajectory L_i. Let the diameter at the beginning of the tube be so small that $\int_{L_i} (\int_{\sigma_i(l)} \omega\, ds) dl \le 2\omega(U_i)$. Then, the set H of points $l \in L_i$, where $\int_{\sigma_i(l)} \omega\, ds > 8\omega(U_i)$, satisfies the inequality $\mu_1 L_i < 1/4$. Thus, for $E = L_i \setminus H$, the inequality $\mu_1 L_i > 1/4$ is valid and

$$\int_{\sigma_i(l)} \omega\, ds < 8\omega(U_i) \quad \text{for } l \in E. \tag{2.24}$$

At the points of the curve L_i, the derivative $\partial f/\partial l$ preserves the sign and therefore

$$\int_E \left| \frac{\partial f}{\partial l} \right| dl \le \int_{L_i} \left| \frac{\partial f}{\partial l} \right| dl < \underset{D'_\delta}{\operatorname{osc}} f. \tag{2.25}$$

Hence, using $\mu_1 L_i > 1/4$ and the mean value theorem, we see that there exists a point $l_0 \in E$ such that $|\partial f/\partial l|_{l=l_0} \le 4 \operatorname{osc} f$. On the other hand, since, by virtue of (2.21), $|\partial f/\partial l|_{l=l_0} \ge \gamma |\nabla f|_{l=l_0}$, we have $|\nabla f|^{p-1}|_{l=l_0} \le (4 \operatorname{osc} f)^{p-1} \gamma^{1-p}$. This together with (2.24) gives the estimate

$$\left(\left| \nabla f \right|^{p-1} \Big|_{l=l_0} \right) \int_{\sigma_i(l_0)} \omega\, ds \le C(p,\gamma)\omega(U_i)(\operatorname{osc} f)^{p-1}. \tag{2.26}$$

Let us now choose a cell diameter so small that

$$\int_{\sigma_i(l_0)} \left| \nabla f \right|^{p-1} \omega\, ds \le 2C(p,\gamma)\omega(U_i)(\operatorname{osc} f)^{p-1}. \tag{2.27}$$

This can be done since the derivatives $\partial f/\partial x_k$, $k = 1, 2, \dots, n$, are uniformly continuous. Therefore,

$$\sum_{i=1}^{s} \int_{\sigma_i(l_0)} \left| \nabla f \right|^{p-1} \omega\, ds \le 4C(p,\gamma)\omega(U_i)(\operatorname{osc} f)^{p-1}. \tag{2.28}$$

Denote by Σ the set-theoretic sum of all nonthrough tubes, all $\sigma_i(l_0)$, all spheres S_i, and the set G on the sphere $S^0_{(1+3/4)r}$. Then, from Lemma 2.1 and (2.22)–(2.28), we obtain

$$\int_{\Sigma} |\nabla f|^{p-2} \left| \frac{\partial f}{\partial v} \right| \omega \, ds \leq C(p,n,\gamma)\omega(U_i)(\operatorname{osc} f)^{p-1}. \tag{2.29}$$

Lemma 2.3 is proved. □

In this paper, we give the complete proof of Theorem 1.7. Some part of the proof of sufficiency is in fact identical to the proof given in [9]. The method of proving Theorem 1.8 is analogous to the method [10], where the nonweight case was considered.

Proof of Theorem 1.7 (Approximation). Let $\Lambda_{\omega}^{-p+(p-1)\alpha}(E) = 0$ and $u \in C^{\alpha}(D)$; let $u \in W^1_{p\omega,\mathrm{loc}}(D \setminus E)$ be a solution of (1.1). Denote by u^j a mean value of the function u with smooth kernel ρ with finite support, $u^{(j)} = \rho_{1/j} * u = j^n \int_{\mathbb{R}^n} \rho((x-y)j)u(y)dy$, $\int_{\mathbb{R}^n} \rho(x)dx = 1$, $j \in \mathbb{N}$. Then, it is obvious that $u^{(j)} \in C^{\infty}(D)$, $j = 1,2,\ldots$. Moreover, $u^j \to u$ uniformly in any subdomain $\overline{G} \subset D$. Also, for any open set $E' \supset E$ contained in G, $u^{(j)} \to u$ in the norm of the space $W^1_{p\omega}(G \setminus E')$ (see [2, 16]). Since, by condition (1.15), we have $\mathrm{mes}_n E = 0$, it can be assumed that $\mathrm{mes}_n E' < \eta$, where $\eta > 0$ is an arbitrary number.

Let $\varepsilon > 0$ be an arbitrary number. Cover the set E by a finite system of balls $\{B_\nu\}_{\nu=1,2,\ldots,N}$, $\bigcup_{\nu=1}^N B_\nu \supset E$ such that $\operatorname{diam} B_\nu < \delta$,

$$\sum_{\nu=1}^{N} r_\nu^{-p+(p-1)\alpha} \omega(B_\nu) < \varepsilon. \tag{2.30}$$

Assume that the number $\delta = \delta(\varepsilon,\eta)$ is so small that the set $\Gamma'' = \bigcup_{\nu=1}^N (4B_\nu)$ lies in E'.

For every ν, there exists, by virtue of Lemma 2.3 and inequality (1.31), a piecewise-smooth surface $\gamma_\nu^{(j)}$, $\nu = 1,2,\ldots,N$, $j = 1,2,\ldots$, separating the surfaces of the spheres $\partial(2B_\nu)$ and $\partial(4B_\nu)$, such that

$$\int_{\gamma_\nu^j} \omega \left| \nabla u^{(j)} \right|^{p-2} \left| \frac{\partial u^{(j)}}{\partial v} \right| ds \leq K r_\nu^{-p} \left(\operatorname*{osc}_{2B_\nu} u^{(j)} \right)^{p-1} \omega(4B_\nu). \tag{2.31}$$

Denote by $\Gamma_\nu^{(j)}$ the interiority of the surface $\gamma_\nu^{(j)}$. Then, $\Gamma^{(j)} = \bigcup_\nu \Gamma_\nu^{(j)} \supset \Gamma' = \bigcup_\nu (2B_\nu)$. Assume that $\sigma_\nu^{(j)} = \Gamma^{(j)} \cap \gamma_\nu$. Let $\sigma_\nu^{(j)} \neq \emptyset$ for some ν. Then, for ν, the inequality (2.31) implies the estimate

$$\int_{\sigma_\nu^{(j)}} \omega \left| \nabla u^{(j)} \right|^{p-2} \left| \frac{\partial u^{(j)}}{\partial v} \right| ds \leq K r_\nu^{-p} \left(\operatorname*{osc}_{2B_\nu} u^{(j)} \right)^{p-1} \omega(4B_\nu). \tag{2.32}$$

It is obvious that the set $G \setminus \overline{\Gamma'}$ is a strictly interior subdomain of the domain $D \setminus E$. Thus, we have the identity

$$\sum_{i,k=1}^{n} \int_{G \setminus \Gamma'} \omega |\nabla u|^{p-2} b_{ik} u_{x_k} \varphi_{x_i} dx = 0, \tag{2.33}$$

for any $\psi \in C_0^1(D \setminus \Gamma')$. From this, by virtue of the convergence $\|u^{(j)} - u\|_{W^1_{p\omega}(G\setminus\Gamma')} \to 0$ as $j \to \infty$ and the fact that the $\|b_{ik}^{(j)}\|$ satisfies (2.16) by some $\lambda > 0$, we find

$$\sum_{i,k=1}^{n} \int\limits_{G\setminus\Gamma'} \omega \left|\nabla u^{(j)}\right|^{p-2} b_{ik}^{(j)} u_{x_k}^{(j)} \psi_{x_i} dx = \delta_j. \tag{2.34}$$

This follows from the fact that the integrand is a system of equi-integrable functions: for any subset $g \subset D \setminus \Gamma'$, we have

$$\begin{aligned}\sum_{i,k=1}^{n} \int_g \omega \left| b_{ik}^{(j)} \left|\nabla u^{(j)}\right|^{p-2} u_{x_i}^{(j)} \psi_{x_k} \right| dx &\le C \int_g \omega \left|\nabla u^{(j)}\right|^{p-1} |\nabla \psi| dx \\ &\le 2C \left(\int_g \omega |\nabla u|^p dx \right)^{1/p'} \left(\int_g \omega |\nabla \psi|^p dx \right)^{1/p} \longrightarrow 0\end{aligned} \tag{2.35}$$

as $\mathrm{mes}_n g \to 0$. Here and in the sequel, speaking in general, we denote by δ_j different sequences tending to zero as $j \to \infty$.

Green's Formulae for Approximations

Let now $\varphi \in C_0^1(D)$ be an arbitrary function. Assume that $\psi = \varphi\xi(d(x)/\tau)$, where $0 \le \xi(s) \le 1$ is an infinitely differentiable function equal to zero for $s \le 0$ and to one for $s \ge 1$ and $\tau > 0$ is a parameter, for all $\varphi \in C_0^1(D)$, $d(x) = \mathrm{dist}(x, \Gamma^j)$. It is obvious that $\psi \in C_0^1(D \setminus \Gamma^j)$. Then, (2.34) implies, for $j = 1, 2, \ldots$,

$$\sum_{i,k=1}^{n} \int_{D\setminus\Gamma'} \omega \left|\nabla u^{(j)}\right|^{p-2} b_{ik}^{(j)} u_{x_k}^{(j)} \varphi_{x_i} \xi \, dx + \sum_{i,k=1}^{n} \frac{1}{\tau} \int_{D\setminus\Gamma'} \omega \left|\nabla u^{(j)}\right|^{p-2} b_{ik}^{(j)} u_{x_k}^{(j)} d_{x_i} \xi' \left(\frac{d(x)}{\tau} \right) \varphi \, dx = \delta_j. \tag{2.36}$$

By virtue of the majorant Lebesgue theorem, for $\tau \to 0$, the first summand in (2.36) tends to the limit $\sum_{i,k=1}^{n} \int_{D\setminus\Gamma^{(j)}} \omega |\nabla u^{(j)}|^{p-2} b_{ik}^{(j)} u_{x_k}^{(j)} \varphi_{x_i} dx$. Let us now find the limit of the second summand. Applying the Federer formula, we have

$$\begin{aligned}&\sum_{i,k=1}^{n} \frac{1}{\tau} \int_{D\setminus\Gamma'} \xi' \left(\frac{d(x)}{\tau} \right) \omega \left|\nabla u^{(j)}\right|^{p-2} b_{ik}^{(j)} u_{x_k}^{(j)} \varphi_{x_i} dx \\ &\quad = \sum_{i,k=1}^{n} \frac{1}{\tau} \int_0^{\tau} \left(\int_{\{d(x)=t\cap(D\setminus\Gamma')\}} \varphi\omega \left|\nabla u^{(j)}\right|^{p-2} \frac{b_{ik}^{(j)} u_{x_k}^{j} d_{x_i}}{|\nabla d|} ds_t \right) \xi' \left(\frac{t}{\tau} \right) dt.\end{aligned} \tag{2.37}$$

Applying the mean value theorem, for some $t_0 \in (0, \tau)$, we obtain

$$
\begin{aligned}
&= \sum_{i,k=1}^{n} \left(\int_{d(x)=t_0} \varphi\omega \left| \nabla u^j \right|^{p-2} \frac{b_{ik}^{(j)} u_{x_k} d_{x_i}}{|\nabla d|} ds_{t_0} \right) \left(\frac{1}{\tau} \int_0^{\tau} \xi' \left(\frac{t}{\tau} \right) dt \right) \\
&\longrightarrow \sum_{i,k=1}^{n} \int_{d(x)=t_0} \varphi\omega \left| \nabla u^{(j)} \right|^{p-2} \frac{b_{ik}^{(j)} u_{x_k}^{(j)} d_{x_i}}{|\nabla d|} ds_{t_0} \qquad (2.38) \\
&\longrightarrow \int_{\partial\Gamma^{(j)}} \varphi\omega \left| \nabla u^{(j)} \right|^{p-2} \frac{\partial u^{(j)}}{\partial v} ds \quad \text{as } \tau \longrightarrow 0.
\end{aligned}
$$

Taking this limit relation into account, from (2.36), we obtain, as $\tau \rightarrow 0$, the following equality which is Green' formula for approximation function u^j:

$$
\sum_{i,k=1}^{n} \int_{D\setminus\Gamma^{(j)}} \omega \left| \nabla u^{(j)} \right|^{p-2} b_{ik}^{(j)} u_{x_k}^{(j)} \varphi_{x_i} dx = \int_{\partial\Gamma^{(j)}} \varphi\omega \left| \nabla u^{(j)} \right|^{p-2} \frac{\partial u^{(j)}}{\partial v} ds + \delta_j. \qquad (2.39)
$$

Whence, in view of the inequality $|\varphi(x)| \le \|\varphi\|_{C(D)}$, $x \in D$, we have

$$
\begin{aligned}
\left| \sum_{i,k=1}^{n} \int_{D\setminus\Gamma^{(j)}} \omega \left| \nabla u^{(j)} \right|^{p-2} b_{ik}^{(j)} u_{x_k}^{(j)} \varphi_{x_i} dx \right| &\le \|\varphi\|_{C(D)} \int_{\partial\Gamma^{(j)}} \omega \left| \nabla u^{(j)} \right|^{p-2} \left| \frac{\partial u^{(j)}}{\partial v} \right| ds + \delta_j \\
&\le \|\varphi\|_{C(D)} \sum_{\nu} \int_{\gamma_\nu} \omega \left| \nabla u^{(j)} \right|^{p-2} \left| \frac{\partial u^{(j)}}{\partial v} \right| ds + \delta_j.
\end{aligned} \qquad (2.40)
$$

Using the convergence $\|u^{(j)} - u\|_{W_{p\omega}^1(G\setminus\Gamma')} \rightarrow 0$ $(j \rightarrow \infty)$, Lemma 1.6, conditions (1.2), and Hölder inequality, we have the estimate for $1 < p < 2$:

$$
\begin{aligned}
&\left| \sum_{i,k=1}^{n} \int_{D\setminus\Gamma^{(j)}} \omega b_{ik}^{(j)} \left(\left| \nabla u^{(j)} \right|^{p-2} u_{x_i}^{(j)} - |\nabla u|^{p-2} u_{x_i} \right) \varphi_{x_k} dx \right| \\
&\quad \le C \int_{D\setminus\Gamma'} \omega \left| \nabla \left(u^{(j)} - u \right) \right|^{p-1} |\nabla\varphi| dx \\
&\quad \le C \left(\int_{D\setminus\Gamma'} \omega \left| \nabla \left(u^{(j)} - u \right) \right|^{p} dx \right)^{1/p'} \left(\int_{D\setminus\Gamma'} \omega |\nabla\varphi|^p dx \right)^{1/p} = \delta_j \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.
\end{aligned} \qquad (2.41)
$$

An analogous estimate for $2 \le p < \infty$ has the form

$$\begin{aligned}
&\left|\sum_{i,k=1}^{n}\int_{D\setminus\Gamma^{(j)}}\omega b_{ik}^{(j)}\left(\left|\nabla u^{(j)}\right|^{p-2}u_{x_i}^{(j)}-|\nabla u|^{p-2}u_{x_i}\right)\varphi_{x_k}dx\right| \\
&\quad\le C\int_{D\setminus\Gamma'}\omega\left|\nabla\left(u^{(j)}-u\right)\right|\left(|\nabla u|^{p-2}+\left|\nabla u^{(j)}\right|^{p-2}\right)|\nabla\varphi|dx \\
&\quad\le C\left(\int_{D\setminus\Gamma'}\omega\left|\nabla\left(u^{(j)}-u\right)\right|^{p}dx\right)^{1/p}\left(\int_{D\setminus\Gamma'}\omega\left(|\nabla u|^{p}+\left|\nabla u^{(j)}\right|^{p}\right)dx\right)^{(p-2)/p} \\
&\qquad\times\left(\int_{D}\omega|\nabla\varphi|^{p}dx\right)^{1/(p-2)} \\
&\quad\le 2C\|\nabla\varphi\|_{L_{p\omega}(D)}^{p'}\|\nabla u\|_{L_{p\omega}(D\setminus\Gamma')}^{p-2}\left\|\nabla\left(u^{(j)}-u\right)\right\|_{L_{p\omega}(D\setminus\Gamma')}=\delta_j\longrightarrow 0\quad\text{as } j\longrightarrow\infty.
\end{aligned}\tag{2.42}$$

The Belongness $u \in W_{p\omega}^1(D)$

Taking into account (2.41) and (2.42), the estimate (2.32), and the uniform convergence $u^{(j)} \to u$ in G, convergence a.e. $b_{ik}^{(j)} \to b_{ik}$, we find

$$\begin{aligned}
\left|\sum_{i,k=1}^{n}\int_{D\setminus\Gamma^{(j)}}\omega|\nabla u|^{p-2}b_{ik}u_{x_k}\varphi_{x_i}dx\right| &\le C\|\varphi\|_{C(D)}\sum_{\nu}r_\nu^{-p}\left(\operatorname*{osc}_{2B_\nu}u^{(j)}\right)^{p-1}\omega(B_\nu)+\delta_j \\
&\le C\|\varphi\|_{C(D)}\sum_{\nu}r_\nu^{-p+(p-1)\alpha}\omega(B_\nu)+\delta_j.
\end{aligned}\tag{2.43}$$

Therefore,

$$\sum_{i,k=1}^{n}\int_{D\setminus\Gamma^{j}}\omega|\nabla u|^{p-2}b_{ik}u_{x_k}\varphi_{x_i}dx=O(\varepsilon)+\delta_j.\tag{2.44}$$

Taking into account the density of the class of functions $C_0^1(D)$ in the space $\dot{W}_{p\omega}^1(D)$ and the fact that $u \in W_{p\omega,\mathrm{loc}}^1(D \setminus E)$, we also come to the same equality (2.44) for any function $\varphi \in \dot{W}_{p\omega}^1(D)$. Assuming now that, in (2.44) $\varphi = u\xi^p$, where $\xi \in C_0^\infty(D)$ is a positive function equal to one in G, since $\Gamma^j \subset \Gamma''$ and the integrand is positive, we obtain

$$\lambda\int_{G\setminus\Gamma''}\omega|\nabla u|^{p}dx\le\sum_{i,k=1}^{n}\int_{D\setminus\Gamma^{j}}\omega\xi|\nabla u|^{p-2}b_{ik}u_{x_i}u_{x_k}dx\le\int_{D\setminus\Gamma^{j}}\omega\xi^{p-1}|\nabla u|^{p-1}|\nabla\xi||u|dx+O(\varepsilon)+\beta_j.\tag{2.45}$$

Whence, by means of Young's inequality, we derive

$$\int_{D\setminus\Gamma''}\omega|\nabla u|^{p}dx\le C\int_{D}\omega|\nabla\xi|^{p}|u|^{p}dx=O(1).\tag{2.46}$$

Then, by virtue of the arbitrariness of ε, η ($\mathrm{mes}_n E = 0$), we obtain $u \in W_{p\omega}^1(D)$.

proof of that $u(x)$ is a solution in D. Let us return to relation (2.44), from which, in view of $u \in W^1_{p\omega}(D)$ and $\Gamma'' \subset E'$, we have

$$\sum_{i,k=1}^{n} \int_{D\setminus E'} \omega |\nabla u|^{p-2} b_{ik} u_{x_k} \varphi_{x_i} dx = O(\varepsilon) \tag{2.47}$$

for any $\varphi \in \dot{W}^1_{p\omega}(D)$. By virtue of the arbitrariness of $\varepsilon,\ \eta$, we find

$$\sum_{i,k=1}^{n} \int_{D} \omega |\nabla u|^{p-2} b_{ik} u_{x_k} \varphi_{x_i} dx = 0, \tag{2.48}$$

that is, the function $u \in W^1_{p\omega}(D)$ is a solution of (1.5) throughout the domain D and thereby of (1.1), too. □

Theorem 1.7 is proved. □

Proof of Theorem 1.8. Let $\Lambda_{\omega}^{-p+(p-1)\alpha}(E) > 0$ for some compact set $E \subset D$. Let us use the recent results for a Frosman type lemma with measure [11, 17] and follow the reasoning of the original paper [3]. We come to the following conclusion. There exists a Radon measure μ with a support on the set E, such that $\mu(E) > 0$ and for any ball $B = B(x, r)$ we have

$$\mu(B) \le C r^{-p+(p-1)\alpha} \omega(B). \tag{2.49}$$

Let $u \in \dot{W}^1_{p\omega}$ be a solution of the equation

$$\operatorname{div}(A(x, u, \nabla u)) = \mu \tag{2.50}$$

in the domain $D = B(0, R)$, where B is a sufficiently large ball. The solution of (2.50) is understood in the sense as follows: the integral identity

$$\int_D A(x, u, \nabla u) \cdot \nabla \varphi \, dx = \int_D \varphi \, d\mu \tag{2.51}$$

is fulfilled for any test function $\varphi \in \dot{W}^1_{p\omega}(D)$. Such a solution exists by virtue of $\mu \in (W^1_{p\omega}(D))^*$. Let us show the latter inclusion.

By virtue of $\omega \in A_p$ for $q > p$ and the fact that q is sufficiently close to p, inequality (2.49) implies for $0 < \rho < r$:

$$r^{1-n} (\mu(B(x,\rho)))^{1/q} (\sigma(B(x,\rho)))^{1/p'} \le C \rho^{\alpha(p-1)/q-(n-1)(1-p/q)} (\sigma(B(x,\rho)))^{(p-1)(1/p-1/q)}, \tag{37'}$$

where $\sigma(B(x,\rho)) = \int_{B(x,\rho)} \omega^{-1/(p-1)}(y)dy$. This inequality defines the constant in the Adams inequality

$$\left(\int_{B(0,r)} |u|^q d\mu\right)^{1/q} \leq C_{p,q}(r)\left(\int_{B(0,r)} \omega|\nabla u|^p dx\right)^{1/p}, \tag{2.52}$$

as

$$C_{p,q}(r) = \sup_{x\in B(0,r),0<\rho<4r} \rho^{1-n}(\mu(B(x,\rho)))^{1/q}(\sigma(B(x,\rho)))^{1/p'}. \tag{2.53}$$

Then, by virtue of (37′), we have

$$C_{p,q}(r) \leq Cr^{\alpha(p-1)/q-(n-1)(1-p/q)}(\sigma(B(0,r)))^{(p-1)(1/p-1/q)}. \tag{37''}$$

Whence, by virtue of Hölder inequality, we obtain

$$\int_{B(0,r)} |u|d\mu \leq C_{p,q}(r)(\mu(B(0,r)))^{1/q'}\left(\int_{B(0,r)} \omega|\nabla u|^p dx\right)^{1/p}. \tag{2.54}$$

Taking into account inequalities (2.49), (37″) and the A_p-condition, for any function $u \in C_0^1(B(0,r))$, we have

$$\int_{B(0,r)} |u|d\mu \leq C(\omega(B(0,r)))^{1/p'}r^{1-p+(p-1)\alpha}\left(\int_{B(0,r)} \omega|\nabla u|^p dx\right)^{1/p}, \tag{38'}$$

whence it follows that $\mu \in (W_{p\omega}^1(D))^*$.

Let us, following ideas of [10], show that $u(x) \in C^\alpha(D)$. Let $h \in W_{p\omega}^1(B(x_0,r))$ be a solution of the equation

$$\operatorname{div}(A(x,h,\nabla h)) = 0 \tag{2.55}$$

with the condition $h - u \in \dot{W}_{p\omega}^1(B(x_0,r))$. Then, for it, we have the integral identity

$$\int_{B(x_0,r)} (A(x,u,\nabla u) - A(x,h,\nabla h)) \cdot \nabla v\, dx = \int_{B(x_0,r)} v\, d\mu, \tag{2.56}$$

where $v = u - h$. The integrand in the left-hand part of (2.56) is positive. Therefore, for $\rho < r/2$, we have

$$\begin{aligned}
&\int_{B(x_0,\rho)} (A(x,u,\nabla u) \cdot \nabla u + A(x,h,\nabla h) \cdot \nabla h)dx \\
&\quad = \int_{B(x_0,\rho)} (A(x,h,\nabla h)\nabla u + A(x,u,\nabla u)\nabla h)dx + \int_{B(x_0,r)} v\, d\mu.
\end{aligned} \tag{2.57}$$

By virtue of Young's inequality and conditions (1.2), (1.3), from (2.57), we find

$$\int_{B(x_0,\rho)} \omega|\nabla u|^p dx \leq C\left[\int_{B(x_0,\rho)} \omega|\nabla h|^p dx + \int_{B(x_0,r)} v\, d\mu\right]; \quad C = C(n,p,\lambda). \tag{2.58}$$

From (2.55), we obtain

$$\int_{B(x_0,r)} A(x,h,\nabla h) \cdot \nabla v\, dx = 0, \tag{2.59}$$

whence by virtue of Hölder inequality, we find

$$\int_{B(x_0,r)} \omega|\nabla h|^p dx \leq C\int_{B(x_0,r)} \omega|\nabla u|^p dx; \quad C = C(n,p,\lambda). \tag{41$'$}$$

For the first summand of (2.58), we have the following estimates. According to [14], there exists a positive number $\kappa = \kappa(n,p,\lambda,C_p) \in (0,1)$ such that for the solution of (1.1) the inequality

$$\operatorname*{osc}_{B(x_0,r_1)} h \leq C\left(\frac{r_1}{r_2}\right)^{\kappa} \operatorname*{osc}_{B(x_0,r_2)} h, \tag{2.60}$$

where $C = C(n,p,C_p,\lambda,\kappa)$, is fulfilled for any $r_1 < r_2$. If we take into account the Caccioppoli type estimate (see [14])

$$\int_{B(x_0,r_1)} \omega|\nabla h|^p dx \leq \frac{C}{(r_2 - r_1)^p}\left(\operatorname*{osc}_{B(x_0,r_2)} h\right)^p \omega(B(x_0,r_2)), \tag{2.61}$$

then, by virtue of Moser's inequality, we obtain

$$\left(\sup_{B(x_0,r_2)} h\right)^p \leq \frac{C}{\omega(B(x_0,2r_2))}\int_{B(x_0,2r_2)} \omega(h - h_{r_2}^-)^p dx. \tag{2.62}$$

From (2.60), we derive the estimate

$$\int_{B(x_0,\rho)} \omega|\nabla h|^p dx \le C\left(\frac{\rho}{r}\right)^{-p+p\kappa} \frac{\omega(B(x_0,\rho))}{\omega(B(x_0,r))} \int_{B(x_0,r)} \omega|\nabla h|^p dx. \tag{2.63}$$

Indeed,

$$\begin{aligned}
\int_{B(x_0,\rho)} \omega|\nabla h|^p dx &\le \frac{C}{\rho}\int_{B(x_0,\rho)} \omega|\nabla h|^{p-1}\left|h-h_\rho^-\right|^p dx \le \frac{C}{\rho^p}\int_{B(x_0,\rho)} \omega\left|h-h_\rho^-\right|^p dx \\
&\le \frac{C}{\rho^p}\left(\operatorname*{osc}_{B(x_0,\rho)} h\right)^p \omega(B(x_0,\rho)) \le \frac{C}{\rho^p}\left(\frac{\rho}{r}\right)^{\kappa p}\left(\operatorname*{osc}_{B(x_0,r/2)} h\right)^p \omega(B(x_0,\rho)) \\
&\le \frac{C}{\rho^p}\left(\frac{\rho}{r}\right)^{\kappa p} \omega(B(x_0,\rho))\left(\frac{1}{\omega(B(x_0,r))}\int_{B(x_0,r)} \left|h-h_r^-\right|^p dx\right) \\
&\le \frac{C}{\rho^p}\left(\frac{\rho}{r}\right)^{-p+p\kappa} \frac{\omega(B(x_0,\rho))}{\omega(B(x_0,r))}\int_{B(x_0,r)} \omega|\nabla h|^p dx,
\end{aligned} \tag{2.64}$$

where h_r^- is the lower bound of the function h in the ball $B(x_0,r)$. Inequality (2.63) is proved.

Using the estimate (2.63) in (2.58), by virtue of (41′), we have for $0<\rho<r/2$

$$\int_{B(x_0,\rho)} \omega|\nabla u|^p dx \le C\left(\frac{\rho}{r}\right)^{-p+p\kappa} \frac{\omega(B(x_0,\rho))}{\omega(B(x_0,r))} \int_{B(x_0,r)} \omega|\nabla u|^p dx + \int_{B(x_0,r)} v\, d\mu. \tag{2.65}$$

Now, let us derive an estimate for the last summand in (2.65). To this end, we use inequality (38′) to obtain

$$\left(\int_{B(x_0,r)} v\, d\mu\right) \le C(p,n,C_p)\left(\int_{B(x_0,r)} \omega|\nabla v|^p dx\right)^{1/p}, \tag{2.66}$$

where for the constant we have the estimate

$$C(p,n,C_p) \le Cr^{1-p+(p-1)\alpha}(\omega(B(x_0,r)))^{1/p'} \tag{2.67}$$

By virtue of (2.65) and (2.67), we find

$$\begin{aligned}
\int_{B(x_0,\rho)} \omega|\nabla u|^p dx &\le C\left(\frac{\rho}{r}\right)^{-p+p\kappa} \frac{\omega(B(x_0,\rho))}{\omega(B(x_0,r))} \int_{B(x_0,r)} \omega|\nabla u|^p dx \\
&\quad + C\left[\frac{\omega(B(x_0,r))}{r^{(1-\alpha)p}}\right]^{1/p'}\left(\int_{B(x_0,r)} \omega|\nabla u|^p dx\right)^{1/p},
\end{aligned} \tag{2.68}$$

whence, by virtue of Young's inequality, we obtain

$$\int_{B(x_0,\rho)} \omega|\nabla u|^p dx \leq C\left[\left(\frac{\rho}{r}\right)^{-p+p\kappa}\frac{\omega(B(x_0,\rho))}{\omega(B(x_0,r))}+\varepsilon\right]\int_{B(x_0,r)} \omega|\nabla u|^p dx + C_\varepsilon \omega(B(x_0,r))r^{-p+p\alpha}. \tag{2.69}$$

Assuming that $0<\alpha<\kappa$, from (2.69) and Lemma 1.5, we obtain the estimate

$$\int_{B(x_0,\rho)} \omega|\nabla u|^p dx \leq C\left(\frac{\rho}{r}\right)^{-p+p\alpha}\frac{\omega(B(x_0,\rho))}{\omega(B(x_0,r))}\int_{B(x_0,r)} \omega|\nabla u|^p dx + C\omega(B(x_0,\rho))\rho^{-p+p\alpha}. \tag{2.70}$$

This inequality implies

$$\begin{aligned}\int_{B(x_0,\rho)} |\nabla u| dx &\leq \left(\int_{B(x_0,\rho)} \omega|\nabla u|^p dx\right)^{1/p}\left(\int_{B(x_0,\rho)} \omega^{-(1/(p-1))} dx\right)^{1/p'} \\ &\leq C\left(\int_{B(x_0,\rho)} \omega\, dx\right)^{1/p}\left(\int_{B(x_0,\rho)} \omega^{-(1/(p-1))} dx\right)^{1/p'}\rho^{-1+\alpha} \leq C\rho^{n-1+\alpha},\end{aligned} \tag{2.71}$$

whence, by virtue of the Poincaré inequality, we obtain

$$\int_{B(x_0,\rho)} \left|u-(u)_\rho\right|^p dx \leq C\rho^{n+\alpha}, \tag{2.72}$$

where $(u)_\rho$ is average of the function u with respect to the ball $B(x_0,\rho)$.

By (2.72) and Campanato's Lemma 1.4, we find $u \in C^\alpha$.

Theorem 1.8 is proved. □

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