Research Article

Two-Parametric Conditionally Oscillatory Half-Linear Differential Equations

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We study perturbations of the nonoscillatory half-linear differential equation $(r(t)\Phi(x'))' + c(t)\Phi(x) = 0$, $\Phi(x) := |x|^{p-2}x$, $p > 1$. We find explicit formulas for the functions $\hat{r}$, $\hat{c}$ such that the equation $[(r(t) + \lambda\hat{r}(t))\Phi(x')]' + [c(t) + \mu\hat{c}(t)]\Phi(x) = 0$ is conditionally oscillatory, that is, there exists a constant γ such that the previous equation is oscillatory if $\mu - \lambda > \gamma$ and nonoscillatory if $\mu - \lambda < \gamma$. The obtained results extend the previous results concerning two-parametric perturbations of the half-linear Euler differential equation.

1. Introduction

Conditionally oscillatory equations play an important role in the oscillation theory of the Sturm-Liouville second-order differential equation

$$(r(t)x')' + c(t)x = 0, \tag{1.1}$$

with positive continuous functions r, c. Equation (1.1) with λc instead of c is said to be *conditionally oscillatory* if there exists $\lambda_0 > 0$, the so-called *oscillation constant* of (1.1), such that this equation is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. A typical example of a conditionally oscillatory equation is the Euler differential equation

$$x'' + \frac{\lambda}{t^2}x = 0, \tag{1.2}$$

which has the oscillation constant $\lambda_0 = 1/4$ as can be verified by a direct computation when looking for solutions of (1.2) in the form $x(t) = t^\alpha$. This leads to the classical Kneser (non)oscillation criterion which states that (1.1) with $r(t) \equiv 1$ is oscillatory provided

$$\liminf_{t \to \infty} t^2 c(t) > \frac{1}{4}, \tag{1.3}$$

and nonoscillatory if

$$\limsup_{t \to \infty} t^2 c(t) < \frac{1}{4}. \tag{1.4}$$

This shows that the potential $c(t) = t^{-2}$ is the border line between oscillation and nonoscillation. Note that the concept of conditional oscillation of (1.1) was introduced in [1].

The linear oscillation theory extends almost verbatim to the half-linear differential equation

$$\left(r(t)\Phi(x')\right)' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \ p > 1, \tag{1.5}$$

including the definition of conditional oscillation. The half-linear version of Euler equation (1.2) is the equation

$$\left(\Phi(x')\right)' + \frac{\lambda}{t^p}\Phi(x) = 0, \tag{1.6}$$

which has the oscillation constant $\lambda_0 = \gamma_p := ((p-1)/p)^p$, and (non)oscillation criteria (1.3), (1.4) extend in a natural way to (1.5) with $r(t) \equiv 1$. A complementary concept to the conditional oscillation is the concept of strong (non)oscillation. Equation (1.5) with λc instead of c is said to be *strongly (non)oscillatory* if it is (non)oscillatory for every $\lambda > 0$. Sometimes, strongly oscillatory equations are regarded as conditionally oscillatory with the oscillation constant $\lambda_0 = 0$ and strongly nonoscillatory as conditionally oscillatory with the oscillation constant $\lambda_0 = \infty$. We refer to [2] for results along this line.

In our paper, we are motivated by a statement presented in [3, 4], where the two-parametric perturbation of the Euler differential equation with the critical coefficient

$$\left(\Phi(x')\right)' + \frac{\gamma_p}{t^p}\Phi(x) = 0 \tag{1.7}$$

is investigated. It is shown there that the equation

$$\left[\left(1 + \frac{\lambda}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right]\Phi(x) = 0 \tag{1.8}$$

is oscillatory if $\mu - \gamma_p \lambda > \mu_p := (1/2)((p-1)/p)^{p-1}$ and nonoscillatory in the opposite case. Note that an important role in proving the results of [4] is played by the fact that we know explicitly the solution $h(t) = t^{(p-1)/p}$ of (1.7).

Here, we treat the problem of conditional oscillation in the following general setting. We suppose that (1.5) is nonoscillatory and that h is its eventually positive solution. We find explicit formulas for the functions $\hat{r}, \hat{c}$ such that the equation

$$\left[(r(t) + \lambda \hat{r}(t))\Phi(x')\right]' + \left[c(t) + \mu \hat{c}(t)\right]\Phi(x) = 0 \tag{1.9}$$

is conditionally oscillatory, that is, there exists a constant γ such that (1.9) is oscillatory if $\mu - \lambda > \gamma$ and nonoscillatory if $\mu - \lambda < \gamma$.

The setup of the paper is as follows. In the next section, we present some statements of the half-linear oscillation theory. Section 3 is devoted to the so-called modified Riccati equation associated with (1.5) and (1.9). The main result of the paper, the construction of the functions $\hat{r}, \hat{c}$ such that (1.9) is two-parametric conditionally oscillatory, is presented in Section 4.

2. Auxiliary Results

As we have already mentioned in the previous section, the linear oscillation theory extends almost verbatim to half-linear equation (1.5). The word "almost" reflects the fact that not all linear methods can be extended to (1.5), some results for (1.5) are the same as those for (1.1), but to prove them, one has to use different methods than in the linear case. A typical method of this kind is the following transformation formula. If $f(t) \neq 0$ is a sufficiently smooth function and functions x, y are related by the formula $x = f(t)y$, then we have the identity

$$f(t)\left[(r(t)x')' + c(t)x\right] = (R(t)y')' + C(t)y, \tag{2.1}$$

where

$$R(t) = r(t)f^2(t), \qquad C(t) = f(t)\left[(r(t)f'(t))' + c(t)f(t)\right]. \tag{2.2}$$

In particular, x is a solution of (1.1) if and only if y is a solution of the equation $(Ry')' + Cy = 0$. The transformation identity (2.1) *does not extend* to (1.5).

To illustrate the meaning of this fact in the conditional oscillation of (1.1) and (1.5), suppose that (1.1) is nonoscillatory and let h be its so-called *principal solution* (see [5, Chapter XI]), that is, a solution such that $\int^\infty r^{-1}(t)h^{-2}(t)dt = \infty$. We would like to find a function $\hat{c}$ such that the equation

$$(r(t)x')' + (c(t) + \mu \hat{c}(t))x = 0 \tag{2.3}$$

is conditionally oscillatory and to find its oscillation constant. The transformation $x = h(t)y$ transforms (1.1) into the one term equation $(r(t)h^2(t)y')' = 0$ and the transformation of independent variable $s = \int^t r^{-1}(\tau)h^{-2}(\tau)d\tau$ further to the equation $d^2y/ds^2 = 0$. Now,

from (1.2), we know that the "right" perturbation term in the last equation is $1/s^2$ with the oscillation constant $1/4$. Substituting back for s, we get the conditionally oscillatory equation

$$(R(t)y')' + \frac{\mu}{R(t)\left(\int^t R^{-1}(s)ds\right)^2}y = 0, \quad R(t) = r(t)h^2(t), \tag{2.4}$$

and the back transformation $y = h^{-1}(t)x$ results in the conditionally oscillatory equation

$$(r(t)x')' + \left[c(t) + \frac{\mu}{h^2(t)R(t)\left(\int^t R^{-1}(s)ds\right)^2}\right]x = 0, \tag{2.5}$$

with the oscillation constant $\mu_0 = 1/4$. The previous result is one of the main statements of [6], but it was proved there by a different method.

In the next section, we will show how to modify this method to be applicable to half-linear equations. At this moment, we present the result of [7] with the classical (i.e., one parametric) conditional oscillation of (1.5). Let h be a positive solution of (1.5) such that $h'(t) \neq 0$ for large t. We denote

$$R(t) := r(t)h^2(t)|h'(t)|^{p-2}, \qquad G(t) := r(t)h(t)\Phi(h'(t)), \tag{2.6}$$

$$\hat{c}(t) = \frac{1}{|h(t)|^p R(t)\left(\int^t R^{-1}(s)ds\right)^2}. \tag{2.7}$$

Theorem 2.1. *Suppose that* (1.5) *possesses a nonoscillatory solution h such that $h'(t) \neq 0$ for large t, and R, G are given by* (2.6). *If*

$$\int^\infty \frac{dt}{R(t)} = \infty, \qquad \liminf_{t\to\infty}|G(t)| > 0, \tag{2.8}$$

then the equation

$$(r(t)\Phi(x'))' + [c(t) + \mu\hat{c}(t)]\Phi(x) = 0 \tag{2.9}$$

is conditionally oscillatory, and its oscillation constant is $\mu_0 = 1/2q$, where q is the conjugate exponent to p, that is, $1/p + 1/q = 1$.

Note that in the linear case $p = 2$, the function $f(t) = h(t)\sqrt{\int^t r^{-1}(\tau)h^{-2}(\tau)d\tau}$ is a solution of (2.9) with $\mu = \mu_0 = 1/4$. In the general half-linear case, an explicit solution of (2.9) is no longer known, but we are able to "estimate" this solution. The next statement, which is also taken from [7], presents a result along this line.

Theorem 2.2. *Suppose that* (2.8) *holds and let* $f(t) = h(t)(\int^t R^{-1}(s)ds)^{1/p}$, *then a solution of* (2.9) *with* $\mu = 1/2q$ *is of the form*

$$x(t) = f(t)\left(1 + O\left(\left(\int^t R^{-1}(s)ds\right)^{-1}\right)\right),\qquad(2.10)$$

and (suppressing the argument t)

$$f\left[(r\Phi(f'))' + \left(c + \frac{1}{2qh^p R\left(\int^t R^{-1}\right)^2}\right)\Phi(f)\right]$$

$$= -\frac{(p-1)(p-2)G'}{G^2\left(\int^t R^{-1}\right)} - \frac{(p-1)(p-2)}{3p^3 G^3\left(\int^t R^{-1}\right)^2}\left[(p-3)G' + 2pr|h'|^p\right]\qquad(2.11)$$

$$+ O\left(G^{-3}\left(\int^t R^{-1}\right)^{-3}\right)\left[\frac{G'}{pG^2} - \frac{(p^3 - 4p^2 + 11p - 6)h'}{2p^3 h} - \frac{1}{qR\left(\int^t R^{-1}\right)}\right],$$

as $t \to \infty$.

The last statement presented in this section is the so-called *reciprocity principle*. Let x be a solution of (1.5) and let $u := r\Phi(x')$ be its *quasiderivative*, then u is a solution of the reciprocal equation

$$\left(c^{1-q}(t)\Phi^{-1}(u')\right)' + r^{1-q}(t)\Phi^{-1}(u) = 0,\qquad(2.12)$$

where $\Phi^{-1}(u) = |u|^{q-2}u$ is the inverse function of Φ.

3. Modified Riccati Equation

Suppose that λ and $\hat{r}$ in (1.9) are such that $r(t) + \lambda\hat{r}(t) > 0$. Let $x(t) \neq 0$ in an interval I be a solution of (1.9), and let $w = (r + \lambda\hat{r})\Phi(x'/x)$. Then, w solves in I the "standard" Riccati equation

$$w' + c(t) + \mu\hat{c}(t) + (p-1)[r(t) + \lambda\hat{r}(t)]^{1-q}|w|^q = 0.\qquad(3.1)$$

More precisely, the following statement holds.

Lemma 3.1 ([8, Theorem 2.2.1]). *The following statements are equivalent:*

  (i) *equation* (1.9) *is nonoscillatory;*

  (ii) *equation* (3.1) *has a solution on an interval* $[T, \infty)$;

(iii) *there exists a continuously differentiable function w such that*

$$w' + c(t) + \mu \widehat{c}(t) + (p-1)[r(t) + \lambda \widehat{r}(t)]^{1-q}(t)|w|^q \le 0 \tag{3.2}$$

on an interval $[T, \infty)$.

In the linear case, if x is a solution of (1.1), $x = f(t)y$, and $v = rf^2y'/y$ is the Riccati variable corresponding to the equation on the right-hand side in (2.1), then $v = f^2(w - w_f)$ where $w = rx'/x$, $w_f = rf'/f$. This suggests to investigate the function $v = f^p(w - w_f)$ in the half-linear case, and this leads to the *modified Riccati equation* introduced in the next statement which is taken from [4] with a modification from [3].

Lemma 3.2. *Suppose that f is a positive differentiable function, $w_f = (r + \lambda \widehat{r})\Phi(f'/f)$, and w is a continuously differentiable function, and put $v = f^p(w - w_f)$, then the following identity holds:*

$$f^p(t)\left[w' + c(t) + \mu \widehat{c}(t) + (p-1)(r(t) + \lambda \widehat{r}(t))^{1-q}|w|^q\right]$$
$$= v' + f(t)\left[\ell(f(t)) + \widehat{\ell}(f(t))\right] + (p-1)(r(t) + \lambda \widehat{r}(t))^{1-q}f^{-q}(t)\mathcal{G}(t,v), \tag{3.3}$$

where

$$\ell(f) = (r(t)\Phi(f'))' + c(t)\Phi(f), \qquad \widehat{\ell}(f) = \lambda(\widehat{r}(t)\Phi(f'))' + \mu \widehat{c}(t)\Phi(f), \tag{3.4}$$

$$\mathcal{G}(t,v) = |v + \Omega(t)|^q - q\Phi^{-1}(\Omega(t))v - |\Omega(t)|^q, \quad \Omega := (r + \lambda \widehat{r})f\Phi(f'). \tag{3.5}$$

In particular, if w is a solution of (3.1), then v is a solution of the modified Riccati equation

$$v' + f(t)\left[\ell(f(t)) + \widehat{\ell}(f(t))\right] + (p-1)(r(t) + \lambda \widehat{r}(t))^{1-q}f^{-q}(t)\mathcal{G}(t,v) = 0. \tag{3.6}$$

Conversely, if v is a solution of (3.6), then $w = w_f + f^{-p}v$ is a solution of (3.1).

Observe that in case $f \equiv 1$, the modified Riccati equation (3.6) reduces to the standard Riccati equation (3.1).

Next, we will investigate the function $\mathcal{G}$ in (3.5). First, we present a result from [4, Lemmas 5 and 6].

Lemma 3.3. *The function $\mathcal{G}$ defined in (3.5) has the following properties.*

(i) $\mathcal{G}(t,v) \ge 0$ *with the equality if and only if $v = 0$.*

(ii) *If $q \ge 2$, one has the inequality*

$$\mathcal{G}(t,v) \ge \frac{q}{2}|\Omega(t)|^{q-2}v^2. \tag{3.7}$$

Now, we concentrate on an estimate of the function $\mathcal{G}$ in case $q < 2$.

Lemma 3.4. *Suppose that $q < 2$ and $\lim_{t \to \infty} |\Omega(t)| = \infty$, then there is a constant $\beta > 0$ such that for $v \in (-\infty, -v_0]$, $v_0 > 0$, and large t*

$$\mathcal{G}(t, v) \geq \beta |\Omega(t)|^{q-2} |v|^q. \tag{3.8}$$

Proof. Consider the function

$$\mathcal{H}(t, v) = \begin{cases} \dfrac{\mathcal{G}(t, v)}{|v|^q}, & \text{for } v \neq 0, \\ 0, & \text{for } v = 0. \end{cases} \tag{3.9}$$

First of all,

$$\lim_{v \to \pm\infty} \mathcal{H}(t, v) = 1, \qquad \lim_{v \to 0} \mathcal{H}(t, v) = 0. \tag{3.10}$$

Now, we compute local extrema of $\mathcal{H}$ with respect to v. We have (suppressing the argument t)

$$
\begin{aligned}
\mathcal{H}_v &= \frac{1}{|v|^{2q}} \left\{ \left[q\Phi^{-1}(v + \Omega) - q\Phi^{-1}(\Omega) \right] |v|^q - q\Phi^{-1}(v) \left[|v + \Omega|^q - q\Phi^{-1}(\Omega)v - |\Omega|^q \right] \right\} \\
&= \frac{q}{v^2 \Phi^{-1}(v)} \left\{ v\Phi^{-1}(v + \Omega) - v\Phi^{-1}(\Omega) - |v + \Omega|^q + q\Phi^{-1}(\Omega)v + |\Omega|^q \right\} \\
&= \frac{q}{v^2 \Phi^{-1}(v)} \left\{ -\Omega\Phi^{-1}(v + \Omega) + (q - 1)\Phi^{-1}(\Omega)v + |\Omega|^q \right\}.
\end{aligned}
\tag{3.11}
$$

Denote $\mathcal{N}(v)$ the function in braces on the last line of the previous computation. We have $\mathcal{N}(0) = 0$,

$$
\begin{aligned}
\mathcal{N}'(v) &= -(q - 1)\Omega |v + \Omega|^{q-2} + (q - 1)\Phi^{-1}(\Omega) \\
&= (q - 1)\Omega \left[-|v + \Omega|^{q-2} + |\Omega|^{q-2} \right] \\
&= 0
\end{aligned}
\tag{3.12}
$$

if and only if $v = 0$ and $v = -2\Omega$, and

$$\mathcal{N}''(v) = -(q - 1)(q - 2)\Omega |v + \Omega|^{q-3} \operatorname{sgn}(v + \Omega). \tag{3.13}$$

This means that $v = 0$ is the local minimum and $v = -2\Omega$ is the local maximum of the function $\mathcal{N}$. Using this result, an examination of the graph of the function $\mathcal{H}$ shows that this function has the local minimum at $v = 0$ and a local maximum in the interval $(-\infty, -2\Omega)$ if $\Omega > 0$,

and this maximum is in $(-2\Omega, \infty)$ if $\Omega < 0$. Next, denote v^* the value for which $\mathcal{H}(t, v^*) = 1$. Consequently, for any $v_0 > 0$, it follows from (3.10) that

$$\inf_{v \in (-\infty, -v_0]} \mathcal{H}(t, v) = \mathcal{H}(t, -\overline{v}) = \frac{1}{|\overline{v}|^q} \left[|\Omega - \overline{v}|^q + q\overline{v}\Phi^{-1}(\Omega) - |\Omega|^q \right], \tag{3.14}$$

where

$$\overline{v} = \begin{cases} -v^* & \text{if } -v_0 < v^* < 0, \\ v_0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Next, we want to investigate the dependence of this infimum on Ω when $|\Omega| \to \infty$. To this end, we investigate the function $F(x) = |x - a|^q + qa\Phi^{-1}(x) - |x|^q$ for $x \to \pm\infty$, $a \in \mathbb{R}$ being a parameter. We have (using the expansion formula for $(1 + x)^\alpha$)

$$F(x) = \Phi^{-1}(x) \left\{ \frac{|x - a|^q - |x|^q}{\Phi^{-1}(x)} + qa \right\} = \Phi^{-1}(x) \left\{ x \left[\left(1 - \frac{a}{x} \right)^q - 1 \right] + qa \right\}$$

$$= \Phi^{-1}(x) \left(\binom{q}{2} \frac{a^2}{x} + o(x^{-1}) \right) = a^2 \binom{q}{2} |x|^{q-2} (1 + o(1)), \tag{3.16}$$

as $|x| \to \infty$. Consequently, if $\lim_{t \to \infty} |\Omega(t)| = \infty$, there exists a constant $\beta > 0$ such that (3.8) holds. $\square$

Now, we are ready to formulate a complement of [9, Theorem 2] which is presented in that paper under the assumption that the function Ω is bounded.

Theorem 3.5. *Let f be a positive continuously differentiable function such that $f'(t) \neq 0$ for large t. Suppose that $\int^\infty \mathcal{R}^{-1}(t)\,\mathrm{d}t = \infty$, where $\mathcal{R} = (r + \lambda\widehat{r}) f^2 |f'|^{p-2}$, $C(t) \geq 0$ for large t, and $\lim_{t \to \infty} |\Omega(t)| = \infty$, then all possible proper solutions (i.e., solutions which exist on some interval of the form $[T, \infty)$) of the equation*

$$v' + C(t) + (p - 1)(r(t) + \lambda\widehat{r}(t))^{1-q} f^{-q}(t) \mathcal{G}(t, v) = 0 \tag{3.17}$$

are nonnegative.

Proof. First consider the case $q < 2$. Let $v_0 > 0$ be arbitrary. By Lemma 3.4, there exists $T_0 \in \mathbb{R}$ and $\beta > 0$ such that for $t \geq T_0$ and $v \in (-\infty, -v_0]$,

$$(p - 1)(r + \lambda\widehat{r})^{1-q} f^{-q} \mathcal{G}(t, v) \geq \beta(p - 1)(r + \lambda\widehat{r})^{1-q} f^{-q} |\Omega|^{q-2} |v|^q = (p - 1)\beta \frac{|v|^q}{\mathcal{R}}. \tag{3.18}$$

Suppose that v is the solution of (3.17) such that $v(t_0) = -v_0$ for some $t_0 \geq T_0$, then

$$v' + C(t) + (p - 1)\beta \frac{|v|^q}{\mathcal{R}(t)} \leq 0, \tag{3.19}$$

for $t \geq t_0$ for which the solution v exists. Now, we use the same argument as in the proof of Theorem 2 in [9]. Consider the equation

$$z' + C(t) + (p-1)\beta\frac{|z|^q}{\mathcal{R}(t)} = 0. \tag{3.20}$$

This is the standard Riccati equation corresponding to the half-linear equation

$$\left(\mathcal{R}^{p-1}(t)\Phi(x')\right)' + \beta^{p-1}C(t)\Phi(x) = 0. \tag{3.21}$$

Assumptions of theorem imply, by [8, Corollary 4.2.1], that all proper solutions of (3.20) are nonnegative. It means that any solution of (3.20) which starts with a negative initial condition blows down to $-\infty$ in a finite time. Inequality (3.19) implies that if z is the solution of (3.20) satisfying $z(t_0) = v(t_0) = -v_0$, that is, z starts with the same initial value as the solution v of (3.17), then v decreases faster than z. In particular, if z blows down to $-\infty$ at a finite time, then v does as well. This means that all proper solutions of (3.17), if any, are nonnegative.

In case $q \geq 2$, we proceed in a similar way. We use (3.7) and we compare (3.17) with the equation

$$z' + C(t) + \frac{p}{2}\frac{z^2}{\mathcal{R}(t)} = 0, \tag{3.22}$$

which is the standard Riccati equation corresponding to the linear equation

$$\tag{3.23}$$
$$\left(\mathcal{R}(t)x'\right)' + \frac{p}{2}C(t)x = 0.$$

Then, reasoning in the same way as in case $q < 2$, we obtain the conclusion that all proper solutions of (3.17) are nonnegative also in this case. □

4. Two-Parametric Conditional Oscillation

Recall that h is a positive solution of (1.5) such that $h'(t) \neq 0$ for large t, $g = r\Phi(h')$ is its quasiderivative, R, G are given by (2.6), and $\hat{c}$ is given by (2.7). Recall also that the quasiderivative g is a solution of the reciprocal equation (2.12), denote by

$$\widetilde{G} := c^{1-q}g\Phi^{-1}(g') = -rh\Phi(h'), \qquad \widetilde{R} := c^{1-q}g^2|g'|^{q-2} = \frac{r^2|h'|^{2p-2}}{ch^{p-2}} \tag{4.1}$$

the "reciprocal" analogues of G and R, and define

$$\hat{r}(t) = \frac{1}{|h'(t)|^p\widetilde{R}(t)\left(\int^t\widetilde{R}^{-1}(s)ds\right)^2}. \tag{4.2}$$

Our main result reads as follows.

Theorem 4.1. *Suppose that conditions* (2.8) *hold. Further, suppose that*

$$\lim_{t \to \infty} \frac{\widehat{r}(t)}{r(t)} = 0, \tag{4.3}$$

and that there exist limits

$$\lim_{t \to \infty} \frac{r(t)|h'(t)|^p}{c(t)h^p(t)}, \qquad \lim_{t \to \infty} \frac{(\widehat{r}(t)\Phi(f'(t)))'}{\widehat{c}(t)\Phi(f(t))}, \tag{4.4}$$

the second one being finite, where $f(t) = h(t)(\int^t R^{-1}(s)ds)^{1/p}$. *If* $\mu - \lambda < 1/2q$, *then* (1.9) *is nonoscillatory; if* $\mu - \lambda > 1/2q$, *then it is oscillatory.*

Proof. First consider the case $\mu = 0$ in (1.9), that is, we consider the equation

$$[(r(t) + \lambda\widehat{r}(t))\Phi(x')]' + c(t)\Phi(x) = 0. \tag{4.5}$$

The quantities $\widetilde{G}$ and $\widetilde{R}$ defined in (4.1) satisfy

$$\widetilde{G} = -rh\Phi(h') = -G\,,$$

$$\widetilde{R} = \frac{r^2|h'|^{2p-2}}{ch^{p-2}} = -\frac{h(r\Phi(h'))^2}{(r\Phi(h'))'}\,, \tag{4.6}$$

hence, integrating by parts,

$$\int^t \widetilde{R}^{-1}(s)ds = -\int^t \frac{1}{h(s)} \frac{[r(s)\Phi(h'(s))]'}{[r(s)\Phi(h'(s))]^2}ds$$

$$= \frac{1}{h(t)r(t)\Phi(h'(t))} + \int^t \frac{h'(s)}{h^2(s)} \frac{1}{r(s)\Phi(h'(s))}ds \tag{4.7}$$

$$= \frac{1}{G(t)} + \int^t R^{-1}(s)ds.$$

Consequently, conditions (2.8) imply that corresponding conditions for $\widetilde{G}$ and $\widetilde{R}$ also hold. This means, in view of Theorem 2.1 (applied to the reciprocal equation (2.12)), that the equation

$$\left(c^{1-q}(t)\Phi^{-1}(u')\right)' + \left[r^{1-q}(t) + \frac{\lambda}{|g(t)|^q\widetilde{R}(t)\left(\int^t \widetilde{R}^{-1}(s)ds\right)^2}\right]\Phi^{-1}(u) = 0 \tag{4.8}$$

is oscillatory for $\lambda > 1/2p$ and nonoscillatory in the opposite case.

The reciprocal equation to (4.5) is the equation

$$\left(c^{1-q}(t)\Phi^{-1}(u')\right)' + (r(t) + \lambda\widehat{r}(t))^{1-q}\Phi^{-1}(u) = 0. \tag{4.9}$$

Since (4.3) holds, we have

$$(r + \lambda\widehat{r})^{1-q} = r^{1-q}\left(1 + \frac{\lambda\widehat{r}}{r}\right)^{1-q} = r^{1-q}\left(1 + \frac{(1-q)\lambda\widehat{r}}{r} + o\left(\frac{\widehat{r}}{r}\right)\right), \tag{4.10}$$

as $t \to \infty$. Hence, we can rewrite (4.9) in the following form:

$$\left(c^{1-q}(t)\Phi^{-1}(u')\right)' + r^{1-q}(t)\left(1 + \frac{(1-q)\lambda\widehat{r}(t)}{r(t)} + o\left(\frac{\widehat{r}(t)}{r(t)}\right)\right)\Phi^{-1}(u) = 0. \tag{4.11}$$

Let $\lambda > -1/2q$ what is equivalent to $\lambda(1-q) < 1/2p$, then, in view of (4.3), there exists $\widetilde{\lambda}$ such that $\lambda(1-q) < \widetilde{\lambda} < 1/2p$, hence, for large t,

$$r^{1-q}\left(1 + \frac{\lambda(1-q)\widehat{r}}{r} + o\left(\frac{\widehat{r}}{r}\right)\right) < r^{1-q}\left(1 + \frac{\widetilde{\lambda}\widehat{r}}{r}\right) = r^{1-q} + \frac{\widetilde{\lambda}}{|g|^q\widetilde{R}\left(\int^t \widetilde{R}^{-1}(s)ds\right)^2}. \tag{4.12}$$

This means that the equation

$$\left(c^{1-q}(t)\Phi^{-1}(u')\right)' + \left[r^{1-q}(t) + \frac{\widetilde{\lambda}}{|g(t)|^q\widetilde{R}(t)\left(\int^t \widetilde{R}^{-1}(s)ds\right)^2}\right]\Phi^{-1}(u) = 0 \tag{4.13}$$

is a majorant of (4.9) and this majorant is nonoscillatory by **Theorem 2.1** applied to (4.8). So (4.9) is also nonoscillatory, and hence (4.5) is nonoscillatory as well. The same argument implies oscillation of (4.5) if $\lambda < -1/2q$.

Now, we turn our attention to the general case $\mu \neq 0$. Let $f := h(\int^t R^{-1}(s)ds)^{1/p}$, and consider the term

$$f\left[\ell(f) + \widehat{\ell}(f)\right] \tag{4.14}$$

appearing in the modified Riccati equation (3.6), where the operators $\ell, \widehat{\ell}$ are defined by (3.4). In order to use the asymptotic formula from **Theorem 2.2**, we write $f[\ell(f) + \widehat{\ell}(f)] = A + B$, where

$$A = f\left[(r\Phi(f'))' + \left(c + \frac{1}{2q}\widehat{c}\right)\Phi(f)\right],$$

$$B = f\left[\lambda(\widehat{r}\Phi(f'))' + \left(\mu - \frac{1}{2q}\right)\widehat{c}\Phi(f)\right]. \tag{4.15}$$

Let $L \in \mathbb{R}$ be the second limit in (4.4), that is,

$$(\hat{r}\Phi(f'))' = L\hat{c}\Phi(f)(1 + o(1)) \quad \text{as } t \longrightarrow \infty. \tag{4.16}$$

The leading term in the expression A is const $G'G^{-2}(\int^t R^{-1}(s)\mathrm{d}s)^{-1}$ by Theorem 2.2, while, concerning the asymptotics of B,

$$B = f\hat{c}\Phi(f)\left[L\lambda + \mu - \frac{1}{2q} + o(1)\right] = \frac{1}{R\left(\int^t R^{-1}(s)\mathrm{d}s\right)}\left[L\lambda + \mu - \frac{1}{2q} + o(1)\right], \tag{4.17}$$

as $t \to \infty$. The existence of the first limit in (4.4) implies that there exists the limit

$$\lim_{t\to\infty} \frac{G'(t)G^{-2}(t)}{R^{-1}(t)} = \lim_{t\to\infty} \frac{r(t)h^2(t)|h'(t)|^{p-2}(r(t)|h'(t)|^p - c(t)h^p(t))}{(r(t)h(t)\Phi(h'(t)))^2}$$

$$= 1 - \lim_{t\to\infty} \frac{c(t)h^p(t)}{r(t)|h'(t)|^p}. \tag{4.18}$$

The limit in (4.18) must be 0, which follows from the l'Hospital rule and the fact that the integral of R^{-1} is divergent, while the integral of $G'G^{-2}$ is convergent by the second assumption in (2.8). This means that the term B dominates A; hence, $A(t) + B(t) > 0$ for large t if $L\lambda + \mu - 1/2q > 0$ and $A(t) + B(t) < 0$ for large t if $L\lambda + \mu - 1/2q < 0$.

Now, it remains to prove that these inequalities imply (non)oscillation of (1.9) and that $L = -1$.

To prove the nonoscillation, let $L\lambda + \mu - 1/2q < 0$, that is, $A(t) + B(t) < 0$ for large t, and let $\mathcal{G}$ be defined by (3.5). By Lemma 3.3(i) $v = 0$ is a solution of the inequality

$$v' + A(t) + B(t) + (p-1)(r(t) + \lambda\hat{r}(t))^{1-q}f^{-q}(t)\mathcal{G}(t,v) \leq 0, \tag{4.19}$$

for large t, and by identity (3.3) in Lemma 3.2 we obtain that $w = (r + \lambda\hat{r})\Phi(f'/f)$ satisfies the Riccati inequality (3.2), that is, (1.9) is nonoscillatory by Lemma 3.1(iii).

To prove the oscillation, let $L\lambda + \mu - 1/2q > 0$, that is, $A(t) + B(t) > 0$ for large t. Observe that for $t \to \infty$

$$\int^t f^p(s)\hat{c}(s)\mathrm{d}s = \int^t \frac{1}{R(s)\left(\int^t R^{-1}(\tau)\mathrm{d}\tau\right)}\mathrm{d}s = \log\left(\int^t R^{-1}(s)\mathrm{d}s\right) \longrightarrow \infty, \tag{4.20}$$

and hence $\int^\infty B(t)\mathrm{d}t = \infty$, which consequently means that $\int^\infty(A(t) + B(t))\mathrm{d}t = \infty$. Here, we have used the fact that the integral of the leading term in A and also integrals of other terms in the asymptotic formula of Theorem 2.2 are convergent, see [7, page 161]. Suppose, on the contrary, that (1.9) is nonoscillatory. Then by Lemma 3.1, there exists a solution w of the associated Riccati equation (3.1) for large t and, by Lemma 3.2, the function $v = f^p(w - w_f)$,

where $w_f = (r + \lambda\widehat{r})\Phi(f'/f)$, is a solution of the modified Riccati equation (3.6) for large t. Integrating (3.6), we get

$$
\begin{aligned}
v(T) - v(t) = {} & \int_T^t (A(s) + B(s))\mathrm{d}s \\
& + (p-1)\int_T^t (r(s) + \lambda\widehat{r}(s))^{1-q} f^{-q}(s)\mathcal{G}(s, v(s))\mathrm{d}s.
\end{aligned}
\tag{4.21}
$$

Now, we use Theorem 3.5. In view of (2.8) and (4.3), we have for $t \to \infty$,

$$
\begin{aligned}
|\Omega(t)| = {} & (r(t) + \lambda\widehat{r}(t))f(t)\left|\Phi(f'(t))\right| \\
= {} & r(t)(1 + o(1))h(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)^{1/p} |\Phi(h'(t))|\left(\int^t R^{-1}(s)\mathrm{d}s\right)^{(p-1)/p} \\
& \times \left(1 + \frac{1}{pG(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)}\right)^{p-1} \\
= {} & |G(t)|\left(\int^t R^{-1}(s)\mathrm{d}s\right)(1 + o(1)) \longrightarrow \infty, \\
\mathcal{R}(t) = {} & (r(t) + \lambda\widehat{r}(t))f^2(t)\left|f'(t)\right|^{p-2} \\
= {} & r(t)(1 + o(1))h^2(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)^{2/p} |h'(t)|^{p-2}\left(\int^t R^{-1}(s)\mathrm{d}s\right)^{(p-2)/p} \\
& \times \left(1 + \frac{1}{pG(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)}\right)^{p-2} \\
= {} & R(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)(1 + o(1)),
\end{aligned}
\tag{4.22}
$$

and hence

$$
\int^t \frac{\mathrm{d}s}{\mathcal{R}(s)} \longrightarrow \infty \quad \text{as } t \longrightarrow \infty.
\tag{4.23}
$$

Consequently, $v(t) \geq 0$ by Theorem 3.5. This means that the left-hand side in (4.21) is bounded above as $t \to \infty$, while the right-hand side tends to ∞ which yields the required contradiction proving that (1.9) is oscillatory if $L\lambda + \mu > 1/2q$.

Finally, consider again the case $\mu = 0$. In that case, we proved in the first part of the proof that (1.9) is oscillatory or nonoscillatory depending on whether $\lambda < -1/2q$ or $\lambda > -1/2q$. This shows that the second limit in (4.4) must be -1. □

Remark 4.2. (i) From the proof of Theorem 4.1, it follows that if the first limit in (4.4) exists, then conditions (2.8) imply that this limit is 1, and the assumptions of the theorem imply that if the second limit in (4.4) exists and is finite, then it is −1.

(ii) Theorem 4.1 can be applied to the Euler equation (1.7), and one can obtain the same result for (1.8) as in [4, Corollary 3]. Indeed, in this case, we have $h(t) = t^{(p-1)/p}$, $r = 1$, $c(t) = \gamma_p t^{-p}$, where $\gamma_p = ((p-1)/p)^p$ and by a direct computation

$$G(t) = \left(\frac{p-1}{p}\right)^{p-1}, \qquad R(t) = \tilde{R}(t) = \left(\frac{p-1}{p}\right)^{p-2} t, \tag{4.24}$$

hence,

$$\hat{c}(t) = \left[\left(t^{(p-1)/p}\right)^p \left(\frac{p-1}{p}\right)^{p-2} t \left[\left(\frac{p}{p-1}\right)^{p-2} \log t\right]^2\right]^{-1} = \left(\frac{p}{p-1}\right)^{2-p} t^{-p} \log^{-2} t,$$

$$\hat{r}(t) = \left[\left(\frac{p-1}{p} t^{-1/p}\right)^p \left(\frac{p-1}{p}\right)^{p-2} t \left[\left(\frac{p}{p-1}\right)^{p-2} \log t\right]^2\right]^{-1} = \left(\frac{p}{p-1}\right)^2 \log^{-2} t, \tag{4.25}$$

which mean that conditions (2.8) and (4.3) are satisfied. Concerning the limits in (4.4), we have

$$r|h'(t)|^p = \left(\frac{p-1}{p}\right)^p t^{-1} = c(t)h^p(t), \tag{4.26}$$

that is, the first limit in (4.4) is 1. Next,

$$f(t) = \left(\frac{p}{p-1}\right)^{(p-2)/p} t^{(p-1)/p} \log^{1/p} t, \tag{4.27}$$

and consequently,

$$\hat{c}(t)\Phi(f(t)) = \left(\frac{p}{p-1}\right)^{2-p} t^{-p} \log^{-2} t \left[\left(\frac{p}{p-1}\right)^{(p-2)/p} t^{(p-1)/p} \log^{1/p} t\right]^{p-1}$$

$$= \left(\frac{p}{p-1}\right)^{-1+2/p} t^{-2+1/p} \log^{-1-1/p} t,$$

$$f'(t) = \left(\frac{p}{p-1}\right)^{(p-2)/p} \left[\frac{p-1}{p} t^{-1/p} \log^{1/p} t + \frac{1}{p} t^{-1/p} \log^{1/p-1} t\right]$$

$$= \left(\frac{p-1}{p}\right)^{2/p} t^{-1/p} \log^{1/p} t \left[1 + \frac{1}{p-1} \log^{-1} t\right]. \tag{4.28}$$

Using this formula,

$$\hat{r}(t)\Phi(f'(t)) = \left(\frac{p}{p-1}\right)^{2/p} t^{-1+1/p}\log^{-1-1/p}t\left[1 + \log^{-1}t + \frac{p-2}{2}\log^{-2}t + O\left(\log^{-2}t\right)\right], \quad (4.29)$$

and hence,

$$
\begin{aligned}
(\hat{r}(t)\Phi(f'(t)))' &= \left(\frac{p}{p-1}\right)^{2/p}\left[-\frac{p-1}{p}t^{-2+1/p}\log^{-1-1/p}t\left(1 + O\left(\log^{-1}t\right)\right)\right.\\
&\qquad - \frac{p+1}{p}t^{-2+1/p}\log^{-2-1/p}t\left(1 + O\left(\log^{-1}t\right)\right)\\
&\qquad \left.+ t^{-2+1/p}\log^{-1-1/p}tO\left(\log^{-2}t\right)\right]\\
&= -\left(\frac{p}{p-1}\right)^{2/p-1}t^{-2+1/p}\log^{-1-1/p}t\left(1 + O\left(\log^{-1}t\right)\right),
\end{aligned}
\tag{4.30}
$$

as $t \to \infty$. This means that the second limit in (4.4) is -1. According to Theorem 4.1, we obtain that the equation

$$\left[\left(1 + \lambda\left(\frac{p}{p-1}\right)^2\frac{1}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \mu\left(\frac{p}{p-1}\right)^{2-p}\frac{1}{t^p\log^2 t}\right]\Phi(x) = 0 \tag{4.31}$$

is nonoscillatory if $\mu - \lambda < 1/2q$ and oscillatory if $\mu - \lambda > 1/2q$. If we denote $\tilde{\lambda} = \lambda(p/(p-1))^2$ and $\tilde{\mu} = \mu(p/(p-1))^{2-p}$, we see that (1.8) (with $\tilde{\lambda}, \tilde{\mu}$ instead of λ, μ, resp.) is nonoscillatory if $\tilde{\mu} - \gamma_p\tilde{\lambda} < (1/2)((p-1)/p)^{p-1}$, and it is oscillatory if $\tilde{\mu} - \gamma_p\tilde{\lambda} > (1/2)((p-1)/p)^{p-1}$, that is, we have the statement from [4].

(iii) In [3], it is proved that (1.8) is nonoscillatory also in the limiting case $\mu - \gamma_p\lambda = \mu_p$. We conjecture that we have also the same situation in the general case, that is, (1.9) is nonoscillatory also in the case $\mu - \lambda = 1/2q$.

Acknowledgments

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