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Research Article

Function Spaces with a Random Variable Exponent

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The spaces with a random variable exponent $L^{p(\omega)}(D\times\Omega)$ and $W^{k,p(\omega)}(D\times\Omega)$ are introduced. After discussing the properties of the spaces $L^{p(\omega)}(D\times\Omega)$ and $W^{k,p(\omega)}(D\times\Omega)$, we give an application of these spaces to the stochastic partial differential equations with random variable growth.

1. Introduction

In the study of some nonlinear problems in natural science and engineering, for example, a class of nonlinear problems with variable exponential growth, variable exponent function spaces play an important role. In recent years, there is a great development in the field of variable exponent analysis. In [1], basic properties of the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ have been discussed by Kováčik and Rákosník. Some theories of variable exponent spaces can also be found in [2, 3]. In [4], Harjulehto et al. present an overview about applications of variable exponent spaces to differential equations with nonstandard growth. In [5], Diening et al. summarize a lot of the existing literature of theory of function spaces with variable exponents and applications to partial differential equations. In [6], Aoyama discusses the properties of Lebesgue spaces with variable exponent on a probability space.

Motivated by [6, 7], we first introduce $L^{p(\omega)}(D\times\Omega)$ and $W^{k,p(\omega)}(D\times\Omega)$ in Section 2, which are function spaces with a random variable exponent. We also discuss some properties of these spaces. Stochastic partial differential equations have many applications in finance, such as option pricing. In Section 3, an application of the random variable exponent spaces to the stochastic partial differential equations with random variable growth is given. We discuss the existence and uniqueness of weak solution for the following equation:

$$\begin{gathered}-\operatorname{div} A(x,\omega,u,\nabla u)+B(x,\omega,u,\nabla u)=f(x,\omega),\quad (x,\omega)\in D\times\Omega,\\ u=0,\quad (x,\omega)\in\partial D\times\Omega.\end{gathered}\tag{1.1}$$

A and B are Carathéodory functions, which are integrable on $D \times \Omega$ and continuous for u and ∇u. $f(x,\omega)$ is an integrable function on $D \times \Omega$. $(\Omega, \mathcal{F}, P)$ is a complete probability space, and D is a bounded open subset of R^n $(n > 1)$. The random variable $p : \Omega \to [1,\infty)$ satisfies $1 < p^- \le p(\omega) \le p^+ < \infty$. Furthermore,

$$A : R^n \times \Omega \times R \times R^n \longrightarrow R^n, \quad B : R^n \times \Omega \times R \times R^n \longrightarrow R, \quad f : R^n \times \Omega \longrightarrow R \tag{1.2}$$

satisfy the following growth conditions:

$(H1)$ $|A(x,\omega,s,\xi)|\beta_1(\omega)|\xi|^{p(\omega)-1} + \beta_2(\omega)|s|^{p(\omega)-1} + K_1(x,\omega)$, $|B(x,\omega,s,\xi)| \le \beta_3(\omega)|\xi|^{p(\omega)-1} + \beta_4(\omega)|s|^{p(\omega)-1} + K_2(x,\omega)$,

$(H2)$ $E((A(x,\omega,s_1,\xi) - A(x,\omega,s_2,\eta))(\xi-\eta) + (B(x,\omega,s_1,\xi) - B(x,\omega,s_2,\eta))(s_1-s_2)) > 0$, $\xi \neq \eta$ or $s_1 \neq s_2$,

$(H3)$ $E(A(x,\omega,s,\xi)\xi + B(x,\omega,s,\xi)s) \ge \beta E(|\xi|^{p(\omega)} + |s|^{p(\omega)})$, a.e. in D,

where $K_1(x,\omega), K_2(x,\omega) \in L^{p'(\omega)}(D\times\Omega)$, $\beta > 0$, $\beta_i(\omega)$ $(i = 1,2,3,4)$ are nonnegative bounded random variables, $f \in L^{p'(\omega)}(D\times\Omega)$, and $1/p(\omega) + 1/p'(\omega) = 1$.

2. Some Properties of Function Spaces with a Random Variable Exponent

Let λ be a product measure on $D \times \Omega$ and $u(x,\omega)$ a Lebesgue measurable function on $D\times\Omega$. In this section, $p : \Omega \to [1,\infty)$ is a random variable.

On the set of all functions defined on $D \times \Omega$, the functional $\rho_{p(\omega)}$ is defined by

$$\rho_{p(\omega)}(u) = E\left(\int_D |u(x,\omega)|^{p(\omega)} dx\right). \tag{2.1}$$

Definition 2.1. The space $L^{p(\omega)}(D\times\Omega)$ is the set of Lebesgue measurable functions u on $D\times\Omega$ such that $\int_{D\times\Omega} |u(x,\omega)|^{p(\omega)} d\lambda < \infty$, and it is endowed with the following norm:

$$\|u\|_{p(\omega)} = \inf\left\{\lambda > 0 : \rho_{p(\omega)}\left(\frac{u}{\lambda}\right) \le 1\right\}. \tag{2.2}$$

Definition 2.2. The space $W^{k,p(\omega)}(D\times\Omega)$ is the set of functions such that $D^\alpha u \in L^{p(\omega)}(D\times\Omega)$, $|\alpha| \le k$, and it is endowed with the following norm:

$$\|u\|_{k,p(\omega)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{p(\omega)}. \tag{2.3}$$

Here $D^\alpha u$ is the derivative of u with respect to x in the distribution sense.

Definition 2.3. The space $W_0^{k,p(\omega)}(D\times\Omega)$ is the closure of $C(D\times\Omega) = \{u \in W^{k,p(\omega)}(D\times\Omega) : u(\cdot,\omega) \in C_0^\infty(D)$ for each $\omega \in \Omega\}$ in $W^{k,p(\omega)}(D\times\Omega)$.

Theorem 2.4. *If $p(\omega)$ satisfies $1 < p^- \le p(\omega) \le p^+ < \infty$, then the inequality*

$$E\left(\int_D |f(x,\omega)g(x,\omega)|dx\right) \le C\|f\|_{p(\omega)}\|g\|_{p'(\omega)} \tag{2.4}$$

holds for every $f \in L^{p(\omega)}(D \times \Omega)$ and $g \in L^{p'(\omega)}(D \times \Omega)$ with the constant C dependent on $p(\omega)$ only.

Proof. By Young inequality, we have

$$\frac{|f(x,\omega)|}{\|f\|_{p(\omega)}}\frac{|g(x,\omega)|}{\|g\|_{p'(\omega)}} \le \frac{1}{p(\omega)}\left(\frac{|f(x,\omega)|}{\|f\|_{p(\omega)}}\right)^{p(\omega)} + \frac{1}{p'(\omega)}\left(\frac{|g(x,\omega)|}{\|g\|_{p'(\omega)}}\right)^{p'(\omega)}. \tag{2.5}$$

Integrating over $D \times \Omega$, we obtain

$$E\left(\int_D \frac{|f(x,\omega)|}{\|f\|_{p(\omega)}}\frac{|g(x,\omega)|}{\|g\|_{p'(\omega)}}dx\right) \le 1 + \frac{1}{p^-} - \frac{1}{p^+}. \tag{2.6}$$

So

$$E\left(\int_D |f(x,\omega)g(x,\omega)|dx\right) \le \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right)\|f\|_{p(\omega)}\|g\|_{p'(\omega)}. \tag{2.7}$$

Now the proof is completed. □

Theorem 2.5. *Suppose that $p(\omega)$ satisfies $1 < p^- \le p(\omega) \le p^+ < \infty$. If u_k, $u \in L^{p(\omega)}(D \times \Omega)$, then*

(1) *if $\|u\|_{p(\omega)} \ge 1$, then $\|u\|_{p(\omega)}^{p^-} \le \rho_{p(\omega)}(u) \le \|u\|_{p(\omega)}^{p^+}$,*

(2) *if $\|u\|_{p(\omega)} \ge 1$, then $\|u\|_{p(\omega)}^{p^+} \le \rho_{p(\omega)}(u) \le \|u\|_{p(\omega)}^{p^-}$,*

(3) $\lim_{k\to\infty}\|u_k\|_{p(\omega)} = 0$ *if and only if* $\lim_{k\to\infty}\rho_{p(\omega)}(u_k) = 0$,

(4) $\lim_{k\to\infty}\|u_k\|_{p(\omega)} = \infty$ *if and only if* $\lim_{k\to\infty}\rho_{p(\omega)}(u_k) = \infty$.

Proof. (1) By $\|u\|_{p(\omega)} \ge 1$ and the definition of the norm,

$$E\left(\int_D \frac{|u|^{p(\omega)}}{\|u\|_{p(\omega)}^{p^+}}dx\right) \le E\left(\int_D \left(\frac{|u|}{\|u\|_{p(\omega)}}\right)^{p(\omega)}dx\right) \le 1. \tag{2.8}$$

So $\rho_{p(\omega)}(u) \le \|u\|_{p(\omega)}^{p^+}$. As

$$\|u\|_{p(\omega)}^{p^-/p(\omega)} \le \|u\|_{p(\omega)}, \tag{2.9}$$

we also have

$$E\left(\int_D \left(\frac{|u|}{\|u\|_{p(\omega)}^{p^-/p(\omega)}}\right)^{p(\omega)} dx\right) \geq 1. \tag{2.10}$$

That is to say $\|u\|_{p(\omega)}^{p^-} \leq \rho_{p(\omega)}(u)$. The proof is completed. (2), (3), and (4) can be easily proved with similar methods. □

Theorem 2.6. *If $p(\omega)$ satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$, the space $L^{p(\omega)}(D \times \Omega)$ is complete.*

Proof. Let u_n be a Cauchy sequence in $L^{p(\omega)}(D \times \Omega)$. Then, by Theorem 2.4,

$$E\left(\int_D |u_m(x,\omega) - u_n(x,\omega)| dx\right) \leq C\|u_m - u_n\|_{p(\omega)} \|\chi_D\|_{p'(\omega)}, \tag{2.11}$$

where C is constant. That is to say u_n is a Cauchy sequence in $L^1(D \times \Omega)$. In view of the completeness of $L^1(D \times \Omega)$, u_n converges in $L^1(D \times \Omega)$. Suppose that $u_n \to u$, $u \in L^1(D \times \Omega)$ and further suppose that $u_n(x,\omega) \to u(x,\omega)$ a.e. in $D \times \Omega$ (subtracting a subsequence if necessary). For each $0 < \varepsilon < 1$, there exists n_0 such that $\|u_m - u_n\| < \varepsilon$ for $m, n \geq n_0$. Fix n, by Fatou's lemma

$$\begin{aligned}
E\left(\int_D \left(\frac{|u_n(x,\omega) - u(x,\omega)|}{\varepsilon}\right)^{p(\omega)} dx\right) &\leq E\left(\limsup_{m\to\infty} \int_D \left(\frac{|u_n(x,\omega) - u_m(x,\omega)|}{\varepsilon}\right)^{p(\omega)} dx\right) \\
&\leq \limsup_{m\to\infty} E\left(\int_D \left(\frac{|u_n(x,\omega) - u_m(x,\omega)|}{\varepsilon}\right)^{p(\omega)} dx\right) \\
&\leq 1.
\end{aligned} \tag{2.12}$$

So $\|u_n - u\|_{p(\omega)} \leq \varepsilon$, and further $\rho_{p(\omega)}(u_n - u) \leq \|u_n - u\|_{p(\omega)}^{p^-} \leq \varepsilon^{p^-}$; that is to say, $u_n - u \in L^{p(\omega)}(D\times\Omega)$. Next as $u_n \in L^{p(\omega)}(D\times\Omega)$, we have $u \in L^{p(\omega)}(D\times\Omega)$. Now the proof is completed. □

Theorem 2.7. *If $p(\omega)$ satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$, then the space $L^{p(\omega)}(D \times \Omega)$ is reflexive.*

Proof. First, suppose that $v \in L^{p'(\omega)}(D \times \Omega)$ is fixed, and let

$$L_v(u) = E\left(\int_D uv\, dx\right), \tag{2.13}$$

for every $u \in L^{p(\omega)}(D \times \Omega)$.

Note that

$$|L_v(u)| \le C\|u\|_{p(\omega)}\|v\|_{p'(\omega)}. \tag{2.14}$$

So $L_v(u) \in [L^{p(\omega)}(D \times \Omega)]'$.

Second, we show that each bounded linear functional on $L^{p(\omega)}(D \times \Omega)$ is of the form $L_v(u) = E(\int_D uv dx)$ for every $v \in L^{p'(\omega)}(D \times \Omega)$. Let $L \in [L^{p(\omega)}(D \times \Omega)]'$ be given. Let S be a subset of $D \times \Omega$. We define

$$\mu(S) = L(\chi_S), \tag{2.15}$$

where χ_S is the characteristic function of S; then

$$\begin{aligned} |\mu(S)| &\le \|L\| \|\chi_S\|_{p(\omega)} \\ &\le \|L\| \max\left\{\rho_{p(\omega)}(\chi_S)^{1/p^-}, \quad \rho_{p(\omega)}(\chi_S)^{1/p^+}\right\} \\ &= \|L\| \max\left\{\text{meas}\,(S)^{1/p^-}, \quad \text{meas}\,(S)^{1/p^+}\right\}, \end{aligned} \tag{2.16}$$

so μ is absolutely continuous with respect to the measure λ. Also we have that μ is σ finite measure. By Radon-Nikodym theorem, there is an integrable function $\tilde{v}$ on $D \times \Omega$ such that

$$\mu(S) = \int_S \tilde{v} d\lambda = E\left(\int_D \tilde{v}\chi_S dx\right). \tag{2.17}$$

Now $L(u) = E(\int_D u\tilde{v} dx)$ holds for simple functions u. If $u \in L^{p(\omega)}(D \times \Omega)$, there is a sequence of simple function u_j, converging a.e. to u and $|u_j(x,\omega)| \le |u(x,\omega)|$ on $D \times \Omega$. By Fatou's lemma

$$\begin{aligned} \left|E\left(\int_D u\tilde{v}\, dx\right)\right| &\le \limsup_{j\to\infty} E\left(\int_D |u_j \tilde{v}| dx\right) \\ &= \limsup_{j\to\infty} L(|u_j| \operatorname{sgn} \tilde{v}) \\ &\le \|L\| \limsup_{j\to\infty} \|u_j\|_{p(\omega)} \\ &\le \|L\| \|u\|_{p(\omega)}. \end{aligned} \tag{2.18}$$

Then

$$L_{\tilde{v}}(u) = E\left(\int_D u\tilde{v}\, dx\right) \tag{2.19}$$

is also a bounded linear functional on $L^{p(\omega)}(D\times\Omega)$. By Lebesgue theorem

$$\lim_{j\to\infty} E\left(\int_D \left|u_j-u\right|^{p(\omega)}dx\right) = E\left(\int_D \lim_{j\to\infty}\left|u_j-u\right|^{p(\omega)}dx\right) = 0. \tag{2.20}$$

By Theorem 2.5, we have that

$$\lim_{j\to\infty}\left\|u_j-u\right\|_{p(\omega)} = 0, \tag{2.21}$$

that is, $u_j \to u$. As $L(u_j) = L_{\tilde{v}}(u_j)$, by letting $j\to\infty$, we have that $L(u) = L_{\tilde{v}}(u)$.

At last, we show that $\tilde{v}\in L^{p'(\omega)}(D\times\Omega)$. Let

$$E_l = \{(x,\omega)\in D\times\Omega : |\tilde{v}(x,\omega)|\le l\}. \tag{2.22}$$

As

$$\int_{D\times\Omega}\left|\tilde{v}\chi_{E_l}\right|^{p'(\omega)}d\lambda \le \int_{E_l}|l|^{p'(\omega)}d\lambda \le \infty, \tag{2.23}$$

$\tilde{v}\chi_{E_l}\in L^{p'(\omega)}(D\times\Omega)$. Suppose that $\|\tilde{v}\chi_{E_l}\|>0$, and take

$$u = \chi_{E_l}\left(\frac{|\tilde{v}|}{\left\|\tilde{v}\chi_{E_l}\right\|_{p'(\omega)}}\right)^{1/(p(\omega)-1)}\operatorname{sgn}\tilde{v}. \tag{2.24}$$

Then

$$\begin{aligned}\left|E\left(\int_D u\tilde{v}\,dx\right)\right| &= \left|E\left(\int_D \chi_{E_l}\left(\frac{|\tilde{v}|}{\left\|\tilde{v}\chi_{E_l}\right\|_{p'(\omega)}}\right)^{p'(\omega)}\left\|\tilde{v}\chi_{E_l}\right\|_{p'(\omega)}dx\right)\right|\\ &\ge \frac{\left\|\tilde{v}\chi_{E_l}\right\|_{p'(\omega)}}{2^{p^+}}E\left(\int_D \chi_{E_l}\left(\frac{|\tilde{v}|}{(1/2)\left\|\tilde{v}\chi_{E_l}\right\|_{p'(\omega)}}\right)^{p'(\omega)}dx\right)\\ &> \frac{\left\|\tilde{v}\chi_{E_l}\right\|_{p'(\omega)}}{2^{p^+}}.\end{aligned} \tag{2.25}$$

As

$$\|u\|_{p(\omega)} = \inf\left\{\lambda>0 : E\left(\int_D \chi_{E_l}\left(\frac{|\tilde{v}|}{\lambda\left\|\tilde{v}\chi_{E_l}\right\|_{p'(\omega)}}\right)^{p'(\omega)}dx\right)\le 1\right\} = 1, \tag{2.26}$$

we have that

$$\begin{gathered}\left\|\widetilde{v}\chi_{E_l}\right\|_{p'(\omega)} \leq 2^{p^+}\|L\|,\\ E\left(\int_D \left(\frac{\left|\widetilde{v}\chi_{E_l}\right|}{2^{p^+}\|L\|}\right)^{p'(\omega)} dx\right) \leq 1.\end{gathered} \tag{2.27}$$

By Fatou's lemma

$$E\left(\int_D \left(\frac{|\widetilde{v}|}{2^{p^+}\|L\|}\right)^{p'(\omega)} dx\right) \leq \limsup_{l\to\infty} E\left(\int_D \left(\frac{\left|\widetilde{v}\chi_{E_l}\right|}{2^{p^+}\|L\|}\right)^{p'(\omega)} dx\right) \leq 1. \tag{2.28}$$

So $\widetilde{v} \in L^{p'(\omega)}(D\times\Omega)$. Now we reach the conclusion that $L^{p'(\omega)}(D\times\Omega) = [L^{p(\omega)}(D\times\Omega)]'$, and, furthermore, $L^{p(\omega)}(D\times\Omega)$ is reflexive. □

Theorem 2.8. *If $p(\omega)$ satisfies $1 < p^- \leq p(\omega) \leq p^+ < \infty$, then the space $W^{k,p(\omega)}(D\times\Omega)$ is a reflexive Banach space.*

Proof. $W^{k,p(\omega)}(D\times\Omega)$ can be treated as a subspace of the product space

$$\prod_m L^{p(\omega)}(D\times\Omega), \tag{2.29}$$

where m is the number of multi-indices α with $|\alpha| \leq k$.

Then we need only to show that $W^{k,p(\omega)}(D\times\Omega)$ is a closed subspace of $\prod_m L^{p(\omega)}(D\times\Omega)$. Let $\{u_n\} \subset W^{k,p(\omega)}(D\times\Omega)$ be a convergent sequence. Then $\{u_n\}$ is a convergent sequence in $L^{p(\omega)}(D\times\Omega)$, so there exists $u \in L^{p(\omega)}(D\times\Omega)$ such that $u_n \to u$ in $L^{p(\omega)}(D\times\Omega)$ by Theorem 2.6.

Similarly, there exists $u_\alpha \in L^{p(\omega)}(D\times\Omega)$ such that $D^\alpha u_n \to u_\alpha$ in $L^{p(\omega)}(D\times\Omega)$ for $|\alpha| \leq k$. As, for $\varphi \in C(D\times\Omega)$,

$$(-1)^{|\alpha|}E\left(\int_D D^\alpha u_n \varphi\, dx\right) = E\left(\int_D u_n D^\alpha \varphi\, dx\right), \tag{2.30}$$

we have

$$(-1)^{|\alpha|}E\left(\int_D u_\alpha \varphi\, dx\right) = E\left(\int_D u D^\alpha \varphi\, dx\right), \tag{2.31}$$

as $n \to \infty$. By the definition of weak derivative, $D^\alpha u = u_\alpha$ holds for each $|\alpha| \leq k$. So $D^\alpha u \in L^{p(\omega)}(D\times\Omega)$. Then $W^{k,p(\omega)}(D\times\Omega)$ is a closed subspace of $\prod_m L^{p(\omega)}(D\times\Omega)$. □

Theorem 2.9. *Suppose that the sequence $u_n \in L^{p(\omega)}(D\times\Omega)$ is bounded in $L^{p(\omega)}(D\times\Omega)$. If $u_n \to u$ a.e. in $D\times\Omega$, then $u_n \rightharpoonup u$ in $L^{p(\omega)}(D\times\Omega)$.*

Proof. By Theorem 2.7, we need only to show that

$$E\left(\int_D u_n g\, dx\right) \longrightarrow E\left(\int_D u g\, dx\right) \tag{2.32}$$

for each $g \in L^{p'(\omega)}\ (D \times \Omega)$.

Let $\|u_n\|_{p(\omega)} \le C$ for each $n \in N$. By Fatou's lemma,

$$E\left(\int_D \left|\frac{u}{C}\right|^{p(\omega)} dx\right) \le \limsup_{n\to\infty} E\left(\int_D \left|\frac{u_n}{C}\right|^{p(\omega)} dx\right) \le 1, \tag{2.33}$$

so $\|u\|_{p(\omega)} \le C$. By the absolute continuity of Lebesgue integral,

$$\lim_{\text{meas}(E)\to 0} \int_{D\times\Omega} |g\chi_E|^{p'(\omega)} d\lambda = 0, \tag{2.34}$$

where $g \in L^{p'(\omega)}(D \times \Omega)$ and $E \subset D \times \Omega$. By Theorem 2.5, $\lim_{\text{meas}(E)\to 0}\|g\chi_E\|_{p'(\omega)} = 0$. So there exists $\delta > 0$, such that

$$\|g\chi_E\|_{p'(\omega)} < \frac{1}{4C}\varepsilon\left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right)^{-1}, \tag{2.35}$$

for meas$(E) < \delta$. By Egorov theorem, there exists a set $B \subset D\times\Omega$ such that $u_n \to u$ uniformly on B with meas$(\Omega \setminus B) < \delta$. Choose n_0 such that $n > n_0$ implies

$$\max_{(x,\omega)\in B} |u - u_n| \|g\|_{p'(\omega)} \|\chi_B\|_{p'(\omega)} \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) < \frac{\varepsilon}{2}. \tag{2.36}$$

Taking $E = D \times \Omega \setminus B$,

$$\begin{aligned}
&\left|E\left(\int_D u g\, dx\right) - E\left(\int_D u_n g\, dx\right)\right| \\
&\quad\le \int_B |u_n - u||g| d\lambda + \int_E |u_n - u||g| d\lambda \\
&\quad\le \max_{(x,\omega)\in B} |u - u_n| E\left(\int_D |g\chi_B| dx\right) + E\left(\int_D |u_n - u||g\chi_E| dx\right) \\
&\quad\le \max_{(x,\omega)\in B} |u - u_n| \|g\|_{p'(\omega)} \|\chi_B\|_{p'(\omega)} \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \\
&\qquad + \|u_n - u\|_{p(\omega)} \|g\chi_E\|_{p'(\omega)} \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \\
&\quad< \varepsilon.
\end{aligned} \tag{2.37}$$

That is to say $u_n \rightharpoonup u$ weakly in $L^{p(\omega)}(D \times \Omega)$. □

3. An Application to Partial Differential Equations with Random Variable Growth

Definition 3.1. $u \in W_0^{1,p(\omega)}(D \times \Omega)$ is said to be the weak solution of (1.1) if

$$E\left(\int_D A(x,\omega,u,\nabla u)\nabla\varphi + B(x,\omega,u,\nabla u)\varphi\, dx\right) = E\left(\int_D f(x,\omega)\varphi\, dx\right) \tag{3.1}$$

for all $\varphi \in W^{1,p(\omega)}(D \times \Omega)$.

Definition 3.2. Let X be a reflexive Banach space with dual X', and let $\langle\cdot,\cdot\rangle$ denote a pairing between X and X'. If $K \subset X$ is a closed convex set, then a mapping $\mathcal{L} : K \to X'$ is called monotone if $\langle \mathcal{L}u - \mathcal{L}v, u - v\rangle \geq 0$ for all $u, v \in K$. Further, $\mathcal{L}$ is called coercive on K if there exists $\phi \in K$ such that

$$\frac{\langle \mathcal{L}u_j - \mathcal{L}\varphi, u_j - \varphi\rangle}{\|u_j - \varphi\|_X} \longrightarrow \infty, \tag{3.2}$$

whenever u_j is sequence in K with $\|u_j - \varphi\|_X \to \infty$.

Theorem 3.3 (see [8]). *Let K be a nonempty closed convex subset of X, and let $\mathcal{L} : K \to X'$ be monotone, coercive and strong-weakly continuous on K. Then there exists an element u in K such that $\langle Lu, v - u\rangle \geq 0$ for all $v \in K$.*

In the following, let $K = W_0^{1,p(\omega)}(D \times \Omega)$. Then it is obvious that K is a closed convex subset of $X = W^{1,p(\omega)}(D \times \Omega)$. Let $\mathcal{L} : K \to X'$,

$$\langle Lu, v\rangle = E\left(\int_D A(x,\omega,u,\nabla u)\nabla v + B(x,\omega,u,\nabla u)v\, dx - \int_D f(x,\omega)v\, dx\right), \tag{3.3}$$

where $v \in X$.

Lemma 3.4. *$\mathcal{L}$ is monotone and coercive on K.*

Proof. In the view of $(H2)$, it is immediate that $\mathcal{L}$ is monotone. Next that $\mathcal{L}$ is coercive is shown. Fixing $\varphi \in K$, for all $u \in K$, by $(H1)$, $(H3)$, and Young Equality,

$$\begin{aligned}
&\langle \mathcal{L}(u) - \mathcal{L}(\varphi), u - \varphi\rangle \\
&\quad = E\left(\int_D A(x,\omega,u,\nabla u)\nabla u + B(x,\omega,u,\nabla u)u dx + \int_D A(x,\omega,\varphi,\nabla\varphi)\nabla\varphi\right. \\
&\qquad\quad + B(x,\omega,\varphi,\nabla\varphi)\varphi dx - \int_D A(x,\omega,u,\nabla u)\nabla\varphi + A(x,\omega,\varphi,\nabla\varphi)\nabla u
\end{aligned}$$

$$
\begin{aligned}
&\quad +B(x,\omega,u,\nabla u)\varphi + B(x,\omega,\varphi,\nabla\varphi)u dx\Big) \\
&\geq (\beta-(4\beta_0+1)\varepsilon)(\rho_{p(\omega)}(|\nabla u|)+\rho_{p(\omega)}(u)) + (\beta-(4\beta_0+1)C(\varepsilon))(\rho_{p(\omega)}(|\nabla\varphi|)+\rho_{p(\omega)}(\varphi)) \\
&\quad -(C(\varepsilon)+\varepsilon)(\rho_{p'(\omega)}(K_1(x,\omega))+\rho_{p'(\omega)}(K_2(x,\omega))),
\end{aligned} \tag{3.4}
$$

where β_0 is the bound of nonnegative bounded random variable $\beta_i(\omega), (i = 1,2,3,4)$, ε is small enough such that $(\beta-(4\beta_0+1)\varepsilon) > 0$, and $C(\varepsilon)$ is constant only dependent on ε. Then

$$
\begin{aligned}
&\frac{\langle \mathcal{L}(u)-\mathcal{L}(\varphi), u-\varphi\rangle}{\|u-\varphi\|_{W_0^{1,p(\omega)}(D\times\Omega)}} \\
&\quad \geq \frac{(\beta-(4\beta_0+1)\varepsilon)(\rho_{p(\omega)}(|\nabla(u)|)+\rho_{p(\omega)}(u))}{\|u\|_{W_0^{1,p(\omega)}(D\times\Omega)}+\|\varphi\|_{W_0^{1,p(\omega)}(D\times\Omega)}} + \frac{C(\varphi,\nabla\varphi,K_1,K_2,\varepsilon)}{\|u-\varphi\|_{W_0^{1,p(\omega)}(D\times\Omega)}}.
\end{aligned} \tag{3.5}
$$

For $\varepsilon_1, \varepsilon_2 > 0$ small enough, we have that

$$
\begin{aligned}
&\rho_{p(\omega)}(|\nabla(u)|)+\rho_{p(\omega)}(u) \\
&= E\left(\int_D\left(\frac{|\nabla u|^{p(\omega)}}{\left(\|\nabla u\|_{p(\omega)}-\varepsilon_1\right)^{p(\omega)}}(\|\nabla u\|_{p(\omega)}-\varepsilon_1)^{p(\omega)} + \frac{|u|^{p(\omega)}}{\left(\|u\|_{p(\omega)}-\varepsilon_2\right)^{p(\omega)}}\left(\|u\|_{p(\omega)}-\varepsilon_2\right)^{p(\omega)}\right)dx\right) \\
&\geq \min\left\{\left(\|\nabla u\|_{p(\omega)}-\varepsilon_1\right)^{p^-}, \left(\|\nabla u\|_{p(\omega)}-\varepsilon_1\right)^{p^+}\right\} + \min\left\{\left(\|u\|_{p(\omega)}-\varepsilon_2\right)^{p^-}, \left(\|u\|_{p(\omega)}-\varepsilon_2\right)^{p^+}\right\}.
\end{aligned} \tag{3.6}
$$

Thus, we have that

$$
\frac{\langle \mathcal{L}(u)-\mathcal{L}(\varphi), u-\varphi\rangle}{\|u-\varphi\|_{W_0^{1,p(\omega)}(D\times\Omega)}} \longrightarrow \infty, \tag{3.7}
$$

as $\|u\|_{W_0^{1,p(\omega)}(D\times\Omega)} \to \infty$. The proof is completed. ☐

Lemma 3.5. *$\mathcal{L}$ is strong-weakly continuous.*

Proof. Let $u_n \in W_0^{1,p(\omega)}(D\times\Omega)$ be a sequence that converges to $u \in W_0^{1,p(\omega)}(D\times\Omega)$. Then $\|u_n\| \leq C$ for some constant C, and there exists a subsequence such that

$$
\begin{aligned}
u_{n_k} &\longrightarrow u, \\
\nabla u_{n_k} &\longrightarrow \nabla u,
\end{aligned} \tag{3.8}
$$

a.e. in $D \times \Omega$. A, B are Caratheodory functions, so

$$\begin{aligned} A(x,\omega,u_{n_k},\nabla u_{n_k}) &\longrightarrow A(x,\omega,u,\nabla u), \\ B(x,\omega,u_{n_k},\nabla u_{n_k}) &\longrightarrow B(x,\omega,u,\nabla u), \end{aligned} \tag{3.9}$$

a.e. in $D \times \Omega$. By Theorem 2.9,

$$\begin{aligned} A(x,\omega,u_{n_k},\nabla u_{n_k}) &\rightharpoonup A(x,\omega,u,\nabla u), \\ B(x,\omega,u_{n_k},\nabla u_{n_k}) &\rightharpoonup B(x,\omega,u,\nabla u), \end{aligned} \tag{3.10}$$

and $A(x,\omega,u_{n_k},\nabla u_{n_k})$ and $B(x,\omega,u_{n_k},\nabla u_{n_k})$ are bounded. For all $\varphi \in W^{1,p(\omega)}(D \times \Omega)$, we have that

$$\begin{aligned} \langle L(u_{n_k}),\varphi\rangle &= E\left(\int_D A(x,\omega,u_{n_k},\nabla u_{n_k})\nabla\varphi + B(x,\omega,u_{n_k},\nabla u_{n_k})\varphi\, dx - \int_D f(x,\omega)\varphi\, dx\right) \\ &\longrightarrow E\left(\int_D A(x,\omega,u,\nabla u)\nabla\varphi + B(x,\omega,u,\nabla u)\varphi\, dx - \int_D f(x,\omega)\varphi\, dx\right) \\ &= \langle L(u),\varphi\rangle, \end{aligned} \tag{3.11}$$

that is to say $\mathcal{L}$ is strong-weakly continuous. □

Theorem 3.6. *Under conditions (H1)–(H3), there exists a unique weak solution $u \in W_0^{1,p(\omega)}(D\times\Omega)$ to* (1.1) *for any $f \in L^{p'(\omega)}(D \times \Omega)$.*

Proof. From Theorem 3.3, Lemmas 3.4, and 3.5, there exists $u \in W_0^{1,p(\omega)}(D \times \Omega)$ such that

$$\langle \mathcal{L}u, v-u\rangle \geq 0, \tag{3.12}$$

for all $v \in W_0^{1,p(\omega)}(D \times \Omega)$. Let $\varphi \in W_0^{1,p(\omega)}(D \times \Omega)$, then $u-\varphi,\quad u+\varphi \in W_0^{1,p(\omega)}(D \times \Omega)$. So

$$\langle \mathcal{L}u, \varphi\rangle = 0. \tag{3.13}$$

That is to say u is the weak solution of (1.1). If $u_1, u_2 \in W_0^{1,p(\omega)}(D\times\Omega)$ are both weak solutions, then

$$\begin{aligned} &E\big(\textstyle\int_D (A(x,\omega,u_1,\nabla u_1) - A(x,\omega,u_2,\nabla u_2))\nabla(u_1-u_2) \\ &\quad +(B(x,\omega,u_1,\nabla u_1) - B(x,\omega,u_2,\nabla u_2))(u_1-u_2)dx\big) = 0, \end{aligned} \tag{3.14}$$

so $u_1 = u_2$, a.e. in $D \times \Omega$. If not, it is contradicting to $(H2)$. Thus, the weak solution is unique. Now the proof is completed. □

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