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Research Article

Optimal Approximate Solutions of Fixed Point Equations

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The main objective of this paper is to present some best proximity point theorems for K-cyclic mappings and C-cyclic mappings in the frameworks of metric spaces and uniformly convex Banach spaces, thereby furnishing an optimal approximate solution to the equations of the form $Tx = x$ where T is a non-self mapping.

1. Introduction

Fixed point theorems delve into the existence of a solution to the equations of the form $Tx = x$ where T is a self-mapping. However, when T is a nonself-mapping, the equation $Tx = x$ does not necessarily have a solution, in which case best approximation theorems explore the existence of an approximate solution whereas best proximity point theorems analyze the existence of an approximate solution that is optimal. Indeed, a classical and well-known best approximation theorem, due to Fan [1], contends that if K is a nonempty convex compact subset of a Hausdorff topological vector space E and T is a continuous non-self mapping from K to E, then there exists an element x in K such that $d(x, Tx) = d(A, B)$. Subsequently, many authors, including Prolla [2], Reich [3], and Sehgal and Singh [4, 5], accomplished several appealing extensions and variants of the preceding best approximation theorem. Further, Vetrivel et al. [6] elicited a more generalized result that unifies and subsumes many such results. Despite the fact that best approximation theorems produce an approximate solution to the equation $Tx = x$, they may not render an approximate solution that is optimal. On the contrary, best proximity point theorems are intended to furnish an approximate solution x that is optimal in the sense that the error $d(x, Tx)$ is minimum. Indeed, in light of the fact that $d(x, Tx)$ is at least $d(A, B)$, a best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution x satisfies the condition $d(x, Tx) = d(A, B)$. Such optimal approximate solutions are called best proximity points of the mapping T.

Eldred et al. [7] have established interesting best proximity point theorems for relatively nonexpansive mappings. A Best proximity point theorem for contractive mapping has been explored in [8]. Best proximity point theorems for various types of contractions have been obtained in [9–13]. Best proximity point theorems for several types of set valued mappings have been derived in [14–25]. Moreover, common best proximity point theorems for pairs of contractions and for pairs of contractive mappings have been elicited in [26].

The main objective of this article is to prove some best proximity point theorems for K-cyclic mappings and C-cyclic mappings in the frameworks of metric spaces and uniformly convex Banach spaces, thereby furnishing an optimal approximate solution to the equations of the form $Tx = x$ where T is a non-self-K-cyclic mapping or a non-self-C-cyclic mapping.

2. Preliminaries

The following notions will be used in the sequel.

Definition 2.1. A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form a *K-Cyclic* mapping between A and B if there exists a nonnegative real number $k < 1/2$ such that

$$d(Tx, Sy) \leq k\left[d(x, Tx) + d(y, Sy)\right] + (1 - 2k)d(A, B), \tag{2.1}$$

for all $x \in A$ and $y \in B$.

Definition 2.2. A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form a *C-Cyclic* mapping between A and B if there exists a non-negative real number $k < 1/2$ such that

$$d(Tx, Sy) \leq k\left[d(x, Sy) + d(y, Tx)\right] + (1 - 2k)d(A, B), \tag{2.2}$$

for all $x \in A$ and $y \in B$.

Definition 2.3. A subset C of a metric space is said to be *boundedly compact* if every bounded sequence in C has a subsequence converging to some element in C.

It is evident that every compact set is boundedly compact but the converse is not true.

3. K-Cyclic Mappings

This section is concerned with best proximity point theorems for K-cyclic non-self mappings.

Lemma 3.1. *Let A and B be two non-empty subsets of a metric space. Suppose that the mappings $T : A \to B$ and $S : B \to A$ form a K-Cyclic map between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Then, $d(x_n, x_{n+1}) \to d(A, B)$.*

Proof. As T and S form a K-Cyclic map,

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Sx_1) \\ &\leq k[d(x_0, Tx_0) + d(x_1, Sx_1)] + (1-2k)d(A,B) \\ &= k[d(x_0, x_1) + d(x_1, x_2)] + (1-2k)d(A,B). \end{aligned} \tag{3.1}$$

So, it follows that $d(x_1, x_2) \leq (k/(1-k))d(x_0, x_1) + [1 - (k/(1-k))]d(A,B)$.

Similarly, it can be seen that

$$d(x_2, x_3) \leq \left(\frac{k}{1-k}\right)^2 d(x_0, x_1) + \left[1 - \left(\frac{k}{1-k}\right)^2\right] d(A,B). \tag{3.2}$$

Hence, it follows by induction that

$$d(x_n, x_{n+1}) \leq \left(\frac{k}{1-k}\right)^n d(x_0, x_1) + \left[1 - \left(\frac{k}{1-k}\right)^n\right] d(A,B). \tag{3.3}$$

Therefore, $d(x_n, x_{n+1}) \longrightarrow d(A,B)$ because of the fact that $k < 1/2$. □

Lemma 3.2. *Let A and B be non-empty closed subsets of a metric space. Let the mappings $T : A \longrightarrow B$ and $S : B \longrightarrow A$ form a K-Cyclic map between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Then, the sequence $\{x_n\}$ is bounded.*

Proof. It follows from Lemma 3.1 that $d(x_{2n-1}, x_{2n})$ is convergent and hence it is bounded. Further, since S and T form a K-cyclic mapping, it follows that

$$d(x_{2n}, Tx_0) \leq k[d(x_{2n-1}, x_{2n}) + d(x_0, Tx_0)] + (1-2k)d(A,B). \tag{3.4}$$

Therefore, the subsequence $\{x_{2n}\}$ is bounded. Similarly, it can be shown that $\{x_{2n+1}\}$ is also bounded. □

Lemma 3.3. *Let A and B be non-empty closed subsets of a metric space. Let the mappings $T : A \longrightarrow B$ and $S : B \longrightarrow A$ form a K-Cyclic map between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.*

Proof. Suppose that a subsequence $\{x_{2n_k}\}$ converges to x in A. It follows from Lemma 3.1 that $d(x_{2n_k-1}, x_{2n_k})$ converges to $d(A,B)$. As S and T form a K-cyclic mapping, it follows that

$$d(A,B) \leq d(x_{2n_k}, Tx) \leq k[d(x_{2n_k-1}, x_{2n_k}) + d(x, Tx)] + (1-2k)d(A,B). \tag{3.5}$$

Therefore, $d(x, Tx) = d(A,B)$. □

The preceding two lemmas yield the following best proximity point theorem for K-cyclic mappings in the setting of metric spaces.

Corollary 3.4. *Let A and B be two non-empty and closed subsets of a metric space. Let the mappings $T : A \to B$ and $S : B \to A$ form a K-Cyclic map between A and B. If A is boundedly compact, then T has a best proximity point.*

The following lemma, due to Eldred and Veeramani [10], will be required subsequently to establish the next best proximity point theorem of this section.

Lemma 3.5. *Let A be a non-empty, closed, and convex subset and B be a non-empty and closed subset of a uniformly convex Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in A and $\{z_n\}$ is a sequence in B satisfying the following conditions:*

(a) $\|y_n - z_n\| \to d(A,B)$,

(b) *for every* $\epsilon > 0$, $\|x_m - z_n\| \le d(A,B) + \epsilon$,

for sufficiently large values of m and n.

Then, for every $\epsilon > 0$, $\|x_m - y_n\| \le \epsilon$ for sufficiently large values of m and n.

The following best proximity point theorem is for K-cyclic mappings in the setting of uniformly convex Banach spaces.

Theorem 3.6. *Let A and B be non-empty, closed, and convex subsets of a uniformly convex Banach space. If the mappings $T : A \to B$ and $S : B \to A$ form a K-Cyclic map between A and B, then there exist a unique element $x \in A$ and a unique element $y \in B$ such that*

$$\begin{aligned} d(x,Tx) &= d(A,B), \\ d(y,Sy) &= d(A,B), \\ d(x,y) &= d(A,B). \end{aligned} \tag{3.6}$$

Further, if x_0 is any fixed element in A, $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$, then the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to the best proximity points x and y, respectively.

Proof. It follows from Lemma 3.1 that

$$\begin{aligned} \|x_{2m-1} - x_{2m}\| &\longrightarrow d(A,B), \\ \|x_{2n} - x_{2n+1}\| &\longrightarrow d(A,B). \end{aligned} \tag{3.7}$$

Therefore, for every $\epsilon > 0$,

$$\begin{aligned} \|x_{2m-1} - x_{2m}\| &< d(A,B) + \frac{\epsilon}{2k}, \\ \|x_{2n} - x_{2n+1}\| &< d(A,B) + \frac{\epsilon}{2k}, \end{aligned} \tag{3.8}$$

for sufficiently large values of m and n. As T and S form a K-cyclic mapping,

$$\begin{aligned}\|x_{2m} - x_{2n+1}\| &\leq k[\|x_{2m-1} - x_{2m}\| + \|x_{2n} - x_{2n+1}\|] + (1-2k)d(A,B) \\ &< d(A,B) + \epsilon,\end{aligned} \tag{3.9}$$

for sufficiently large values of m and n. Thus, $\{x_{2n}\}$ is a Cauchy sequence by Lemma 3.5. Since the space is complete, $\{x_{2n}\}$ converges to some element $x \in A$, which becomes a best proximity point of the mapping T by Lemma 3.3. Similarly, $\{x_{2n+1}\}$ converges to some element $y \in B$, which is a best proximity point of the mapping S. Further, $d(Tx, Sy) \leq k[d(x,Tx) + d(y,Sy)] + (1-2k)d(A,B) = d(A,B)$. Therefore, $d(Tx,Sy) = d(A,B)$. By strict convexity of the space, Tx and y should be identical, and Sy and x should be identical. Consequently, $d(x,y) = d(A,B)$. To prove the uniqueness, let us suppose that there exists another element x^* such that

$$\|x^* - Tx^*\| = d(A,B). \tag{3.10}$$

Then, $\|Tx^* - STx^*\| \leq k[\|x^* - Tx^*\| + \|Tx^* - STx^*\|] + (1-2k)d(A,B)$. Consequently, $\|Tx^* - STx^*\| = d(A,B)$. By strict convexity of the space, $STx^* = x^*$. Moreover,

$$\begin{aligned}\|Tx - x^*\| &= \|Tx - STx^*\| \\ &\leq k[\|x - Tx\| + \|Tx^* - STx^*\|] + (1-2k)d(A,B) \\ &= d(A,B).\end{aligned} \tag{3.11}$$

Therefore, $\|Tx - x^*\| = d(A,B)$. By strict convexity of the space, x and x^* are identical. This completes the proof of the theorem. □

The following example illustrates Lemma 3.3. Further, it shows that uniqueness of best proximity point is not feasible.

Example 3.7. Consider the nonuniformly convex Banach space R^2 with the norm $\|(x,y)\| = \max\{|x|,|y|\}$.

Let

$$\begin{aligned}A &:= \{(x,0) : 0 \leq x \leq 1\}, \\ B &:= \{(x,1) : 0 \leq x \leq 1\}.\end{aligned} \tag{3.12}$$

Then, $d(A,B) = 1$ and $d(u,v) = 1$ for all u in A and v in B. Let $T : A \to B$ and $S : B \to A$ be defined as

$$\begin{aligned}T((x,0)) &= (x,1), \\ S((x,1)) &= (x,0).\end{aligned} \tag{3.13}$$

For any positive number k,

$$\begin{aligned} &\|T((x_1,0)) - S((x_2,1))\| \\ &\quad = 1 \\ &\quad = k[\|(x_1,0) - T((x_1,0))\| + \|(x_2,1) - S((x_2,1))\|] + (1-2k)d(A,B). \end{aligned} \tag{3.14}$$

So, the mappings S and T form a K-cyclic mapping. Further, it can be observed that every element of A is a best proximity point of the mapping T.

4. C-Cyclic Mappings

This section is concerned with best proximity point theorems for C-cyclic non-self mappings.

Lemma 4.1. *Let A and B be two non-empty subsets of a metric space. Suppose that the mappings $T : A \to B$ and $S : B \to A$ form a C-cyclic mapping between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Then, $d(x_n, x_{n+1}) \to d(A,B)$.*

Proof. Since T and S form a C-cyclic mapping,

$$\begin{aligned} d(x_1,x_2) &= d(Tx_0, Sx_1) \\ &\leq k[d(x_1,Tx_0) + d(x_0,Sx_1)] + (1-2k)d(A,B) \\ &= kd(x_0,x_2) + (1-2k)d(A,B) \\ &\leq k[d(x_0,x_1) + d(x_1,x_2)] + (1-2k)d(A,B). \end{aligned} \tag{4.1}$$

So, it follows that $d(x_1,x_2) \leq (k/(1-k))d(x_0,x_1) + [1-(k/(1-k))]d(A,B)$.

Similarly, $d(x_2,x_3) \leq (k/(1-k))^2 d(x_0,x_1) + [1-(k/(1-k))^2]d(A,B)$.

It can be shown by induction that

$$d(x_n,x_{n+1}) \leq \left(\frac{k}{1-k}\right)^n d(x_0,x_1) + \left[1-\left(\frac{k}{1-k}\right)^n\right]d(A,B). \tag{4.2}$$

Therefore, $d(x_n,x_{n+1}) \to d(A,B)$ because of the fact that $k < 1/2$. □

Lemma 4.2. *Let A and B be non-empty closed subsets of a metric space. Let the mappings $T : A \to B$ and $S : B \to A$ form a C-cyclic map between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.*

Proof. Suppose that a subsequence $\{x_{2n_k}\}$ converges to x in A. Then, it follows from Lemma 4.1 that $d(x_{2n_k-1}, x_{2n_k}) \to d(A,B)$. Further, we have

$$\begin{aligned} d(x_{2n_k}, Tx) &= d(Sx_{2n_k-1}, Tx) \\ &\le k[d(x_{2n_k-1}, Tx) + d(x, Sx_{2n_k-1})] + (1-2k)d(A,B) \\ &\le k[d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k}, Tx) + d(x, x_{2n_k})] + (1-2k)d(A,B). \end{aligned} \tag{4.3}$$

So, it follows that

$$d(A,B) \le d(x_{2n_k}, Tx) \le \left(\frac{k}{1-k}\right)[d(x_{2n_k-1}, x_{2n_k}) + d(x, x_{2n_k})] + \left[1 - \left(\frac{k}{1-k}\right)\right]d(A,B). \tag{4.4}$$

Letting $k \to \infty$, $d(x,Tx) = d(A,B)$. This completes the proof of the Lemma. □

Lemma 4.3. *Let A and B be non-empty closed subsets of a metric space. Let the mappings $T : A \to B$ and $S : B \to A$ form a C-cyclic map between A and B. For a fixed element x_0 in A, let $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$. Then, the sequence $\{x_n\}$ is bounded.*

Proof. By Lemma 4.1, $d(x_{2n-1}, x_{2n})$ is convergent and hence it is bounded. Further, we have

$$\begin{aligned} d(x_{2n}, Tx_0) &= d(Sx_{2n-1}, Tx_0) \\ &\le k[d(x_{2n-1}, Tx_0) + d(x_0, Sx_{2n-1})] + (1-2k)d(A,B) \\ &\le k[d(x_{2n-1}, x_{2n}) + 2d(x_{2n}, Tx_0) + d(x_0, Tx_0) + (1-2k)d(A,B). \end{aligned} \tag{4.5}$$

Therefore, $d(x_{2n}, Tx_0) \le (k/(1-2k))[d(x_{2n-1}, x_{2n}) + d(x_0, Tx_0)] + d(A,B)$.

Therefore, the subsequence $\{x_{2n}\}$ is bounded. Similarly, it can be shown that $\{x_{2n+1}\}$ is also bounded. □

The preceding two lemmas give rise to the following best proximity point theorem for C-cyclic mappings in the setting of metric spaces.

Corollary 4.4. *Let A and B be two non-empty and closed subsets of a metric space. Let the mappings $T : A \to B$ and $S : B \to A$ form a C-cyclic map between A and B. If A is boundedly compact, then T has a best proximity point.*

The following best proximity point theorem is for C-cyclic mappings in the setting of uniformly convex Banach spaces.

Theorem 4.5. *Let A and B be non-empty, closed, and convex subsets of a uniformly convex Banach space. Let the mappings $T : A \to B$ and $S : B \to A$ form a C-Cyclic map between A and B. If x_0 is any fixed element in A, $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$, then the sequence $\{x_{2n}\}$ converges to a best proximity x of T and the sequence $\{x_{2n+1}\}$ converges to a best proximity point y of S such that $d(x,y) = d(A,B)$.*

Proof. It follows from Lemma 4.1 that

$$\begin{aligned} \|x_{2m-1} - x_{2m}\| &\longrightarrow d(A,B), \\ \|x_{2n} - x_{2n+1}\| &\longrightarrow d(A,B). \end{aligned} \tag{4.6}$$

Therefore, for every $\epsilon > 0$,

$$\begin{aligned} \|x_{2m-1} - x_{2m}\| &< d(A,B) + \frac{\epsilon(1-2k)}{2k}, \\ \|x_{2n} - x_{2n+1}\| &< d(A,B) + \frac{\epsilon(1-2k)}{2k}, \end{aligned} \tag{4.7}$$

for sufficiently large values of m and n. As T and S form a K-cyclic mapping,

$$\begin{aligned} \|x_{2m} - x_{2n+1}\| &= \|Sx_{2m-1} - Tx_{2n}\| \\ &\le k[\|x_{2m-1} - Tx_{2n}\| + \|x_{2n} - Sx_{2m-1}\|] + (1-2k)d(A,B) \\ &\le k[\|x_{2m-1} - x_{2m}\| + \|x_{2m} - x_{2n+1}\| + \|x_{2n} - x_{2n+1}\| + \|x_{2n+1} - x_{2m}\|] \\ &\quad + (1-2k)d(A,B). \end{aligned} \tag{4.8}$$

Thus, it follows that

$$\|x_{2m} - x_{2n+1}\| \le \left(\frac{k}{1-2k}\right)[\|x_{2m-1} - x_{2m}\| + \|x_{2n} - x_{2n+1}\|] + d(A,B). \tag{4.9}$$

Therefore, it can be concluded that

$$\|x_{2m} - x_{2n+1}\| < d(A,B) + \epsilon, \tag{4.10}$$

for sufficiently large values of m and n. Thus, $\{x_{2n}\}$ is a Cauchy sequence by Lemma 3.5. Since the space is complete, $\{x_{2n}\}$ converges to some element $x \in A$, which becomes a best proximity point of the mapping T by Lemma 4.2. Similarly, $\{x_{2n+1}\}$ converges to some element $y \in B$, which is a best proximity point of the mapping S. Further, $d(x_{2n}, x_{2n+1}) \to d(x,y)$. However, by Lemma 4.1, $d(x_{2n}, x_{2n+1}) \to d(A,B)$. Consequently, $d(x,y) = d(A,B)$. This completes the proof of the theorem. □

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