Research Article

Existence and Uniqueness of Positive Solution for a Boundary Value Problem of Fractional Order

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We are concerned with the existence and uniqueness of positive solutions for the following nonlinear fractional boundary value problem: $D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0$, $0 \leq t \leq 1$, $3 < \alpha \leq 4$, $u(0) = u'(0) = u''(0) = u''(1) = 0$, where D_{0+}^{α} denotes the standard Riemann-Liouville fractional derivative. Our analysis relies on a fixed point theorem in partially ordered sets. Some examples are also given to illustrate the results.

1. Introduction

Differential equations of fractional order occur more frequently in different research and engineering areas such as physics, chemistry, economics, and control of dynamical. Indeed, we can find numerous applications in viscoelasticity, electrochemistry control, porous media, and electromagnetism. (see, e.g., [1–7]).

For an extensive collection of results about this type of equations we refer the reader to the monograph by Kilbas and Trujillo [8], Samko et al. [9], Miller and Ross [10], and Podlubny [11].

On the other hand, some basic theory for the initial value problems of fractional differential equations involving the Riemann-Liouville differential operator has been discussed by Lakshmikantham and Vatsala [12], Lakshmikantham [13], El-Sayed and El-Maghrabi [14], Bai [15], Bai and Ge [16], Bai and Lü [17], Zhang [18], and Kempfle et al. [19, 20].

In [17] the authors studied the following two-point boundary value problem of fractional order:

$$D_{0^+}^\alpha u(t) + a(t)f(t, u(t)) = 0, \quad 0 < t < 1, \ 1 < \alpha \leq 2,$$
$$u(0) = u(1) = 0, \tag{1.1}$$

and they proved the existence of positive solutions by means of the Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem.

In [18] the author investigated the existence of solutions of

$$^cD_{0^+}^\alpha u(t) = f(t, u(t)), \quad 0 < t < 1, \ 1 < \alpha \leq 2,$$
$$u(0) = \nu \neq 0, \quad u(1) = \rho \neq 0. \tag{1.2}$$

Since boundary values are nonzero, the Riemann-Liouville fractional derivative $D_{0^+}^\alpha$ is not suitable and the author used the Caputo fractional derivative $^cD_{0^+}^\alpha$.

Motivated by these works, in this paper we discuss the existence and uniqueness of positive solutions for the following nonlinear boundary value problem of fractional order:

$$D_{0^+}^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \ 3 < \alpha \leq 4,$$
$$u(0) = u'(0) = u''(0) = u''(1) = 0. \tag{1.3}$$

This problem was studied in [21], where the authors use lower and upper solution method and the Schauder fixed point theorem which cannot ensure the uniqueness of the solution. The practical relevance of $3 < \alpha \leq 4$ appears in problems related with other areas as physics and economics which can be modeled by these fractional boundary values problems. Particularly, these problems appear in the Hamiltonian formulation for the lagrangians depending on fractional derivatives of coordinates when the systems are nonconservative (see, e.g., [7]).

Our main interest in this paper is to give an alternative answer to the main results of the paper [21].

The main tool used in our study is a fixed point theorem in partially ordered sets which gives us uniqueness of the solution.

2. Preliminaries and Previous Results

For the convenience of the reader, we present here some definitions, lemmas and basic results that will be used in the proofs of our theorems.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$ and where $\Gamma(\alpha)$ denotes the gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f :$ $(0, \infty) \to \mathbb{R}$ is given by

$$D_{0^+}^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t - s)^{\alpha - n + 1}} ds, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

The following two lemmas can be found in [17, 22].

Lemma 2.3. *Let $\alpha > 0$ and $u \in C(0, 1) \cap L^1(0, 1)$. Then fractional differential equation*

$$D_{0^+}^\alpha u(t) = 0 \tag{2.3}$$

has

$$u(t) = c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2} + \cdots + c_n t^{\alpha - n} \tag{2.4}$$

for some $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots n)$ and $n = [\alpha] + 1$ as unique solution.

Lemma 2.4. *Assume that $u \in C(0, 1) \cap L^1(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L^1(0, 1)$. Then*

$$I_{0^+}^\alpha D_{0^+}^\alpha u(t) = u(t) + c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2} + \cdots + c_n t^{\alpha - n}, \tag{2.5}$$

for some $c_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ and $n = [\alpha] + 1$.

Using **Lemma 2.4**, in [21] the following result is proved.

Lemma 2.5. *Given $f \in C[0, 1]$ and $f(t) \geq 0$, the unique nonnegative solution for*

$$D_{0^+}^\alpha u(t) + f(t) = 0, \quad 0 < t < 1, \ 3 < \alpha \leq 4,$$
$$u(0) = u'(0) = u''(0) = u''(1) = 0 \tag{2.6}$$

is

$$u(t) = \int_0^1 G(t, s) f(s) ds, \tag{2.7}$$

where

$$G(t, s) = \begin{cases} \dfrac{t^{\alpha - 1}(1 - s)^{\alpha - 3} - (t - s)^{\alpha - 1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{\alpha - 1}(1 - s)^{\alpha - 3}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.8}$$

In the sequel, we present the fixed-point theorems which we will use later. These results appear in [23].

Theorem 2.6. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies the following condition*

$$if \ \{x_n\} \ is \ a \ nondecreasing \ sequence \ in \ X \ such \ that \ x_n \to x, \ then \ x_n \leq x \ \forall n \in \mathbb{N}. \quad (2.9)$$

Let $T : X \to X$ be a nondecreasing mapping such that

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \quad for \ x \geq y, \quad (2.10)$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t \to \infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \leq T(x_0)$ then T has a fixed point.

Moreover, if $(X, \leq)$ satisfies the following condition:

$$for \ x, \ y \in X \ there \ exists \ z \in X \ which \ is \ comparable \ to \ x \ and \ y, \quad (2.11)$$

which appears in [24], the following result is proved [23].

Theorem 2.7. *Adding condition (2.11) to the hypotheses of Theorem 2.6 one obtains the uniqueness of the fixed point.*

Remark 2.8. In Theorems 2.6 and 2.7 the condition $\lim_{t \to \infty} \psi(t) = \infty$ is redundant.

In our considerations we will work in the Banach space $C[0, 1] = \{x : [0, 1] \to \mathbb{R},$ continuous$\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$.

Notice that this space can be equipped with a partial order given by

$$x, y \in C[0, 1], \ x \leq y \Longleftrightarrow x(t) \leq y(t), \quad for \ t \in [0, 1]. \quad (2.12)$$

In [24] it is proved that $(C[0, 1], \leq)$ with the classical metric given by

$$d(x, y) = \sup_{0 \leq t \leq 1} \{|x(t) - y(t)|\} \quad (2.13)$$

satisfies condition (2.9) of Theorem 2.6. Moreover, for $x, y \in C[0, 1]$, as the function $\max(x, y) \in C[0, 1]$, $(C[0, 1], \leq)$ satisfies condition (2.11).

Finally, by $\mathcal{F}$ we denote the class of functions $\psi : [0, \infty) \to [0, \infty)$ continuous, nondecreasing, positive in $(0, \infty)$ and $\psi(0) = 0$.

By $\mathcal{Q}$ we denote the class of functions $\varphi : [0, \infty) \to [0, \infty)$ continuous, nondecreasing, satisfying that $I - \varphi \in \mathcal{F}$, where I denotes the identity mapping on $[0, \infty)$.

3. Main Result

The main result of the paper is the following.

Theorem 3.1. *Problem* (1.3) *has a unique positive solution $u(t)$ if the following conditions are satisfied.*

(H1) *$f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with respect to the second argument.*

(H2) *There exists $t_0 \in [0,1]$ such that $f(t_0, 0) > 0$.*

(H3) *There exists $0 < \lambda \leq (\alpha - 2)\Gamma(\alpha + 1)/2$ such that, for $x, y \in [0,\infty)$ with $y \geq x$ and $t \in [0,1]$,*

$$f(t, y) - f(t, x) \leq \lambda \cdot \psi(y - x), \tag{3.1}$$

where $\psi \in \mathcal{Q}$.

Before the proof of Theorem 3.1, we will need some properties of Green's function appearing in Lemma 2.5.

Lemma 3.2. *$G(t, s) \geq 0$, and G is a continuous function on $[0,1] \times [0,1]$.*

Proof. The continuity of G is easily checked. In order to prove the nonnegativness of $G(t, s)$, for $0 \leq t \leq s \leq 1$, it is obvious that

$$G(t, s) = \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} \geq 0. \tag{3.2}$$

In the case of $0 \leq s \leq t \leq 1$ with $t \neq 0$, we have

$$\begin{aligned}
G(t, s) &= \frac{1}{\Gamma(\alpha)}\left(t^{\alpha-1}(1-s)^{\alpha-3} - (t-s)^{\alpha-1}\right) \\
&= \frac{1}{\Gamma(\alpha)}\left(t^{\alpha-1}\left[(1-s)^{\alpha-3} - \left(1 - \frac{s}{t}\right)^{\alpha-1}\right]\right).
\end{aligned} \tag{3.3}$$

As $s \leq s/t$, we have $1 - s \geq 1 - (s/t)$ and, consequently,

$$(1-s)^{\alpha-3} \geq \left(1 - \frac{s}{t}\right)^{\alpha-3}. \tag{3.4}$$

Taking into account that the function $g(\alpha) = t^\alpha$ with $\alpha > 0$ and $t \in (0,1)$ is decreasing we have

$$(1-s)^{\alpha-3} \geq \left(1 - \frac{s}{t}\right)^{\alpha-3} \geq \left(1 - \frac{s}{t}\right)^{\alpha-1}. \tag{3.5}$$

The last inequality and (3.3) give us $G(t, s) \geq 0$ with $t \neq 0$. Finally, notice that $G(0, s) = 0$, and this finishes the proof. □

Lemma 3.3. *One has*

$$\sup_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{2}{(\alpha-2)\Gamma(\alpha+1)}. \tag{3.6}$$

Proof. Since

$$
\begin{aligned}
\int_0^1 G(t,s)ds &= \int_0^t G(t,s)ds + \int_t^1 G(t,s)ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t \left(t^{\alpha-1}(1-s)^{\alpha-3} - (t-s)^{\alpha-1} \right)ds + \frac{1}{\Gamma(\alpha)} \int_t^1 t^{\alpha-1}(1-s)^{\alpha-3}ds \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{t^{\alpha-1}}{\alpha-2} - \frac{1}{\alpha}t^{\alpha} \right)
\end{aligned}
\tag{3.7}
$$

and if we put $\varphi(t) = \int_0^1 G(t,s)ds = (1/\Gamma(\alpha))((t^{\alpha-1})/(\alpha-2) - (1/\alpha)t^{\alpha})$, then, as

$$\varphi'(t) = \frac{1}{\Gamma(\alpha)} \left(\frac{\alpha-1}{\alpha-2}t^{\alpha-2} - t^{\alpha-1} \right) > 0, \quad \text{for } t > 0, \tag{3.8}$$

we deduce that $\varphi(t) = \int_0^1 G(t,s)ds$ is strictly increasing and, consequently,

$$
\begin{aligned}
\sup_{t\in[0,1]} \int_0^1 G(t,s)ds &= \int_0^1 G(1,s)ds = \frac{1}{\Gamma(\alpha)} \left(\frac{1}{\alpha-2} - \frac{1}{\alpha} \right) \\
&= \frac{2}{\alpha(\alpha-2)\Gamma(\alpha)} = \frac{2}{(\alpha-2)\Gamma(\alpha+1)}.
\end{aligned}
\tag{3.9}
$$

$\square$

In the sequel, we give the proof of Theorem 3.1.

Proof of Theorem 3.1. Consider the cone

$$P = \{u \in C[0,1] : u(t) \geq 0\}. \tag{3.10}$$

Obviously, P is a closed set of $C[0,1]$, and, thus, P is a complete metric space with the distance given by $d(u,v) = \sup_{t\in[0,1]}\{|u(t) - v(t)|\}$. P can be equipped with a partial order defined by

$$x, y \in P, \quad x \leq y \Longleftrightarrow x(t) \leq y(t), \text{ for } t \in [0,1]. \tag{3.11}$$

Using a similar argument to that in [24], it can be proved that $(P,\leq)$ satisfies condition (2.9) of Theorem 2.6. Moreover, as for $x, y \in P$ the function $\max(x,y) \in P$, $(P,\leq)$ satisfies condition (2.11).

Now, we consider the operator T defined on P and given by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s)) ds, \quad \text{for } u \in P. \tag{3.12}$$

By (H1) and Lemma 3.2, T applies P into itself.

In the sequel we check that T satisfies the assumptions of Theorem 2.6.

Firstly, we prove that T is a nondecreasing operator. In fact, by (H1), for $u, v \in P$ with $u \geq v$ and $t \in [0,1]$, we have

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s)) ds \geq \int_0^1 G(t,s) f(s,v(s)) ds = (Tv)(t). \tag{3.13}$$

Now, we prove that T satisfies the contractive condition appearing in Theorem 2.6.

In fact, for $u, v \in P$ and $u \geq v$, taking into account assumption (H3), we get

$$
\begin{aligned}
d(Tu, Tv) &= \sup_{t \in [0,1]} \{|(Tu)(t) - (Tv)(t)|\} = \sup_{t \in [0,1]} ((Tu)(t) - (Tv)(t)) \\
&= \sup_{t \in [0,1]} \int_0^1 G(t,s)\big(f(s,u(s)) - f(s,v(s))\big) ds \\
&\leq \sup_{t \in [0,1]} \int_0^1 G(t,s) \lambda \cdot \psi(u(s) - v(s)) ds.
\end{aligned}
\tag{3.14}
$$

As $\psi \in \mathcal{Q}$ and, thus, ψ is nondecreasing and by Lemma 3.3, from the last inequality we obtain

$$
\begin{aligned}
d(Tu, Tv) &\leq \lambda \psi(d(u,v)) \cdot \sup_{t \in [0,1]} \int_0^1 G(t,s) ds \\
&= \lambda \cdot \psi(d(u,v)) \cdot \frac{2}{(\alpha - 2)\Gamma(\alpha + 1)}.
\end{aligned}
\tag{3.15}
$$

Using the fact that $\lambda \leq 2/(\alpha - 2)\Gamma(\alpha + 1)$ (assumption (H3)), we have

$$d(Tu, Tv) \leq \psi(d(u,v)) = d(u,v) - (d(u,v) - \psi(d(u,v))). \tag{3.16}$$

Put $\varphi(x) = x - \psi(x)$, As $\psi \in \mathcal{Q}$, this means that $\varphi \in \mathcal{F}$. The last inequality gives us

$$d(Tu, Tv) \leq d(u,v) - \varphi(d(u,v)). \tag{3.17}$$

This proves that T satisfies the contractive condition of Theorem 2.6.

Finally, as $G(t,s) \geq 0$ (Lemma 3.2) and $f \geq 0$ (assumption (H1)), we have

$$(T0)(t) = \int_0^1 G(t,s) f(s,0) ds \geq 0, \tag{3.18}$$

where 0 denotes the zero function.

Now, Theorem 2.6 shows that problem (1.3) has at least one nonnegative solution. As $(P, \leq)$ satisfies condition (2.11), we obtain the uniqueness of the solution.

In what follows, we will prove that this solution is positive (this means that $x(t) > 0$, for $t \in (0, 1)$).

Finally, we will prove that the zero function is not the solution for problem (1.3). In fact, in contrary case, the zero function is a fixed point of T and, thus, we have

$$0 = \int_0^1 G(0, s) f(s, 0) ds, \quad \text{for } t \in [0, 1]. \tag{3.19}$$

The nonnegative character of the functions G and f and the last expression give us

$$G(t, s) \cdot f(s, 0) = 0 \quad \text{a.e. } (s), \text{ for } t \in [0, 1]. \tag{3.20}$$

This and the fact that $G(t, s) \neq 0$ a.e. (s) for any $t \in [0, 1]$ because $G(t, s)$ is given by a polynomial implies

$$f(s, 0) = 0 \quad \text{a.e. } (s). \tag{3.21}$$

Taking into account assumption (H2), $f(t_0, 0) > 0$ for certain $t_0 \in [0, 1]$. By the continuity of f we can find a set $A \subset [0, 1]$ with $t_0 \in A$ and $\mu(A) > 0$, where μ is the Lebesgue measure, such that $f(t, 0) > 0$ for $t \in A$. This contradicts (3.21).

This proves that the zero function is not the solution for problem (1.3). Now, we will prove that the solution x is positive.

In the contrary case, we find $0 < t^* < 1$ such that $x(t^*) = 0$. As the solution x is a fixed point of the operator T, this means that

$$x(t^*) = \int_0^1 G(t^*, s) f(s, x(s)) ds = 0. \tag{3.22}$$

Since $x \in P$ and, thus, $x \geq 0$ and by the fact that f is nondecreasing in the second variable and $G(t, s) \geq 0$, we can get

$$0 = x(t^*) = \int_0^1 G(t^*, s) f(s, x(s)) ds \geq \int_0^1 G(t^*, s) f(s, 0) ds \geq 0, \tag{3.23}$$

and this inequality implies

$$x(t^*) = \int_0^1 G(t^*, s) f(s, 0) ds = 0. \tag{3.24}$$

Using a similar reasoning to the one above used we obtain a contradiction.

Therefore, $x(t) > 0$, for $t \in [0, 1]$.

This finishes the proof.   □

Remark 3.4. In Theorem 3.1, condition (H2) seems to be a strong condition in order to obtain a positive solution for problem (1.3), but when there is uniqueness of solution one will see that this condition is a very adjusted one. More precisely, under the assumption that problem (1.3) has a unique nonnegative solution $x(t)$ one has

$$f(t_0, 0) > 0 \text{ for certain } t_0 \in [0, 1] \text{ if and only if } x(t) \text{ is a positive solution.} \tag{3.25}$$

In fact, if $f(t_0, 0) > 0$ for certain $t_0 \in [0, 1]$ the argument used in the proof of Theorem 3.1 give us that $x(t)$ is a positive solution.

For the other implication, suppose that $f(t, 0) = 0$ for any $t \in [0, 1]$. Under this assumption, our problem (1.3) admits as solutions the function $x(t)$ and the zero function and this contradicts the hypothesis about uniqueness of solution to problem (1.3). Therefore, $f(t_0, 0) > 0$ for certain $t_0 \in [0, 1]$.

Remark 3.5. Notice that the assumptions in Theorem 3.1 are invariant by additive perturbations. More precisely, if $f(t, 0) = 0$ for any $t \in [0, 1]$ and f satisfies (H1) and (H3) of Theorem 3.1, then $g(t, u) = a(t) + f(t, u)$, with $a : [0, 1] \to [0, \infty)$ a nondecreasing continuous function with $a(t_0) \neq 0$ for certain $t_0 \in [0, 1]$, satisfies (H1), (H2), and (H3) of Theorem 3.1 and the following nonlinear boundary value problem of fractional order:

$$D_{0^+}^\alpha u(t) + g(t, u(t)) = 0, \quad 0 < t < 1, \ 3 < \alpha \leq 4,$$
$$u(0) = u'(0) = u''(0) = u''(1) = 0 \tag{3.26}$$

has a unique positive solution (by Theorem 3.1).

In the sequel we present an example where the results can be applied.

Example 3.6. Consider the fractional boundary value problem

$$D_{0^+}^{7/2} u(t) + \left(t^2 + 1\right) \ln(2 + u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = u''(0) = u''(1) = 0. \tag{3.27}$$

In this case, $f(t, u) = (t^2 + 1) \ln(2 + u)$ for $(t, u) \in [0, 1] \times [0, \infty)$. Obviously, f is a continuous function and $f(t, 0) = (t^2 + 1) \ln 2 \neq 0$ for $t \in [0, 1]$. As $\partial f / \partial u = (t^2 + 1)(1/(2 + u)) > 0$ for $u \in [0, \infty)$, f is nondecreasing with respect to the second variable.

Besides, for $u \geq v$ and $t \in [0, 1]$, we have

$$\left(t^2 + 1\right)[\ln(2 + u) - \ln(2 + v)] = \left(t^2 + 1\right) \cdot \ln\left(\frac{2 + u}{2 + v}\right)$$
$$= \left(t^2 + 1\right) \ln\left(\frac{2 + v + u - v}{2 + v}\right) = \left(t^2 + 1\right) \ln\left(1 + \frac{u - v}{2 + v}\right) \tag{3.28}$$
$$\leq \left(t^2 + 1\right) \ln(1 + (u - v)) \leq 2 \ln(1 + (u - v)).$$

A straightforward calculation gives us that $\psi(x) = \ln(1 + x)$ satisfies that $\psi \in \mathcal{D}$.

Moreover, in this case $\lambda = 2$, $\alpha = 7/2$ and we have

$$\frac{(\alpha - 2)\Gamma(\alpha + 1)}{2} = \frac{(7/2 - 2)\Gamma(7/2 + 1)}{2} = \frac{3}{4}\Gamma\left(\frac{7}{2} + 1\right) = \frac{3}{4} \cdot \frac{7}{2} \cdot \Gamma\left(\frac{7}{2}\right) \approx 8.7228 > 2 = \lambda.$$

(3.29)

Finally, Theorem 3.1 proves the existence and uniqueness of a positive solution for problem (3.27).

4. A Final Remark

In connection with problem (1.3), the main result in [21] is the following.

Theorem 4.1 (see [21, Theorem 3.1]). *Problem* (1.3) *has a positive solution* $u(t)$ *if the following conditions are satisfied:*

(H_f) $f(t, u) \in \mathcal{C}([0, 1] \times [0, \infty), \mathbb{R}^+)$ *is nondecreasing relative to* u, $f(t, p(t)) \neq 0$ *for* $t \in (0, 1)$, *where* $p(t) = \int_0^1 G(t, s)ds = (1/\Gamma(\alpha))((t^{\alpha - 1})/(\alpha - 2) - (1/\alpha)t^\alpha)$, *and there exists a positive constant* $\mu < 1$ *such that*

$$k^\mu f(t, u) \leq f(t, ku), \quad \forall \, 0 \leq k \leq 1.$$

(4.1)

In the sequel, we present an example which can be treated by Theorem 3.1 and it cannot be covered by Theorem 4.1.

Example 4.2. Consider the fractional boundary value problem

$$D_{0^+}^{7/2}u(t) + \left(t^2 + 1\right)(\rho u(t) + c) = 0, \quad 0 \leq t \leq 1,$$

$$u(0) = u'(0) = u''(0) = u''(1) = 0,$$

(4.2)

with $c > 0$ and $0 < \rho < 1$.

In this case, $f(t, u) = (t^2 + 1)(\rho u + c)$, for $(t, u) \in [0, 1] \times [0, \infty)$. Obviously, f is continuous and nondecreasing with respect to the second variable since $\partial f / \partial u = \rho(t^2 + 1) > 0$.

Besides, if $u \geq v$ and $t \in [0, 1]$, we have

$$f(t, u) - f(t, v) = \left(t^2 + 1\right)\left[\rho u + c - (\rho v + c)\right]$$

$$= \left(t^2 + 1\right)\rho(u - v) \leq 2\rho(u - v).$$

(4.3)

In this case, $\psi(x) = \rho x$ and it is easily seen that $\varphi(x) = x - \psi(x) = (1 - \rho)x$ belongs to $\mathcal{F}$.

Moreover, in this case, $\lambda = 2$ and, as $\alpha = 7/2$, we have

$$
\begin{aligned}
\frac{(\alpha - 2)\Gamma(\alpha + 1)}{2} &= \frac{(7/2 - 2)\Gamma(7/2 + 1)}{2} = \frac{3}{4}\Gamma\left(\frac{7}{2} + 1\right) \\
&= \frac{3}{4} \cdot \frac{7}{2} \cdot \Gamma\left(\frac{7}{2}\right) \approx 8.7228 > 2 = \lambda.
\end{aligned}
\tag{4.4}
$$

As $f(t, 0) = c(t^2 + 1) > 0$ for any $t \in [0, 1]$, Theorem 3.1 gives us the existence and uniqueness of positive solution for problem (4.2).

On the other hand, we will show that $f(t, u) = (t^2 + 1)(\rho u + c)$ with $0 < \rho < 1$ and $c > 0$ does not satisfy (H_f) of Theorem 4.1. In fact, suppose that there exists $0 < \mu < 1$ such that

$$
k^\mu f(t, u) \leq f(t, ku), \quad \text{for any } 0 \leq k \leq 1.
\tag{4.5}
$$

This implies that

$$
k^\mu \leq \frac{f(t, ku)}{f(t, u)} = \frac{(t^2 + 1)(\rho ku + c)}{(t^2 + 1)(\rho u + c)} = \frac{\rho ku + c}{\rho u + c}.
\tag{4.6}
$$

Notice that $\lim_{u \to \infty}(\rho ku + c/\rho u + c) = k$, and, consequently, taking limit as $u \to \infty$ in the last inequality, we get

$$
k^\mu \leq k,
\tag{4.7}
$$

this is false because $0 < \mu < 1$ and the function $h(\alpha) = k^\alpha$ is decreasing when $0 < k < 1$. Therefore, problem (4.2) can be covered by Theorem 3.1 and it cannot be studied by Theorem 4.1.

5. Conclusions

Our main contribution in this paper is to prove under certain assumptions the existence and the uniqueness of positive solution for problem (1.3) which was treated in [21]. In [21] the question of uniqueness of solution was not considered. Moreover, we present an example which can be covered by the results of this paper and cannot be treated by the ones obtained in [21].

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