

## Research Article

# Strong Convergence Algorithms for Hierarchical Fixed Points Problems and Variational Inequalities

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We introduce a new iterative scheme that converges strongly to a common fixed point of a countable family of nonexpansive mappings in a Hilbert space such that the common fixed point is a solution of a hierarchical fixed point problem. Our results extend the ones of Moudafi, Xu, Cianciaruso et al., and Yao et al.

## 1. Introduction

Let  $H$  be a real Hilbert space and  $C$  a nonempty closed convex subset of  $H$ . A mapping  $T : C \rightarrow C$  is called nonexpansive if one has

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1.1)$$

If there exists a point  $x \in C$  such that  $x = Tx$ , then  $x$  is said to be a fixed point of  $T$ . We denote the set of all fixed points of  $T$  by  $F(T)$ . It is well known that  $F(T)$  is closed and convex if  $T$  is nonexpansive.

Let  $S : C \rightarrow H$  be a mapping. The following problem is called a hierarchical fixed point problem: find  $x^* \in F(T)$  such that

$$\langle x^* - Sx^*, x^* - x \rangle \leq 0, \quad \forall x \in F(T). \quad (1.2)$$

It is known that the hierarchical fixed point problem (1.2) links with some monotone variational inequalities and convex programming problems (see [1]).

In order to solve the hierarchical fixed point problem (1.2), Moudafi [2] introduced the following Krasnoselski-Mann algorithm:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \quad \forall n \geq 0, \quad (1.3)$$

where  $\{\alpha_n\}$  and  $\{\sigma_n\}$  are two sequences in  $(0, 1)$ , and he proved that  $\{x_n\}$  converges weakly to a fixed point of  $T$  which is a solution of the problem (1.2).

Let  $f : C \rightarrow C$  be a mapping. The mapping  $f$  is called a contraction if there exists a constant  $\lambda \in [0, 1)$  such that  $\|fx - fy\| \leq \lambda\|x - y\|$  for all  $x, y \in C$ . For obtaining a strong convergence result, Mainge and Moudafi in [3] and Marino and Xu in [4] introduced the following algorithm:

$$x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \quad \forall n \geq 0, \quad (1.4)$$

where  $S : C \rightarrow C$  is a nonexpansive mapping and  $\{\alpha_n\}$  and  $\{\sigma_n\}$  are two sequences in  $(0, 1)$ , and they proved that  $\{x_n\}$  converges strongly to a fixed point of  $T$  which is a solution of the problem (1.2). Recently, for solving the hierarchical fixed point problem (1.2), Cianciaruso et al. [5] also studied the following iterative scheme:

$$\begin{aligned} y_n &= \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Ty_n, \quad \forall n \geq 0, \end{aligned} \quad (1.5)$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are two sequences in  $[0, 1]$ . The authors proved some strong convergence results. Very recently, Yao et al. [1] introduced the following strong convergence iterative algorithm to solve the problem (1.2):

$$\begin{aligned} y_n &= \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} &= P_C[\alpha_n f(x_n) + (1 - \alpha_n)Ty_n], \quad \forall n \geq 0, \end{aligned} \quad (1.6)$$

where  $f : C \rightarrow H$  is a contraction and  $\{\alpha_n\}$  and  $\{\beta_n\}$  are two sequences in  $(0, 1)$ . Under some certain restrictions on parameters, the authors proved that the sequence  $\{x_n\}$  generated by (1.6) converges strongly to  $z \in F(T)$ , which is the unique solution of the following variational inequality:

$$\langle (I - f)z, x - z \rangle \geq 0, \quad \forall x \in F(T). \quad (1.7)$$

By changing the restrictions on parameters, the authors obtained another result on the iterative scheme (1.6), that is, the sequence  $\{x_n\}$  generated by (1.6) converges strongly to

a point  $x^* \in F(T)$ , which is the unique solution of the following variational inequality:

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, y - x^* \right\rangle \geq 0, \quad \forall y \in F(T), \quad (1.8)$$

where  $\tau \in (0, \infty)$  is a constant.

Let  $S : C \rightarrow H$  be a nonexpansive mapping and  $\{T_i\}_{i=1}^{\infty} : C \rightarrow C$  a countable family of nonexpansive mappings. In this paper, motivated and inspired by the results of Yao et al. [1] and Marino and Xu [4], we introduce and study the following iterative scheme:

$$\begin{aligned} y_n &= P_C [\beta_n Sx_n + (1 - \beta_n)x_n], \\ x_{n+1} &= P_C \left[ \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i y_n \right], \quad \forall n \geq 1, \end{aligned} \quad (1.9)$$

where  $\alpha_0 = 1$ ,  $\{\alpha_n\}$  is a strictly decreasing sequence in  $(0, 1)$  and  $\{\beta_n\}$  is a sequence in  $(0, 1)$ . Under some certain conditions on parameters, we first prove that the sequence  $\{x_n\}$  generated by (1.9) converges strongly to  $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$ , which is the unique solution of the following variational inequality:

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i). \quad (1.10)$$

By changing the restrictions on parameters, we also prove that the sequence  $\{x_n\}$  converges strongly to  $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$ , which is the unique solution of the following variational inequality:

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, y - x^* \right\rangle \geq 0, \quad \forall y \in \bigcap_{i=1}^{\infty} F(T_i), \quad (1.11)$$

where  $\tau \in (0, \infty)$  is a constant. It is easy to see that, if  $T_i = T$  for each  $i \geq 1$  and  $S$  is a self-mapping of  $C$  into itself, then our algorithm (1.9) is reduced to (1.6) of Yao et al. [1] Also, our results extend the corresponding ones of Moudafi [6], Xu [7], and Cianciaruso et al. [5].

## 2. Preliminaries

Let  $H$  be a Hilbert space and  $C$  a nonempty closed convex subset of  $H$ . Let  $T$  be a nonexpansive mapping of  $C$  into itself such that  $F(T) \neq \emptyset$ . For all  $\hat{x} \in F(T)$  and all  $x \in C$ , we have

$$\begin{aligned} \|x - \hat{x}\|^2 &\geq \|Tx - T\hat{x}\|^2 = \|Tx - \hat{x}\|^2 = \|Tx - x + (x - \hat{x})\|^2 \\ &= \|Tx - x\|^2 + \|x - \hat{x}\|^2 + 2\langle Tx - x, x - \hat{x} \rangle \end{aligned} \quad (2.1)$$

and hence

$$\|Tx - x\|^2 \leq 2\langle x - Tx, x - \hat{x} \rangle, \quad \forall \hat{x} \in F(T), x \in C. \quad (2.2)$$

Let  $x \in H$  be an arbitrary point. There exists a unique nearest point in  $C$ , denoted by  $P_C x$ , such that

$$\|P_C x - x\| \leq \|y - x\|, \quad \forall y \in C. \quad (2.3)$$

Moreover, we have the following:

$$z = P_C x \iff \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \quad (2.4)$$

Let  $I$  denote the identity operator of  $H$ , and let  $\{x_n\}$  be a sequence in a Hilbert space  $H$  and  $x \in H$ . Throughout this paper,  $x_n \rightarrow x$  denotes that  $\{x_n\}$  strongly converges to  $x$  and  $x_n \rightharpoonup x$  denotes that  $\{x_n\}$  weakly converges to  $x$ .

The following lemmas will be used in the next section.

**Lemma 2.1** (see [8]). *Let  $T : C \rightarrow C$  be a nonexpansive mapping with  $F(T) \neq \emptyset$ . If  $\{x_n\}$  is a sequence in  $C$  weakly converging to a point  $x \in C$  and  $\{(I - T)x_n\}$  converges strongly to a point  $y \in C$ , then  $(I - T)x = y$ . In particular, if  $y = 0$ , then  $x \in F(T)$ .*

**Lemma 2.2** (see [9]). *Let  $f : C \rightarrow H$  be a contraction with coefficient  $\lambda \in [0, 1)$  and  $T : C \rightarrow C$  a nonexpansive mapping. Then one has the following.*

(1) *The mapping  $(I - f)$  is strongly monotone with coefficient  $(1 - \lambda)$ , that is,*

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \lambda)\|x - y\|^2, \quad \forall x, y \in C. \quad (2.5)$$

(2) *The mapping  $I - T$  is monotone, that is,*

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq 0, \quad \forall x, y \in C. \quad (2.6)$$

**Lemma 2.3** (see [10]). *Let  $\{s_n\}$ ,  $\{c_n\}$  be the sequences of nonnegative real numbers, and let  $\{a_n\} \subset (0, 1)$ . Suppose that  $\{b_n\}$  is a real number sequence such that*

$$s_{n+1} \leq (1 - a_n)s_n + b_n + c_n, \quad \forall n \geq 0. \quad (2.7)$$

*Assume that  $\sum_{n=0}^{\infty} c_n < \infty$ . Then the following results hold.*

(1) *If  $b_n \leq \beta a_n$ , where  $\beta \geq 0$ , then  $\{s_n\}$  is a bounded sequence.*

(2) *If one has*

$$\sum_{n=0}^{\infty} a_n = \infty, \quad \limsup_{n \rightarrow \infty} \frac{b_n}{a_n} \leq 0, \quad (2.8)$$

*then  $\lim_{n \rightarrow \infty} s_n = 0$ .*

### 3. Main Results

Now, we give the main results in this paper.

**Theorem 3.1.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $f : C \rightarrow H$  be a  $\lambda$ -contraction with  $\lambda \in [0, 1)$ . Let  $S : C \rightarrow H$  be a nonexpansive mapping and  $\{T_i\}_{i=1}^{\infty}$  a countable family of nonexpansive mappings of  $C$  into itself such that  $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ . Set  $\alpha_0 = 1$ , and let  $\{\alpha_n\} \subset (0, 1)$  be a strictly decreasing sequence and  $\{\beta_n\} \subset (0, 1)$  a sequence satisfying the following conditions:*

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (b)  $\lim_{n \rightarrow \infty} (\beta_n / \alpha_n) = 0$ ,
- (c)  $\sum_{n=1}^{\infty} (\alpha_{n-1} - \alpha_n) < \infty$  and  $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$ .

Then the sequence  $\{x_n\}$  generated by (1.9) converges strongly to a point  $x^* \in F$ , which is the unique solution of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F. \quad (3.1)$$

*Proof.* First,  $P_F f$  is a contraction from  $C$  into itself with a constant  $\lambda$  and  $C$  is complete, and there exists a unique  $x^* \in C$  such that  $x^* = P_F f(x^*)$ . From (2.4), it follows that  $x^*$  is the unique solution of the problem (3.1).

Now, we prove that  $\{x_n\}$  converges strongly to  $x^*$ . To this end, we first prove that  $\{x_n\}$  is bounded. Take  $p \in F$ . Then it follows from (1.9) that

$$\begin{aligned} \|y_n - p\| &= \|P_C [\beta_n Sx_n + (1 - \beta_n)x_n] - P_C p\| \\ &\leq \|\beta_n Sx_n + (1 - \beta_n)x_n - p\| \\ &\leq \beta_n \|Sx_n - Sp\| + \beta_n \|Sp - p\| + (1 - \beta_n) \|x_n - p\| \\ &\leq \beta_n \|x_n - p\| + \beta_n \|Sp - p\| + (1 - \beta_n) \|x_n - p\| \\ &= \|x_n - p\| + \beta_n \|Sp - p\| \end{aligned} \quad (3.2)$$

and hence (note that  $\{\alpha_n\}$  is strictly decreasing)

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| P_C \left[ \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i y_n \right] - P_C p \right\| \\ &\leq \left\| \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i y_n - p \right\| \\ &= \left\| \alpha_n (f(x_n) - p) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (T_i y_n - p) \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|T_i y_n - p\| \\
&\leq \alpha_n \lambda \|x_n - p\| + \alpha_n \|f(p) - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|y_n - p\| \\
&\leq \alpha_n \lambda \|x_n - p\| + \alpha_n \|f(p) - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (\|x_n - p\| + \beta_n \|Sp - p\|) \\
&= \alpha_n \lambda \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) (\|x_n - p\| + \beta_n \|Sp - p\|).
\end{aligned} \tag{3.3}$$

By condition (b), we can assume that  $\beta_n \leq \alpha_n$  for all  $n \geq 1$ . Hence, from above inequality, we get

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \lambda \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) (\|x_n - p\| + \alpha_n \|Sp - p\|) \\
&\leq \alpha_n \lambda \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| + \alpha_n \|Sp - p\| \\
&= [1 - \alpha_n(1 - \lambda)] \|x_n - p\| + \alpha_n (\|f(p) - p\| + \|Sp - p\|).
\end{aligned} \tag{3.4}$$

For each  $n \geq 1$ , let  $a_n = \alpha_n(1 - \lambda)$ ,  $b_n = \alpha_n(\|f(p) - p\| + \|Sp - p\|)$ , and  $c_n = 0$ . Then  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  satisfy the condition of Lemma 2.3(1). Hence, it follows from Lemma 2.3(1) that  $\{x_n\}$  is bounded and so are  $\{f(x_n)\}$ ,  $\{y_n\}$ ,  $\{T_i x_n\}$ , and  $\{T_i y_n\}$  for all  $i \geq 1$ . Set  $u_n = \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i y_n$  for each  $n \geq 1$ . From (1.9), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|P_C u_n - P_C u_{n-1}\| \leq \|u_n - u_{n-1}\| \\
&= \left\| \alpha_n (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1}) f(x_{n-1}) \right. \\
&\quad \left. + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (T_i y_n - T_i y_{n-1}) + (\alpha_{n-1} - \alpha_n) T_n y_{n-1} \right\| \\
&\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|y_n - y_{n-1}\| \\
&\quad + (\alpha_{n-1} - \alpha_n) (\|f(x_{n-1})\| + \|T_n y_{n-1}\|) \\
&\leq \alpha_n \lambda \|x_n - x_{n-1}\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|y_n - y_{n-1}\| + (\alpha_{n-1} - \alpha_n) M \\
&= \alpha_n \lambda \|x_n - x_{n-1}\| + (1 - \alpha_n) \|y_n - y_{n-1}\| + (\alpha_{n-1} - \alpha_n) M,
\end{aligned} \tag{3.5}$$

where  $M$  is a constant such that

$$\sup_{n \geq 1} \{ \|f(x_{n-1})\| + \|T_n y_{n-1}\| + \|Sx_{n-1}\| + \|x_{n-1}\| \} \leq M. \tag{3.6}$$

From (1.9), we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|P_C(\beta_n Sx_n + (1 - \beta_n)x_n) - P_C(\beta_{n-1} Sx_{n-1} + (1 - \beta_{n-1})x_{n-1})\| \\
&\leq \|(\beta_n Sx_n + (1 - \beta_n)x_n) - (\beta_{n-1} Sx_{n-1} + (1 - \beta_{n-1})x_{n-1})\| \\
&= \|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} + (1 - \beta_n)(x_n - x_{n-1}) + (\beta_{n-1} - \beta_n)x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M.
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.5), we get that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|u_n - u_{n-1}\| \\
&\leq \alpha_n \lambda \|x_n - x_{n-1}\| + (1 - \alpha_n) [\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M] + (\alpha_{n-1} - \alpha_n)M \tag{3.8} \\
&\leq (1 - (1 - \lambda)\alpha_n)\|x_n - x_{n-1}\| + M[(\alpha_{n-1} - \alpha_n) + |\beta_n - \beta_{n-1}|].
\end{aligned}$$

Let  $a_n = (1 - \lambda)\alpha_n$ ,  $b_n = 0$ , and  $c_n = (\alpha_{n-1} - \alpha_n) + |\beta_n - \beta_{n-1}|$ . Then conditions (a) and (c) imply that  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  satisfy the condition of Lemma 2.3(2). Thus, by Lemma 2.3(2), we can conclude that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Since  $T_i x_n \in C$  for each  $i \geq 1$  and  $\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) + \alpha_n = 1$ , we have

$$\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i x_n + \alpha_n z \in C, \quad \forall z \in C. \tag{3.10}$$

Now, fixing a  $z \in F$ , from (1.9) we have

$$\begin{aligned}
&\sum_{i=1}^n (\alpha_{i-1} - \alpha_i)(x_n - T_i x_n) \\
&= P_C u_n + (1 - \alpha_n)x_n - \left( \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i x_n + \alpha_n z \right) + \alpha_n z - x_{n+1} \tag{3.11} \\
&= P_C u_n - P_C \left[ \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i x_n + \alpha_n z \right] + (1 - \alpha_n)(x_n - x_{n+1}) + \alpha_n(z - x_{n+1}).
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
& \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle x_n - T_i x_n, x_n - p \rangle \\
&= \left\langle P_C u_n - P_C \left[ \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i x_n + \alpha_n z \right], x_n - p \right\rangle \\
&\quad + \langle (1 - \alpha_n)(x_n - x_{n+1}), x_n - p \rangle + \alpha_n \langle z - x_{n+1}, x_n - p \rangle \\
&\leq \left\| u_n - \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i x_n - \alpha_n z \right\| \|x_n - p\| + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - p\| \\
&\quad + \alpha_n \|z - x_{n+1}\| \|x_n - p\| \\
&= \left\| \alpha_n (f(x_n) - z) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (T_i y_n - T_i x_n) \right\| \|x_n - p\| \\
&\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - p\| + \alpha_n \|z - x_{n+1}\| \|x_n - p\| \tag{3.12} \\
&\leq \alpha_n \|f(x_n) - z\| \|x_n - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|y_n - x_n\| \|x_n - p\| \\
&\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - p\| + \alpha_n \|z - x_{n+1}\| \|x_n - p\| \\
&\leq \alpha_n \|f(x_n) - z\| \|x_n - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \beta_n \|Sx_n - x_n\| \|x_n - p\| \\
&\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - p\| + \alpha_n \|z - x_{n+1}\| \|x_n - p\| \\
&= \alpha_n \|f(x_n) - z\| \|x_n - p\| + (1 - \alpha_n) \beta_n \|Sx_n - x_n\| \|x_n - p\| \\
&\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - p\| + \alpha_n \|z - x_{n+1}\| \|x_n - p\| \\
&\leq (2\alpha_n + \beta_n) M' + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - p\|,
\end{aligned}$$

where

$$M' = \sup_{n \geq 1} \{ \|f(x_n) - z\| \|x_n - p\|, \|Sx_n - x_n\| \|x_n - p\|, \|z - x_{n+1}\| \|x_n - p\| \}. \tag{3.13}$$

Now, from (2.2) and (3.12), we get

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|x_n - T_i x_n\|^2 &\leq \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle x_n - T_i x_n, x_n - p \rangle \\
&\leq (2\alpha_n + \beta_n) M' + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - p\|.
\end{aligned} \tag{3.14}$$



From (a), (b), (3.9), and (3.14), we have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|x_n - T_i x_n\| = 0. \quad (3.15)$$

Since  $(\alpha_{i-1} - \alpha_i) \|x_n - T_i x_n\| \leq \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|x_n - T_i x_n\|$  for each  $i \geq 1$  and  $\{\alpha_n\}$  is strictly decreasing, one has

$$\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0, \quad \forall i \geq 1. \quad (3.16)$$

Next, we prove that

$$\limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_n - x^* \rangle \leq 0. \quad (3.17)$$

Since  $\{x_n\}$  is bounded, we can take a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \rightharpoonup x'$  and

$$\limsup_{n \rightarrow \infty} \langle f(x^*) - x^*, x_n - x^* \rangle = \lim_{k \rightarrow \infty} \langle f(x^*) - x^*, x_{n_k} - x^* \rangle. \quad (3.18)$$

From (3.16) and Lemma 2.1, we conclude that  $x' \in F(T_i)$  for each  $i \geq 1$ , that is,  $x' \in \bigcap_{i=1}^{\infty} F(T_i)$ . Then

$$\lim_{k \rightarrow \infty} \langle f(x^*) - x^*, x_{n_k} - x^* \rangle = \langle f(x^*) - x^*, x' - x^* \rangle \leq 0. \quad (3.19)$$

By (2.4), we have

$$\langle P_C u_n - u_n, P_C u_n - x^* \rangle \leq 0. \quad (3.20)$$

Also,

$$\begin{aligned} & \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle T_i y_n - x^*, x_{n+1} - x^* \rangle \\ & \leq \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|T_i y_n - x^*\| \|x_{n+1} - x^*\| \\ & \leq \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|y_n - x^*\| \|x_{n+1} - x^*\| \\ & = (1 - \alpha_n) \|y_n - x^*\| \|x_{n+1} - x^*\| \\ & = (1 - \alpha_n) \|P_C(\beta_n S x_n + (1 - \beta_n) x_n) - P_C x^*\| \|x_{n+1} - x^*\| \\ & \leq (1 - \alpha_n) \|\beta_n S x_n + (1 - \beta_n) x_n - x^*\| \|x_{n+1} - x^*\| \end{aligned}$$

$$\begin{aligned}
&= (1 - \alpha_n) \|\beta_n(Sx_n - Sx^*) + \beta_n(Sx^* - x^*) + (1 - \beta_n)(x_n - x^*)\| \|x_{n+1} - x^*\| \\
&\leq (1 - \alpha_n) [\beta_n \|x_n - x^*\| + \beta_n \|Sx^* - x^*\| + (1 - \beta_n) \|x_n - x^*\|] \|x_{n+1} - x^*\| \\
&= (1 - \alpha_n) \|x_n - x^*\| \|x_{n+1} - x^*\| + (1 - \alpha_n) \beta_n \|Sx^* - x^*\| \|x_{n+1} - x^*\|.
\end{aligned} \tag{3.21}$$

Thus,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle P_C u_n - u_n, P_C u_n - x^* \rangle + \langle u_n - x^*, x_{n+1} - x^* \rangle \\
&\leq \langle u_n - x^*, x_{n+1} - x^* \rangle = \alpha_n \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle \\
&\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle T_i y_n - x^*, x_{n+1} - x^* \rangle + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\leq \alpha_n \lambda \|x_n - x^*\| \|x_{n+1} - x^*\| + (1 - \alpha_n) \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + (1 - \alpha_n) \beta_n \|Sx^* - x^*\| \|x_{n+1} - x^*\| + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&= [1 - \alpha_n(1 - \lambda)] \|x_n - x^*\| \|x_{n+1} - x^*\| + (1 - \alpha_n) \beta_n \|Sx^* - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\leq \frac{1 - \alpha_n(1 - \lambda)}{2} [\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2] + (1 - \alpha_n) \beta_n \|Sx^* - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle.
\end{aligned} \tag{3.22}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \left[ 1 - \frac{2(1 - \lambda)\alpha_n}{1 + (1 - \lambda)\alpha_n} \right] \|x_n - x^*\|^2 + \frac{2(1 - \alpha_n)\beta_n}{1 + (1 - \lambda)\alpha_n} \|Sx^* - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \frac{2\alpha_n}{1 + (1 - \lambda)\alpha_n} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&= \left[ 1 - \frac{2(1 - \lambda)\alpha_n}{1 + (1 - \lambda)\alpha_n} \right] \|x_n - x^*\|^2 + \frac{2(1 - \lambda)\alpha_n}{1 + (1 - \lambda)\alpha_n} \\
&\quad \times \left\{ \frac{(1 - \alpha_n)\beta_n}{(1 - \lambda)\alpha_n} \|Sx^* - x^*\| \|x_{n+1} - x^*\| + \frac{1}{1 - \lambda} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \right\}.
\end{aligned} \tag{3.23}$$

Let  $a_n = 2(1 - \lambda)\alpha_n / (1 + (1 - \lambda)\alpha_n)$ ,  $b_n = (2(1 - \lambda)\alpha_n / (1 + (1 - \lambda)\alpha_n)) \{ (1 - \alpha_n)\beta_n / (1 - \lambda)\alpha_n \|Sx^* - x^*\| \|x_{n+1} - x^*\| + (1/1 - \lambda) \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \}$ , and  $c_n = 0$  for all  $n \geq 1$ . Since

$$\limsup_{n \rightarrow \infty} \left\{ \frac{(1 - \alpha_n)\beta_n}{(1 - \lambda)\alpha_n} \|Sx^* - x^*\| \|x_{n+1} - x^*\| + \frac{1}{1 - \lambda} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \right\} \leq 0, \tag{3.24}$$

$\sum_{n=1}^{\infty} \alpha_n = \infty$ , and  $2(1 - \lambda)\alpha_n / (1 + (1 - \lambda)\alpha_n) \geq (1 - \lambda)\alpha_n$ , we have

$$\sum_{n=1}^{\infty} a_n = \infty, \quad \limsup_{n \rightarrow \infty} \frac{b_n}{a_n} = 0, \quad \sum_{n=1}^{\infty} c_n = 0. \quad (3.25)$$

Therefore, it follows from Lemma 2.3(2) that

$$\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0. \quad (3.26)$$

This completes the proof.  $\square$

*Remark 3.2.* In (1.9), if  $f = 0$ , then it follows that  $x_n \rightarrow x^* = P_F 0$ . In this case, from (3.1), it follows that

$$\langle x^*, x^* - x \rangle \leq 0, \quad \forall x^* \in F, \quad (3.27)$$

that is,

$$\|x^*\|^2 \leq \langle x^*, x \rangle \leq \|x^*\| \|x\|, \quad \forall x \in F. \quad (3.28)$$

Therefore, the point  $x^*$  is the unique solution to the quadratic minimization problem

$$x^* = \arg \min_{x \in F} \|x\|^2. \quad (3.29)$$

In Theorem 3.1, if  $T_i = T$  for all  $i \geq 1$ , then we get the following result.

**Corollary 3.3.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $f : C \rightarrow H$  be a  $\lambda$ -contraction with  $\lambda \in [0, 1)$ . Let  $S : C \rightarrow H$  be a nonexpansive mapping and  $T : C \rightarrow C$  a nonexpansive mapping such that  $F(T) \neq \emptyset$ . Let  $x_1 \in C$  and define a sequence  $\{x_n\}$  by*

$$\begin{aligned} y_n &= P_C [\beta_n S x_n + (1 - \beta_n) x_n], \\ x_{n+1} &= P_C [\alpha_n f(x_n) + (1 - \alpha_n) T y_n], \quad \forall n \geq 1, \end{aligned} \quad (3.30)$$

where  $\{\alpha_n\} \subset (0, 1)$  and  $\{\beta_n\} \subset (0, 1)$  are two sequences satisfying the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (b)  $\lim_{n \rightarrow \infty} (\beta_n / \alpha_n) = 0$ ,
- (c)  $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$  and  $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in F(T)$ , which is the unique solution of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \quad (3.31)$$

In Corollary 3.3, if  $S$  is a self-mapping of  $C$  into itself, then we get the following result.

**Corollary 3.4.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $f : C \rightarrow H$  be a  $\lambda$ -contraction with  $\lambda \in [0, 1)$ . Let  $S : C \rightarrow C$  be a nonexpansive mapping and  $T : C \rightarrow C$  a nonexpansive mapping such that  $F(T) \neq \emptyset$ . Let  $x_1 \in C$  and define a sequence  $\{x_n\}$  by*

$$\begin{aligned} y_n &= \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} &= P_C[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n], \quad \forall n \geq 1, \end{aligned} \quad (3.32)$$

where the sequences  $\{\alpha_n\} \subset (0, 1)$  and  $\{\beta_n\} \subset (0, 1)$  are two sequences satisfying the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (b)  $\lim_{n \rightarrow \infty} (\beta_n / \alpha_n) = 0$ ,
- (c)  $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$  and  $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in F(T)$ , which is the unique solution of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \quad (3.33)$$

By changing the restrictions on parameters in Theorem 3.1, we obtain the following.

**Theorem 3.5.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $f : C \rightarrow H$  be a  $\lambda$ -contraction with  $\lambda \in [0, 1)$ . Let  $S : C \rightarrow C$  be a nonexpansive mapping and  $\{T_i\}_{i=1}^{\infty} : C \rightarrow C$  a countable family of nonexpansive mappings such that  $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ . Set  $\alpha_0 = 1$ . Let  $x_1 \in C$  and define a sequence  $\{x_n\}$  by*

$$\begin{aligned} y_n &= \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} &= P_C \left[ \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i y_n \right], \quad \forall n \geq 1, \end{aligned} \quad (3.34)$$

where  $\{\alpha_n\} \subset (0, 1)$  is a strictly decreasing sequence and  $\{\beta_n\} \subset (0, 1)$  is a sequence satisfying the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (b)  $\lim_{n \rightarrow \infty} (\beta_n / \alpha_n) = \tau \in (0, \infty)$ ,
- (c)  $\sum_{n=1}^{\infty} (\alpha_{n-1} - \alpha_n) < \infty$  and  $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$ ,
- (d)  $\lim_{n \rightarrow \infty} ((\alpha_{n-1} - \alpha_n) + |\beta_n - \beta_{n-1}|) / \alpha_n \beta_n = 0$ ,
- (e) there exists a constant  $K > 0$  such that  $(1/\alpha_n)|1/\beta_n - 1/\beta_{n-1}| \leq K$ .

Then the sequence  $\{x_n\}$  generated by (3.34) converges strongly to a point  $x^* \in F$ , which is the unique solution of the variational inequality

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F. \quad (3.35)$$

*Proof.* First, the proof of Theorem 3.2 of [1] shows that (3.35) has the unique solution. By a similar argument as in that of Theorem 3.1, we can conclude that  $\{x_n\}$  is bounded,  $\|x_{n+1} - x_n\| \rightarrow 0$  and  $\|x_n - T_i x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Note that conditions (a) and (b) imply that  $\beta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hence we have

$$\|y_n - x_n\| = \beta_n \|Sx_n - x_n\| \rightarrow 0 \quad (n \rightarrow \infty). \quad (3.36)$$

It follows that, for all  $i \geq 1$ ,

$$\|y_n - T_i x_n\| \leq \|y_n - x_n\| + \|x_n - T_i x_n\| \rightarrow 0 \quad (n \rightarrow \infty). \quad (3.37)$$

Now, it follows from (3.36) and (3.37) that, for all  $i \geq 1$ ,

$$\|y_n - T_i y_n\| \leq \|y_n - T_i x_n\| + \|T_i x_n - T_i y_n\| \leq \|y_n - T_i x_n\| + \|x_n - y_n\| \rightarrow 0 \quad (n \rightarrow \infty). \quad (3.38)$$

From (3.8), we get

$$\begin{aligned} \frac{\|x_{n+1} - x_n\|}{\beta_n} &\leq \frac{\|u_n - u_{n-1}\|}{\beta_n} \\ &\leq (1 - (1 - \lambda)\alpha_n) \frac{\|x_n - x_{n-1}\|}{\beta_n} + M \left[ \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{\alpha_{n-1} - \alpha_n}{\beta_n} \right] \\ &= (1 - (1 - \lambda)\alpha_n) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + (1 - (1 - \lambda)\alpha_n) \|x_n - x_{n-1}\| \left( \frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right) \\ &\quad + M \left[ \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{\alpha_{n-1} - \alpha_n}{\beta_n} \right]. \end{aligned} \quad (3.39)$$

Note that

$$(1 - (1 - \lambda)\alpha_n) \left( \frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right) \leq \alpha_n \frac{1}{\alpha_n} \left| \frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right| \leq \alpha_n K. \quad (3.40)$$

Hence, from (3.39), we have

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\beta_n} &\leq \frac{\|u_n - u_{n-1}\|}{\beta_n} \leq (1 - (1 - \lambda)\alpha_n) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \alpha_n K \|x_n - x_{n-1}\| \\
&\quad + M \left[ \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{\alpha_{n-1} - \alpha_n}{\beta_n} \right] \\
&\leq (1 - (1 - \lambda)\alpha_n) \frac{\|u_{n-1} - u_{n-2}\|}{\beta_{n-1}} + \alpha_n K \|x_n - x_{n-1}\| \\
&\quad + M \left[ \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{\alpha_{n-1} - \alpha_n}{\beta_n} \right].
\end{aligned} \tag{3.41}$$

Let  $a_n = (1 - \lambda)\alpha_n$  and  $b_n = \alpha_n K \|x_n - x_{n-1}\| + M[|\beta_n - \beta_{n-1}|/\beta_n + (\alpha_{n-1} - \alpha_n)/\beta_n]$ . From conditions (a) and (d), we have

$$\sum_{n=1}^{\infty} a_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = 0. \tag{3.42}$$

By Lemma 2.3(2), we get

$$\lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x_n\|}{\beta_n} = 0, \quad \lim_{n \rightarrow \infty} \frac{\|u_n - u_{n-1}\|}{\beta_n} = \lim_{n \rightarrow \infty} \frac{\|u_n - u_{n-1}\|}{\alpha_n} = 0. \tag{3.43}$$

From (3.34), we have

$$x_{n+1} = P_C u_n - u_n + \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (T_i y_n - y_n) + (1 - \alpha_n) y_n. \tag{3.44}$$

Hence it follows that

$$\begin{aligned}
x_n - x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n x_n \\
&\quad - \left[ P_C u_n - u_n + \alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (T_i y_n - y_n) + (1 - \alpha_n) y_n \right] \\
&= (1 - \alpha_n)\beta_n(x_n - Sx_n) + (u_n - P_C u_n) \\
&\quad + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (y_n - T_i y_n) + \alpha_n (x_n - f(x_n))
\end{aligned} \tag{3.45}$$

and hence

$$\begin{aligned} \frac{x_n - x_{n+1}}{(1 - \alpha_n)\beta_n} &= x_n - Sx_n + \frac{1}{(1 - \alpha_n)\beta_n}(u_n - P_C u_n) \\ &+ \frac{1}{(1 - \alpha_n)\beta_n} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)(y_n - T_i y_n) + \frac{\alpha_n}{(1 - \alpha_n)\beta_n}(x_n - f(x_n)). \end{aligned} \quad (3.46)$$

Let  $v_n = (x_n - x_{n+1}) / (1 - \alpha_n)\beta_n$ . For any  $z \in F$ , we have

$$\begin{aligned} \langle v_n, x_n - z \rangle &= \frac{1}{(1 - \alpha_n)\beta_n} \langle u_n - P_C u_n, P_C u_{n-1} - z \rangle + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \langle (I - f)x_n, x_n - z \rangle \\ &+ \langle x_n - Sx_n, x_n - z \rangle + \frac{1}{(1 - \alpha_n)\beta_n} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle y_n - T_i y_n, x_n - z \rangle. \end{aligned} \quad (3.47)$$

By Lemma 2.2, we have

$$\begin{aligned} \langle x_n - Sx_n, x_n - z \rangle &= \langle (I - S)x_n - (I - S)z, x_n - z \rangle + \langle (I - S)z, x_n - z \rangle \\ &\geq \langle (I - S)z, x_n - z \rangle, \end{aligned} \quad (3.48)$$

$$\begin{aligned} \langle (I - f)x_n, x_n - z \rangle &= \langle (I - f)x_n - (I - f)z, x_n - z \rangle + \langle (I - f)z, x_n - z \rangle \\ &\geq (1 - \lambda)\|x_n - z\|^2 + \langle (I - f)z, x_n - z \rangle, \end{aligned} \quad (3.49)$$

$$\begin{aligned} \langle y_n - T_i y_n, x_n - z \rangle &= \langle (I - T_i)y_n - (I - T_i)z, x_n - y_n \rangle + \langle (I - T_i)y_n - (I - T_i)z, y_n - z \rangle \\ &\geq \langle (I - T_i)y_n - (I - T_i)z, x_n - y_n \rangle \\ &= \beta_n \langle (I - T_i)y_n, x_n - Sx_n \rangle, \quad i \geq 1. \end{aligned} \quad (3.50)$$

By (2.4), we have

$$\begin{aligned} \langle u_n - P_C u_n, P_C u_{n-1} - z \rangle &= \langle u_n - P_C u_n, P_C u_{n-1} - P_C u_n \rangle \\ &+ \langle u_n - P_C u_n, P_C u_n - z \rangle \\ &\geq \langle u_n - P_C u_n, P_C u_{n-1} - P_C u_n \rangle. \end{aligned} \quad (3.51)$$

Now, from (3.47)–(3.51) it follows that

$$\begin{aligned} \langle v_n, x_n - z \rangle &\geq \frac{1}{(1 - \alpha_n)\beta_n} \langle u_n - P_C u_n, P_C u_{n-1} - P_C u_n \rangle + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \langle (I - f)z, x_n - z \rangle \\ &+ \langle (I - S)z, x_n - z \rangle + \frac{1}{(1 - \alpha_n)} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle (I - T_i)y_n, x_n - Sx_n \rangle \\ &+ \frac{(1 - \lambda)\alpha_n}{(1 - \alpha_n)\beta_n} \|x_n - z\|^2. \end{aligned} \quad (3.52)$$

Observe that (3.52) implies that

$$\begin{aligned} \|x_n - z\|^2 &\leq \frac{(1 - \alpha_n)\beta_n}{(1 - \lambda)\alpha_n} [\langle v_n, x_n - z \rangle - \langle (I - S)z, x_n - z \rangle] - \frac{1}{1 - \lambda} \langle (I - f)z, x_n - z \rangle \\ &\quad - \frac{\beta_n}{(1 - \lambda)\alpha_n} \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle (I - T_i)y_n, x_n - Sx_n \rangle + \frac{\|u_{n-1} - u_n\|}{(1 - \lambda)\alpha_n} \|u_n - P_C u_n\|. \end{aligned} \quad (3.53)$$

Since  $v_n \rightarrow 0$ ,  $y_n - T_i y_n \rightarrow 0$  for all  $i \geq 1$ , and  $\|u_{n-1} - u_n\|/\alpha_n \rightarrow 0$  as  $n \rightarrow \infty$ , every weak cluster point of  $\{x_n\}$  is also a strong cluster point. Since  $\{x_n\}$  is bounded, there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  converging to a point  $x^* \in C$ . Note that  $x_n - T_i x_n \rightarrow 0$  as  $n \rightarrow \infty$  for all  $i \geq 1$ . By the demiclosed principle for a nonexpansive mapping, we have  $x^* \in F(T_i)$  for all  $i \geq 1$  and so  $x^* \in F = \bigcap_{i=1}^{\infty} F(T_i)$ . From (3.47), (3.48), (3.50), and (3.51), it follows that, for all  $z \in F$ ,

$$\begin{aligned} &\langle (I - f)x_{n_k}, x_{n_k} - z \rangle \\ &= \frac{(1 - \alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}} \langle v_{n_k}, x_{n_k} - z \rangle - \frac{1}{\alpha_{n_k}} \langle u_{n_k} - P_C u_{n_k}, P_C u_{n_k-1} - z \rangle \\ &\quad - \frac{(1 - \alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}} \langle x_{n_k} - Sx_{n_k}, x_{n_k} - z \rangle - \frac{1}{\alpha_{n_k}} \sum_{i=1}^{n_k} (\alpha_{i-1} - \alpha_i) \langle y_{n_k} - T_i y_{n_k}, x_{n_k} - z \rangle \\ &\leq \frac{(1 - \alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}} \langle v_{n_k}, x_{n_k} - z \rangle - \frac{1}{\alpha_{n_k}} \|u_{n_k} - P_C u_{n_k}\| \|u_{n_k-1} - u_{n_k}\| \\ &\quad - \frac{(1 - \alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}} \langle (I - S)z, x_{n_k} - z \rangle - \frac{\beta_{n_k}}{\alpha_{n_k}} \sum_{i=1}^{n_k} (\alpha_{i-1} - \alpha_i) \langle (I - T_i)y_{n_k}, x_{n_k} - Sx_{n_k} \rangle. \end{aligned} \quad (3.54)$$

Since  $v_n \rightarrow 0$ ,  $(I - T_i)y_n \rightarrow 0$  for all  $i \geq 1$ , and  $\|u_n - u_{n-1}\|/\alpha_n = 0$ , letting  $k \rightarrow \infty$  in (3.54), we obtain

$$\langle (I - f)x^*, x^* - z \rangle \leq -\tau \langle (I - S)z, x^* - z \rangle, \quad \forall z \in F. \quad (3.55)$$

Since (3.35) has the unique solution, it follows that  $\omega_w(x_n) = \{\bar{x}\}$ . Since every weak cluster point of  $\{x_n\}$  is also a strong cluster point, we conclude that  $x_n \rightarrow \bar{x}$  as  $n \rightarrow \infty$ . This completes the proof.  $\square$

In Theorem 3.5, if  $T_i = T$  for each  $i \geq 1$ , then we have the following result, which is Theorem 3.2 of Yao et al. [1].

**Corollary 3.6.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $f : C \rightarrow H$  be a  $\lambda$ -contraction with  $\lambda \in [0, 1)$ . Let  $S : C \rightarrow C$  be a nonexpansive mapping and  $T : C \rightarrow C$  a nonexpansive mapping such that  $F(T) \neq \emptyset$ . Let  $x_1 \in C$  and define a sequence  $\{x_n\}$  by*

$$\begin{aligned} y_n &= \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} &= P_C [\alpha_n f(x_n) + (1 - \alpha_n)Ty_n], \quad \forall n \geq 1, \end{aligned} \quad (3.56)$$



where  $\{\alpha_n\} \subset (0, 1)$  and  $\{\beta_n\} \subset (0, 1)$  are the sequences satisfying the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (b)  $\lim_{n \rightarrow \infty} (\beta_n / \alpha_n) = \tau \in (0, \infty)$ ,
- (c)  $\sum_{n=1}^{\infty} (\alpha_{n-1} - \alpha_n) < \infty$  and  $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$ ,
- (d)  $\lim_{n \rightarrow \infty} ((\alpha_{n-1} - \alpha_n) + |\beta_n - \beta_{n-1}|) / \alpha_n \beta_n = 0$ ,
- (e) there exists a constant  $K > 0$  such that  $(1/\alpha_n)|(1/\beta_n) - 1/\beta_{n-1}| \leq K$ .

Then the sequence  $\{x_n\}$  generated by (3.56) converges strongly to a point  $x^* \in F(T)$ , which is the unique solution of the variational inequality

$$\left\langle \frac{1}{\tau} (I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(T). \quad (3.57)$$

*Remark 3.7.* In (1.9), if  $S = I$  and  $f$  is a self-contraction of  $C$ , then we get

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Ty_n, \quad \forall n \geq 0, \quad (3.58)$$

which is well known as the viscosity method studied by Moudafi [6] and Xu [7]. If  $S$  and  $f$  are both self-mappings of  $C$  in (1.9), then we get the algorithm of Cianciaruso et al. [5].

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