Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 147130, 8 pages
doi:10.1155/2011/147130

Research Article

On the Fuzzy Convergence

Abdul Hameed Q. A. Al-Tai

College of Qu'ranic Studies, University of Babylon, Babylon, Iraq

Correspondence should be addressed to Abdul Hameed Q. A. Al-Tai, ahbabil@yahoo.com

Received 9 July 2011; Accepted 19 September 2011

Academic Editor: Ch Tsitouras

The aim of this paper is to introduce and study the fuzzy neighborhood, the limit fuzzy number, the convergent fuzzy sequence, the bounded fuzzy sequence, and the Cauchy fuzzy sequence on the base which is adopted by Abdul Hameed (every real number r is replaced by a fuzzy number $\overline{r}$ (either triangular fuzzy number or singleton fuzzy set (fuzzy point))). And then, we will consider that some results respect effect of the upper sequence on the convergent fuzzy sequence, the bounded fuzzy sequence, and the Cauchy fuzzy sequence.

1. Introduction

Zadeh [1] introduced the concept of fuzzy set in 1965. Kramosil and Michálek [2] introduced the concept of fuzzy metric space using continuous t-norms in 1975.

Matloka [3] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties in 1986. Sequences of fuzzy numbers also were discussed by Nanda [4], Kwon [5], Esi [6], and many others.

In 2010, Al-Tai defined the fuzzy metric space, the fuzzy sequence, and many other concepts, on other base, that the family of fuzzy real numbers is $\overline{\overline{R}} = \overline{\overline{Z}} \cup \overline{\overline{Q}} \cup \overline{\overline{Q}}'$. $\overline{\overline{Z}}$ is the family of fuzzy integer numbers, where every $\overline{r} \in \overline{\overline{Z}}$ $(r \in Z)$ is a singleton fuzzy set (fuzzy point) (see [7–12]). $\overline{\overline{Q}}$ and $\overline{\overline{Q}}'$ are the family of fuzzy rational numbers and the family of fuzzy irrational numbers, respectively, where every $\overline{r} \in \overline{\overline{Q}}$ $(r \in Q)$ or $\overline{r}' \in \overline{\overline{Q}}'$ $(r' \in Q')$ is a triangular fuzzy number [13] and by using the representation Theorem (the resolution principle) [14].

In this research, we will consider the fuzzy neighborhood, the limit fuzzy number, the convergent fuzzy sequence, the bounded fuzzy sequence, and the Cauchy fuzzy sequence, and then we will introduce some results about properties of the above concepts on the base which depended by Al-Tai to define the fuzzy metric space.

Definition 1.1. Let $(\overline{\overline{X}}, \overline{\overline{d}})$ be a fuzzy metric space. If $\overline{x} \in \overline{\overline{X}}$, then a neighborhood of $\overline{x}$ is a family of fuzzy numbers $\overline{\overline{N}}_{\overline{\varepsilon}}(\overline{x})$ consisting of all fuzzy numbers $\overline{y} \in \overline{\overline{X}}$, such that $\overline{d}(\overline{x}, \overline{y}) < \overline{\varepsilon}$ $(\overline{\varepsilon} > \overline{0})$. The fuzzy number $\overline{\varepsilon}$ will be called the fuzzy radius of $\overline{\overline{N}}_{\overline{\varepsilon}}(\overline{x})$.

Definition 1.2. A fuzzy number $\overline{x}$ is a limit fuzzy number of the family of the fuzzy numbers $\overline{\overline{E}}$, if every fuzzy neighborhood of $\overline{x}$ contains a fuzzy number $\overline{y} \neq \overline{x}$, such that $\overline{y} \in \overline{\overline{E}}$.

Definition 1.3 (Al-Tai [13]). If we have the family of the fuzzy numbers $\overline{\overline{X}}$ and the family of the fuzzy natural numbers $\overline{\overline{N}}$, the fuzzy sequence $\langle \overline{x}_n \rangle$ in $\overline{\overline{X}}$ is a fuzzy function from $\overline{\overline{N}}$ to $\overline{\overline{X}}$.

A fuzzy sequence $\langle \overline{x}_n \rangle$ consists of all ordered tuples (sequences) at degree α, for all $\alpha \in]0,1]$; $\langle x_{j,n,\alpha} \rangle = \langle x_{j,1,\alpha}, x_{j,2,\alpha}, \ldots \rangle \in \langle \overline{x}_1[\alpha], \overline{x}_2[\alpha], \ldots \rangle = \langle \overline{x}_1, \overline{x}_2, \ldots \rangle[\alpha]$ is its α-cut, $j \in N$. The fuzzy value of the fuzzy function at $\overline{n} \in \overline{\overline{N}}$ is denoted by $\overline{x}_n$.

Definition 1.4. A fuzzy sequence $\langle \overline{x}_n \rangle$ in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ is said to be convergent, if there is a fuzzy number $\overline{x} \in \overline{\overline{X}}$, such that for every $\overline{\varepsilon} > \overline{0}$, there is a fuzzy natural number $\overline{N}$, with $\overline{n} \geq \overline{N}$, which implies that

$$\overline{d}(\overline{x}_n, \overline{x}) < \overline{\varepsilon}. \tag{1.1}$$

That is, for every $\varepsilon_{j,\alpha} \in \overline{\varepsilon}\ [\alpha]$ α-cut of $\overline{\varepsilon}$, there is a natural number N, with $n \geq N$, such that

$$d(x_{j,n,\alpha}, x_{j,\alpha}) < \varepsilon_{j,\alpha}, \tag{1.2}$$

where $x_{j,n,\alpha} \in \overline{x}_n[\alpha]$ α-cut of $\overline{x}_n$, $x_{j,\alpha} \in \overline{x}[\alpha]$ α-cut of $\overline{x}$, $N \in \overline{N}$, and $n \in \overline{n}$, for all $\alpha \in]0,1]$. $\overline{x}$ will be called the fuzzy limit of $\langle \overline{x}_n \rangle$, and we write

$$\lim_{n \to \infty} \overline{x}_n = \overline{x}. \tag{1.3}$$

If $\langle x_{j,n,\alpha} \rangle$ is not convergent at any $\alpha \in]0,1]$, then $\langle \overline{x}_n \rangle$ is said to be divergent.

Definition 1.5. The family of the fuzzy numbers $\overline{\overline{E}}$ in the fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ is bounded, if there is a fuzzy real number $\overline{r} > \overline{0}$ and a fuzzy number $\overline{x} \in \overline{\overline{X}}$, such that

$$\overline{d}(\overline{y}, \overline{x}) < \overline{r} \tag{1.4}$$

for all $\overline{y} \in \overline{\overline{X}}$. That is, if $r_\alpha \in \overline{r}[\alpha]$ α-cut of $\overline{r}$ and $x_\alpha \in \overline{x}[\alpha]$ α-cut of $\overline{x} \in \overline{X}$, then

$$d(x_\alpha, y_\alpha) < r_\alpha, \tag{1.5}$$

where $y_\alpha \in \overline{y}[\alpha]$ α-cut of all $\overline{y} \in \overline{\overline{X}}$, for all $\alpha \in]0,1]$.

Definition 1.6. The fuzzy sequence $\langle \overline{x}_n \rangle$ in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ is said to be bounded if its fuzzy range is bounded.

Definition 1.7. A fuzzy sequence $\langle \overline{x}_n \rangle$ in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ is said to be a Cauchy fuzzy sequence if for every $\overline{\varepsilon} > \overline{0}$, there is a fuzzy natural number $\overline{N}$, such that

$$\overline{d}(\overline{x}_n, \overline{x}_m) < \overline{\varepsilon}, \tag{1.6}$$

whenever $\overline{n} \geq \overline{N}$ and $\overline{m} \geq \overline{N}$. In other words, every $x_{j,n,\alpha} \in \overline{x}_n[\alpha]$ α-cut of $\overline{x}_n$ is a Cauchy sequence; that is, for every $\varepsilon_{j,\alpha} \in \overline{\varepsilon}[\alpha]$ α-cut of $\overline{\varepsilon}$, there is a natural number N, such that

$$d\left(x_{j,n,\alpha}, x_{j,m,\alpha}\right) < \varepsilon_{j,\alpha} \tag{1.7}$$

for every $n \geq N,\ m \geq N$, where $x_{j,n,\alpha} \in \overline{x}_n[\alpha]$ α-cut of $\overline{x}_n$, $x_{j,m,\alpha} \in \overline{x}_m[\alpha]$ α-cut of $\overline{x}_m$, $n \in \overline{n}$, $m \in \overline{m}$, and $N \in \overline{N}$, for all $\alpha \in]0,1]$.

Examples 1.8. (1) $\langle 1/(n+1), 1/n, 1/(n-1) \rangle$ converges to $\overline{0}$, since for all $\langle x_{n,\alpha} \rangle \in [(n+\alpha)/n(n+1), (n-\alpha)/n(n-1)]$ α-cut of $\langle 1/(n+1), 1/n, 1/(n-1) \rangle$, for all $\varepsilon_\alpha > 0$, $\exists N \in \overline{N} \in \overline{\overline{N}}$: $d_\alpha(x_{n,\alpha}, 0) < \varepsilon_\alpha$, for all $n > N$, for all $\alpha \in]0,1]$.

Therefore,

$$\lim_{n \to \infty} \left\langle \frac{1}{n+1}, \frac{1}{n}, \frac{1}{n-1} \right\rangle = \overline{0}. \tag{1.8}$$

(2) $\langle 1/e^{n+1}, 1/e^n, 1/e^{n-1} \rangle$ converges to $\overline{0}$, since for all $\langle x_{n,\alpha} \rangle \in [((e-1)\alpha+1)/e^{n+1}, ((1-e)\alpha+e)/e^n]$ α-cut of $\langle 1/e^{n+1}, 1/e^n, 1/e^{n-1} \rangle$, for all $\varepsilon_\alpha > 0$, $\exists N \in \overline{N} \in \overline{\overline{N}}$: $d_\alpha(x_{n,\alpha}, 0) < \varepsilon_\alpha$, for all $n > N$, for all $\alpha \in]0,1]$.

Therefore,

$$\lim_{n \to \infty} \left\langle \frac{1}{e^{n+1}}, \frac{1}{e^n}, \frac{1}{e^{n-1}} \right\rangle = \overline{0}. \tag{1.9}$$

(3) $\overline{\sin}(\overline{n} \cdot (x_1, x, x_2))/\overline{\sqrt{n}} = (\sin(nx_1), \sin(nx), \sin(nx_2))/((\sqrt{n}\,)_1, \sqrt{n}, (\sqrt{n})_2)$ converges to $\overline{0}$, where α-cut of $(\sin(nx_1), \sin(nx), \sin(nx_2))$ is $[(\sin(nx) - \sin(nx_1))\alpha + \sin(nx_1), (\sin(nx) - \sin(nx_2))\alpha + \sin(nx_2)] = [a_1(\alpha), a_2(\alpha)]$.

And α-cut of $((\sqrt{n})_1, \sqrt{n}, (\sqrt{n})_2)$ is

$$\left[\left(\sqrt{n} - \left(\sqrt{n}\right)_1\right)\alpha + \left(\sqrt{n}\right)_1, \left(\sqrt{n} - \left(\sqrt{n}\right)_2\right)\alpha + \left(\sqrt{n}\right)_2\right] = [b_1(\alpha), b_2(\alpha)]. \tag{1.10}$$

Since

$$\forall \langle x_{n,\alpha} \rangle \in \left[\min\left(\frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)}\right), \max\left(\frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)}\right)\right], \\ 0 \notin [b_1(\alpha), b_2(\alpha)], \tag{1.11}$$

α-cut of $(\sin(nx_1), \sin(nx), \sin(nx_2))/((\sqrt{n})_1, \sqrt{n}, (\sqrt{n})_2)$, for all $\varepsilon_\alpha > 0$, $\exists N \in \overline{N} \in \overline{\overline{N}}$: $d_\alpha(x_{n,\alpha}, 0) < \varepsilon_\alpha$, for all $n > N$, for all $\alpha \in]0,1]$.

Therefore,

$$\lim_{n\to\infty} \frac{(\sin(nx_1), \sin(nx), \sin(nx_2))}{((\sqrt{n})_1, \sqrt{n}, (\sqrt{n})_2)} = \overline{0}. \tag{1.12}$$

(4) $\langle (1/2, 1/3, 1/4) \rangle$ converges to $(1/2, 1/3, 1/4)$.

(5) $\langle \overline{n} \rangle$ is divergent and unbounded.

(6) $\overline{\cos}(\overline{n} \cdot (x_1, x, x_2)) = (\cos(nx_1), \cos(nx), \cos(nx_2))$ is divergent, since for all $\langle x_{n,\alpha} \rangle \in [(\cos(nx) - \cos(nx_1))\alpha + \cos(nx_1), (\cos(nx) - \cos(nx_2))\alpha + \cos(nx_2)]$ α-cut of $\overline{\cos}(\overline{n} \cdot (x_1, x, x_2))$ is divergent. And it is bounded.

Theorem 1.9. *Let $\langle \overline{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α-cut of $\langle \overline{x}_n \rangle$ is bounded for all $\alpha \in]0,1]$, then $\langle \overline{x}_n \rangle$ is bounded.*

Proof. Let $x_{j,\alpha}$, be a supremum of a sequence $\langle x_{j,n,\alpha} \rangle$, $j = 1, 2, \ldots$, for all $\alpha \in]0,1]$. We have from the hypothesis, the upper sequence $\langle x_{1,n,\alpha} \rangle$ is bounded for all $\alpha \in]0,1]$; that is, there is a real number $r_{1,\alpha} \in \overline{r}[\alpha]$ α-cut of $\overline{r} \in \overline{\overline{R}}$, such that

$$d(x_{1,i,\alpha}, 0) < r_{1,\alpha} \tag{1.13}$$

for all $x_{1,i,\alpha} \in \langle x_{1,n,\alpha} \rangle$, $i \in N$, for all $\alpha \in]0,1]$.

When $j = 2$,

$$\begin{aligned} d(x_{2,i,\alpha}, 0) &\le d(x_{2,i,\alpha}, x_{1,\alpha}) + d(x_{1,\alpha}, 0) \\ &< r'_{2,\alpha} + r_{1,\alpha} = r_{2,\alpha}, \end{aligned} \tag{1.14}$$

where $r'_{2,\alpha} = \max\{d(x_{2,i,\alpha}, x_{1,\alpha}), x_{2,i,\alpha} \in \langle x_{2,n,\alpha} \rangle\}$. That is, there is a real number $r_{2,\alpha} \in \overline{r}[\alpha]$ α-cut of $\overline{r} \in \overline{\overline{R}}$, such that

$$d(x_{2,i,\alpha}, 0) < r_{2,\alpha} \tag{1.15}$$

for all $x_{2,i,\alpha} \in \langle x_{2,n,\alpha} \rangle$, $i \in N$, for all $\alpha \in]0,1]$.

When $j = 3$,

$$\begin{aligned} d(x_{3,i,\alpha}, 0) &\le d(x_{3,i,\alpha}, x_{2,\alpha}) + d(x_{2,\alpha}, 0) \\ &< r'_{3,\alpha} + r_{2,\alpha} = r_{3,\alpha}, \end{aligned} \tag{1.16}$$

where $r'_{3,\alpha} = \max\{d(x_{3,i,\alpha}, x_{2,\alpha}),\ x_{3,i,\alpha} \in \langle x_{3,n,\alpha} \rangle\}$. That is, there is a real number $r_{3,\alpha} \in \overline{r}[\alpha]$ α-cut of $\overline{r} \in \overline{\overline{R}}$, such that

$$d(x_{3,i,\alpha}, 0) < r_{3,\alpha} \tag{1.17}$$

for all $x_{3,i,\alpha} \in \langle x_{3,n,\alpha} \rangle$, $i \in N$, for all $\alpha \in]0,1]$.

And so on, when $j = m + 1$,

$$\begin{aligned} d(x_{m+1,i,\alpha}, 0) &\le d(x_{m+1,i,\alpha}, x_{m,\alpha}) + d(x_{m,\alpha}, 0) \\ &< r'_{m+1,\alpha} + r_{m,\alpha} = r_{m+1,\alpha}, \end{aligned} \tag{1.18}$$

where $r'_{m+1,\alpha} = \max\{d(x_{m+1,i,\alpha}, x_{m,\alpha}),\ x_{m+1,i,\alpha} \in \langle x_{m+1,n,\alpha}\rangle\}$. That is, there is a real number $r_{m+1,\alpha} \in \overline{r}[\alpha]$ α-cut of $\overline{r} \in \overline{\overline{R}}$, such that

$$d(x_{m+1,i,\alpha}, 0) < r_{m+1,\alpha} \tag{1.19}$$

for all $x_{m+1,i,\alpha} \in \langle x_{m+1,n,\alpha}\rangle,\ i \in N,\ m \in N$, for all $\alpha \in]0,1]$. By Definitions 1.5 and 1.6, $\langle \overline{x}_n\rangle$ is bounded. □

Theorem 1.10. *Let $\langle \overline{x}_n\rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α-cut of $\langle \overline{x}_n\rangle$ is bounded for all $\alpha \in]0,1]$, then $\langle \overline{x}_n\rangle$ is convergent.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha}\rangle$ of the α-cut of $\langle \overline{x}_n\rangle$ is bounded for all $\alpha \in]0,1]$. By Theorem 1.9, $\langle \overline{x}_n\rangle$ is bounded; that is, every sequence $\langle x_{j,n,\alpha}\rangle$, $j = 2,3,\ldots$, of the α-cut of $\langle \overline{x}_n\rangle$ is bounded for all $\alpha \in]0,1]$, but every sequence $\langle x_{j,n,\alpha}\rangle$, $j = 1,2,\ldots$, of the α-cut of $\langle \overline{x}_n\rangle$ is monotonic, for all $\alpha \in]0,1]$; then, every sequence $\langle x_{j,n,\alpha}\rangle$, $j = 1,2,\ldots$, of the α-cut of $\langle \overline{x}_n\rangle$ is convergent, for all $\alpha \in]0,1]$ (see [15, Theorem 3.14]). By Definition 1.4, $\langle \overline{x}_n\rangle$ is convergent. □

Theorem 1.11. *Let $\langle \overline{x}_n\rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α-cut of $\langle \overline{x}_n\rangle$ is convergent for all $\alpha \in]0,1]$, then $\langle \overline{x}_n\rangle$ is convergent.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha}\rangle$ of the α-cut of $\langle \overline{x}_n\rangle$ is convergent for all $\alpha \in]0,1]$, so it is bounded (see [15, Theorem 3.2-c]). By Theorem 1.10, $\langle \overline{x}_n\rangle$ is convergent. □

Theorem 1.12. *Every convergent fuzzy sequence $\langle \overline{x}_n\rangle$ in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ is bounded.*

Proof. Let $\langle \overline{x}_n\rangle$ be a convergent fuzzy sequence in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$. By Definition 1.4, every sequence $\langle x_{j,n,\alpha}\rangle$, $j = 1,2,\ldots$, of the α-cut of $\langle \overline{x}_n\rangle$ is convergent for all $\alpha \in]0,1]$; that is, the upper sequence $\langle x_{1,n,\alpha}\rangle$ of $\langle \overline{x}_n\rangle$ is bounded for all $\alpha \in]0,1]$ (see [15, Theorem 3.2-c]). By Theorem 1.9, $\langle \overline{x}_n\rangle$ is bounded. □

Theorem 1.13. *Every bounded fuzzy sequence $\langle \overline{x}_n\rangle$ in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$ is convergent.*

Proof. Let $\langle \overline{x}_n\rangle$ be a bounded fuzzy sequence in a $(\overline{\overline{R}}, \overline{\overline{d}})$. By Definition 1.5, we get that the upper sequence $\langle x_{1,n,\alpha}\rangle$ of $\langle \overline{x}_n\rangle$ is bounded for all $\alpha \in]0,1]$. By Theorem 1.10, $\langle \overline{x}_n\rangle$ is convergent. □

Theorem 1.14. *Let $\langle \overline{x}_n\rangle$ be a fuzzy sequence in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$; then $\langle \overline{x}_n\rangle$ converges to $\overline{x} \in \overline{\overline{X}}$ if and only if every fuzzy neighborhood of $\overline{x}$ contains all but finitely many of the terms of $\langle \overline{x}_n\rangle$.*

Proof. Suppose that $\overline{\overline{N}}_{\overline{\varepsilon}}(\overline{x})$ is a fuzzy neighborhood of $\overline{x}$. For $\overline{\varepsilon} > \overline{0}$, $\overline{d}(\overline{y},\overline{x}) < \overline{\varepsilon}$, $\overline{y} \in \overline{\overline{X}}$, imply $\overline{y} \in \overline{\overline{N}}_{\overline{\varepsilon}}(\overline{x})$. We have from the hypothesis $\langle \overline{x}_n \rangle$ converges to $\overline{x}$, so for same $\overline{\varepsilon} > \overline{0}$, there exists $\overline{N} \in \overline{\overline{N}}$, such that $\overline{n} \geq \overline{N}$, implying $\overline{d}(\overline{x}_n,\overline{x}) < \overline{\varepsilon}$. That is, $\overline{x}_n \in \overline{\overline{N}}_{\overline{\varepsilon}}(\overline{x})$.

Conversely, suppose that every fuzzy neighborhood of $\overline{x}$ contains all but finitely many of $\langle \overline{x}_n \rangle$. Fix $\overline{\varepsilon} > \overline{0}$, and let $\overline{\overline{N}}_{\overline{\varepsilon}}(\overline{x})$ be the set of all $\overline{y} \in \overline{\overline{X}}$, such that $\overline{d}(\overline{y},\overline{x}) < \overline{\varepsilon}$.

By assumption, there exists $\overline{N} \in \overline{\overline{N}}$ corresponding to $\overline{\overline{N}}_{\overline{\varepsilon}}(\overline{x})$, such that $\overline{x}_n \in \overline{\overline{N}}_{\overline{\varepsilon}}(\overline{x})$ if $\overline{n} \geq \overline{N}$. Thus $\overline{d}(\overline{x}_n,\overline{x}) < \overline{\varepsilon}$ if $\overline{n} \geq \overline{N}$. Hence $\langle \overline{x}_n \rangle$ converges to $\overline{x}$. □

Theorem 1.15. *In any fuzzy metric space $(\overline{\overline{X}},\overline{\overline{d}})$, every convergent fuzzy sequence is a Cauchy fuzzy sequence.*

Proof. Let $\langle \overline{x}_n \rangle$ be a convergent fuzzy sequence in a fuzzy metric space $(\overline{\overline{X}},\overline{\overline{d}})$. By Definition 1.4, every sequence $\langle x_{j,n,\alpha} \rangle$, $j = 1,2,\ldots$, of the α-cut of $\langle \overline{x}_n \rangle$ is convergent for all $\alpha \in]0,1]$; then every sequence $\langle x_{j,n,\alpha} \rangle$, $j = 1,2,\ldots$, of the α-cut of $\langle \overline{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0,1]$ (see [15, Theorem 3.11-a]); by Definition 1.7, $\langle \overline{x}_n \rangle$ is a Cauchy fuzzy sequence. □

Theorem 1.16. *Let $\langle \overline{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}},\overline{\overline{d}})$. If the upper sequence of the α-cut of $\langle \overline{x}_n \rangle$ is bounded for all $\alpha \in]0,1]$,then $\langle \overline{x}_n \rangle$ is a Cauchy fuzzy sequence.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α-cut of $\langle \overline{x}_n \rangle$ is bounded for all $\alpha \in]0,1]$. By Theorem 1.10, $\langle \overline{x}_n \rangle$ is convergent. By Theorem 1.15, $\langle \overline{x}_n \rangle$ is a Cauchy fuzzy sequence. □

Theorem 1.17. *In the fuzzy metric space $(\overline{\overline{R}},\overline{\overline{d}})$, every bounded fuzzy sequence is a Cauchy fuzzy sequence.*

Proof. Let $\langle \overline{x}_n \rangle$ be a bounded fuzzy sequence in $(\overline{\overline{R}},\overline{\overline{d}})$. By Theorem 1.13, $\langle \overline{x}_n \rangle$ is a convergent fuzzy sequence. By Theorem 1.15, $\langle \overline{x}_n \rangle$ is a Cauchy fuzzy sequence. □

Theorem 1.18. *Let $\langle \overline{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}},\overline{\overline{d}})$. If the upper sequence of the α-cut of $\langle \overline{x}_n \rangle$ is convergent for all $\alpha \in]0,1]$, then $\langle \overline{x}_n \rangle$ is a bounded fuzzy sequence.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α-cut of $\langle \overline{x}_n \rangle$ is convergent for all $\alpha \in]0,1]$. By Theorem 1.11, $\langle \overline{x}_n \rangle$ is a convergent fuzzy sequence. By Theorem 1.12, $\langle \overline{x}_n \rangle$ is a bounded fuzzy sequence. □

Theorem 1.19. *Let $\langle \overline{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}},\overline{\overline{d}})$. If the upper sequence of the α-cut of $\langle \overline{x}_n \rangle$ is convergent for all $\alpha \in]0,1]$, then $\langle \overline{x}_n \rangle$ is a Cauchy fuzzy sequence.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α-cut of $\langle \overline{x}_n \rangle$ is convergent for all $\alpha \in]0,1]$. By Theorem 1.11, $\langle \overline{x}_n \rangle$ is a convergent fuzzy sequence. By Theorem 1.15, $\langle \overline{x}_n \rangle$ is a Cauchy fuzzy sequence. □

Theorem 1.20. *Every Cauchy fuzzy sequence in $\overline{\overline{R}}^n$, the n-dimensional fuzzy Euclidean space, is convergent.*

Proof. Let $\langle \overline{x}_m \rangle$ be a Cauchy fuzzy sequence in $\overline{\overline{R}}^n$. By Definitions 1.7 and 1.3, $\overline{x}_m$ consists of all Cauchy sequences at degree α, $\langle x_{m,\alpha} \rangle = \langle x_{1,\alpha}, x_{2,\alpha}, \ldots \rangle \in \langle \overline{x}_1, \overline{x}_2, \ldots \rangle[\alpha]$ α-cut of $\langle \overline{x}_m \rangle$ for all $\alpha \in]0,1]$, where every element $x_{i,\alpha} \in \langle x_{m,\alpha} \rangle$ is $(x_{i,1,\alpha}, x_{i,2,\alpha}, \ldots, x_{i,n,\alpha}) \in (\overline{x}_{i,1}, \overline{x}_{i,2}, \ldots, \overline{x}_{i,n})[\alpha]$ α-cut of $\overline{x}_i = (\overline{x}_{i,1}, \overline{x}_{i,2}, \ldots, \overline{x}_{i,n})$ for all $\alpha \in]0,1]$ (Al-Tai [13]). By Theorem 3.11-c (see [15]), $\langle x_{m,\alpha} \rangle$ is a convergent sequence for all $\alpha \in]0,1]$. By Definition 1.4, $\langle \overline{x}_m \rangle$ is a convergent fuzzy sequence. □

Theorem 1.21. *Let $\langle \overline{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α-cut of $\langle \overline{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0,1]$; then $\langle \overline{x}_n \rangle$ is convergent.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α-cut of $\langle \overline{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0,1]$, so it is convergent (see [15, Theorem 3.11-c]). By Theorem 1.11, $\langle \overline{x}_n \rangle$ is convergent. □

Theorem 1.22. *Every Cauchy fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$ is bounded.*

Proof. Let $\langle \overline{x}_n \rangle$ be a Cauchy fuzzy sequence in $(\overline{\overline{R}}, \overline{\overline{d}})$. By Theorem 1.20, $\langle \overline{x}_n \rangle$ is a convergent fuzzy sequence. By Theorem 1.12, $\langle \overline{x}_n \rangle$ is bounded. □

Theorem 1.23. *Let $\langle \overline{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α-cut of $\langle \overline{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0,1]$, then $\langle \overline{x}_n \rangle$ is bounded.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α-cut of $\langle \overline{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0,1]$. By Theorem 1.21, $\langle \overline{x}_n \rangle$ is convergent. By Theorem 1.12, $\langle \overline{x}_n \rangle$ is bounded. □

Theorem 1.24. *Let $\langle \overline{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α-cut of $\langle \overline{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0,1]$, then $\langle \overline{x}_n \rangle$ is Cauchy.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α-cut of $\langle \overline{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0,1]$; then it is convergent (see [15, Theorem 3.11-c]). By Theorem 1.19, $\langle \overline{x}_n \rangle$ is Cauchy. □

Theorem 1.25. *The limit fuzzy number of a convergent fuzzy sequence $\langle \overline{x}_n \rangle$ in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ is unique.*

Proof. Let $\langle \overline{x}_n \rangle$ be a fuzzy sequence in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ which converges to $\overline{x} \in (\overline{\overline{X}}, \overline{\overline{d}})$. By Definition 4 and Theorem 3.2-b [15], we have that every sequence $\langle x_{n,\alpha} \rangle$ of the α-cut of $\langle \overline{x}_n \rangle$ has a unique limit point $x_\alpha \in \overline{x}[\alpha]$ α-cut of $\overline{x}$ for all $\alpha \in]0,1]$, and hence, $\langle \overline{x}_n \rangle$ converges to a unique limit fuzzy number $\overline{x}$. □

Theorem 1.26. *If $\langle \overline{y}_n \rangle$ is a convergent fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$, then a fuzzy sequence $\langle \overline{x}_n \rangle$ is convergent for all $\overline{n} \geq \overline{N}$, as $\overline{0} \leq \langle \overline{x}_n \rangle \leq \langle \overline{y}_n \rangle$.*

Proof. Let $\langle y_{1,n,\alpha} \rangle$ be the upper sequence of the α-cut of $\langle \overline{y}_n \rangle$ for all $\alpha \in]0,1]$. Since $\overline{0} \leq \langle \overline{x}_n \rangle \leq \langle \overline{y}_n \rangle$, then $0 \leq \langle x_{1,n,\alpha} \rangle \leq \langle y_{n,\alpha} \rangle$ [16], where $\langle x_{1,n,\alpha} \rangle$ is an upper sequence of the α-cut of $\langle \overline{x}_n \rangle$ for all $\alpha \in]0,1]$. By Theorem 7.4.4 (Comparison Test) [17], $\langle x_{1,n,\alpha} \rangle$ is convergent for all $\alpha \in]0,1]$. By Theorem 1.11, $\langle \overline{x}_n \rangle$ is convergent. □

2. Conclusion

Fuzzy neighborhood, limit fuzzy point, convergent fuzzy sequence, bounded fuzzy sequence, and Cauchy fuzzy sequence can be defined on the dependent base by Al-Tai, these concepts help us to show that the fuzzy sequence in $(\overline{\overline{R}}, \overline{\overline{d}})$ is convergent, if an upper sequence of the α-cut of it is convergent, bounded, or Cauchy for all $\alpha \in]0,1]$. The fuzzy sequence in $(\overline{\overline{R}}, \overline{\overline{d}})$ is bounded, if an upper sequence of the α-cut of it is convergent, bounded, or Cauchy for all $\alpha \in]0,1]$. The fuzzy sequence in $(\overline{\overline{R}}, \overline{\overline{d}})$ is Cauchy, if an upper sequence of the α-cut of it is convergent, bounded, or, Cauchy for all $\alpha \in]0,1]$. The limit fuzzy number of a convergent fuzzy sequence is unique. And other results about above concepts.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.
[2] I. Kramosil and J. Michálek, "Fuzzy metrics and statistical metric spaces," *Kybernetika*, vol. 11, no. 5, pp. 336–344, 1975.
[3] M. Matloka, "Sequences of fuzzy numbers," *BUSEFAL*, vol. 28, pp. 28–37, 1986.
[4] S. Nanda, "On sequences of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 123–126, 1989.
[5] J.-S. Kwon, "On statistical and p-Cesaro convergence of fuzzy numbers," *Journal of Computational and Applied Mathematics*, vol. 7, no. 1, pp. 757–764, 2003.
[6] A. Esi, "On some new paranormed sequence spaces of fuzzy numbers defined by Orlicz functions and statistical convergence," *Mathematical Modelling and Analysis*, vol. 11, no. 4, pp. 379–388, 2006.
[7] J. Zhan, Y. B. Jun, and W. A. Dudek, "On $(\in, \in Vq)$-fuzzy filters of pseudo-BL algebras," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 33, no. 1, pp. 57–67, 2010.
[8] M. N. Mukherjee and S. P. Sinha, "On some weaker forms of fuzzy continuous and fuzzy open mappings on fuzzy topological spaces," *Fuzzy Sets and Systems*, vol. 32, no. 1, pp. 103–114, 1989.
[9] P. Dheena and S. Coumaressane, "Generalization of $(\in, \in Vq)$-fuzzy subnear-rings and ideals," *Iranian Journal of Fuzzy Systems*, vol. 5, no. 1, pp. 79–97, 2008.
[10] P. M. Pu and Y. M. Liu, "Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence," *Journal of Mathematical Analysis and Applications*, vol. 76, no. 2, pp. 571–599, 1980.
[11] S. Jafari and R. M. Latif, "Applications of fuzzy points via fuzzy $\theta-$open sets and fuzzy $\theta-$closure operator," Tech. Rep. number TR 392, 2008.
[12] L. Zadeh, "A fuzzy set theoretic interpretation of linguistic hedges," Memorandum number ERLM335, University of California, Berkeley, Calif, USA, 1972.
[13] A.-H. Q. A. Al-Tai, "On the fuzzy metric spaces," *European Journal of Scientific Research*, vol. 47, no. 2, pp. 214–229, 2010.
[14] S. Chandra and C. Bector, *Fuzzy Mathematical Programming and Fuzzy Matrix Games*, vol. 1st, Springer, New York, NY, USA, 2005.
[15] W. Rudin, *Principles of Mathematics Analysis*, McGraw-Hill, New York, NY, USA, 3rd edition, 1953.
[16] G. J. George and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall PTR, New Jersey, NJ, USA, 1st edition, 1995, Theory and Applications.
[17] J. E. Hutchinson, *Introduction to Mathematical Analysis*, 1994, Revised by R. J. Loy, 1995.