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Research Article

Generalized Second-Order Mixed Symmetric Duality in Nondifferentiable Mathematical Programming

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This paper is concerned with a pair of second-order mixed symmetric dual programs involving nondifferentiable functions. Weak, strong, and converse duality theorems are proved for aforementioned pair using the notion of second-order F-convexity/pseudoconvexity assumptions.

1. Introduction

Duality is a fruitful theory in mathematical programming and is useful both theoretically and practically. Duality as used in our daily life means the sort of harmony of two opposite or complementary parts through which they integrate into a whole. Symmetry is bound up with duality and, in particular, is significant in mathematics. The problem of optimizing a numerical function of one or more variables subject to constraints on the variables is called the mathematical programming, or constrained optimization, problem. When either the objective function or one or more of the constraints are nonlinear, the programming problem is called a nonlinear programming problem, a discipline playing an increasingly imperative role in such diverse fields as operations research and management science, engineering, economics, system analysis, and computer science.

Dantzig et al. [1], Mond [2], and Bazaraa and Goode [3] studied symmetric duality in nonlinear programming. Later, Chandra and Husain [4] formulated a pair of Wolfe-type nondifferentiable symmetric dual programs and proved duality results under convexity/concavity assumptions. Subsequently, Chandra et al. [5] weakened these assumptions to pseudoconvexity/pseudoconcavity. Mond and Schechter [6] presented two symmetric dual pairs involving nondifferentiable functions. Kumar and Bhatia [7] discussed multiobjective symmetric duality by using a nonlinear vector-valued function of two variables corresponding to various objectives.

Mangasarian [8] presented a dual problem associated with a primal nonlinear programming problem that involves second derivatives of the function constituting the primal problem. The study of second-order duality is significant, as it can provide a lower bound to the infimum of a primal optimization problem when it is difficult to find a feasible solution for the first-order dual. Bector and Chandra [9] achieved duality results for a pair of Mond-Weir-type second-order symmetric dual nonlinear programs. Hou and Yang [10] formulated a pair of second-order symmetric dual nondifferentiable programs and established duality theorems under second-order F-pseudoconvexity assumptions.

Chandra et al. [11] and Yang et al. [12] discussed a mixed symmetric dual formulation for a nonlinear programming problem and for a class of nondifferentiable nonlinear programming problems, respectively. Later on, Ahmad [13] formulated mixed type symmetric dual in multiobjective programming problems ignoring nonnegativity restrictions of Bector et al. [14].

In this paper, a pair of second-order mixed symmetric dual programs is presented for a class of nondifferentiable nonlinear programming problems. Weak, strong, and converse duality theorems are proved using the notion of second-order F-convexity/pseudoconvexity assumptions. These results generalize the known work in [6, 10–12, 15–17].

2. Preliminaries

In this section, we presented some of the basic definitions used in the paper.

Definition 2.1. Let C be a compact convex set in R^n. The support function of C is defined by

$$S(x \mid C) = \max\left\{x^T y : y \in C\right\}. \tag{2.1}$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in R^n$ such that

$$S(y \mid C) \geq S(x \mid C) + z^T (y - x), \quad \forall y \in C. \tag{2.2}$$

The subdifferential of $S(x \mid C)$ is given by

$$\partial S(x \mid C) = \left\{z \in C : z^T x = S(x \mid C)\right\}. \tag{2.3}$$

For any set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \left\{y \in R^n : y^T (z - x) \leq 0, \ \forall z \in S\right\}. \tag{2.4}$$

It can be easily seen that for a compact convex set C, y is in $N_C(x)$ if and only if $S(y \mid C) = x^T y$, or equivalently, x is in $\partial S(y \mid C)$.

Definition 2.2. A functional $F : X \times X \times R^n \mapsto R$ (where $X \subseteq R^n$) is sublinear with respect to the third variable if for all $(x, u) \in X \times X$,

(i) $F(x, u; a_1 + a_2) \leq F(x, u; a_1) + F(x, u; a_2)$ for all $a_1, a_2 \in R^n$,

(ii) $F(x, u; \alpha a) = \alpha F(x, u; a)$, for all $\alpha \in R_+$ and for all $a \in R^n$.

Let $\psi : X \mapsto R$ be a real-valued twice differentiable function.

Definition 2.3. ψ is said to be second-order F-convex at $u \in X$ with respect to $q \in R^n$, if for all $x \in X$,

$$\psi(x) - \psi(u) + \frac{1}{2} q^T \nabla_{xx} \psi(u) q \geq F\left(x, u; \nabla_x \psi(u) + \nabla_{xx} \psi(u) q\right). \tag{2.5}$$

Definition 2.4. ψ is said to be second-order F-pseudoconvex at $u \in X$ with respect to $q \in R^n$, if for all $x \in X$,

$$F\left(x, u; \nabla_x \psi(u) + \nabla_{xx} \psi(u) q\right) \geq 0 \Longrightarrow \psi(x) \geq \psi(u) - \frac{1}{2} q^T \nabla_{xx} \psi(u) q. \tag{2.6}$$

ψ is second-order F-concave/pseudoconcave at $u \in X$ with respect to $q \in R^n$ if $-\psi$ is second-order F-convex/pseudoconvex at $u \in X$ with respect to $q \in R^n$.

3. Second-Order Mixed Nondifferentiable Symmetric Dual Programs

For $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, let $J_1 \subseteq N$, $K_1 \subseteq M$, $J_2 = N \setminus J_1$, and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in J_1. The other symbols $|J_2|$, $|K_1|$ and $|K_2|$ are defined similarly. Let $x^1 \in R^{|J_1|}$, $x^2 \in R^{|J_2|}$. Then, any $x \in R^n$ can be written as (x^1, x^2). Similarly, for $y^1 \in R^{|K_1|}$, $y^2 \in R^{|K_2|}$, $y \in R^m$ can be written as (y^1, y^2). It may be noted here that if $J_1 = \emptyset$, then $|J_1| = 0$, $J_2 = N$, and therefore $|J_2| = n$. In this case, $R^{|J_1|}$, $R^{|J_2|}$ and $R^{|J_1|} \times R^{|K_1|}$ will be zero-dimensional, n-dimensional and $|K_1|$-dimensional Euclidean spaces, respectively. The other situations are $J_2 = \emptyset$, $K_1 = \emptyset$ or $K_2 = \emptyset$.

Now we formulate the following pair of mixed nondifferentiable second-order symmetric dual programs and discuss their duality results.

Primal problem (SMNP)

minimize

$$\begin{aligned} &G\left(x^1, y^1, x^2, y^2, z^2, p, r\right) \\ &\quad = f\left(x^1, y^1\right) + S\left(x^1 \mid C_1\right) + g\left(x^2, y^2\right) + S\left(x^2 \mid C_2\right) - \left(y^2\right)^T z^2 \\ &\quad - \left(y^1\right)^T \left[\nabla_{y^1} f\left(x^1, y^1\right) + \nabla_{y^1 y^1} f\left(x^1, y^1\right) p\right] - \frac{1}{2} p^T \nabla_{y^1 y^1} f\left(x^1, y^1\right) p \\ &\quad - \frac{1}{2} r^T \nabla_{y^2 y^2} g\left(x^2, y^2\right) r, \end{aligned} \tag{3.1}$$

subject to

$$\nabla_{y^1} f\left(x^1, y^1\right) - z^1 + \nabla_{y^1 y^1} f\left(x^1, y^1\right) p \leq 0, \tag{3.2}$$

$$\nabla_{y^2} g\left(x^2, y^2\right) - z^2 + \nabla_{y^2 y^2} g\left(x^2, y^2\right) r \leq 0, \tag{3.3}$$

$$\left(y^2\right)^T \left[\nabla_{y^2} g\left(x^2, y^2\right) - z^2 + \nabla_{y^2 y^2} g\left(x^2, y^2\right) r\right] \geq 0, \tag{3.4}$$

$$z^1 \in D_1, \quad z^2 \in D_2. \tag{3.5}$$

Dual problem (SMND)

maximize

$$\begin{aligned} &H\left(u^1, v^1, u^2, v^2, w^2, q, s\right) \\ &\quad = f\left(u^1, v^1\right) - S\left(v^1 \mid D_1\right) + g\left(u^2, v^2\right) - S\left(v^2 \mid D_2\right) + \left(u^2\right)^T w^2 \\ &\quad - \left(u^1\right)^T \left[\nabla_{x^1} f\left(u^1, v^1\right) + \nabla_{x^1 x^1} f\left(u^1, v^1\right) q\right] - \frac{1}{2} q^T \nabla_{x^1 x^1} f\left(u^1, v^1\right) q \\ &\quad - \frac{1}{2} s^T \nabla_{x^2 x^2} g\left(u^2, v^2\right) s, \end{aligned} \tag{3.6}$$

subject to

$$\nabla_{x^1} f\left(u^1, v^1\right) + w^1 + \nabla_{x^1 x^1} f\left(u^1, v^1\right) q \geq 0, \tag{3.7}$$

$$\nabla_{x^2} g\left(u^2, v^2\right) + w^2 + \nabla_{x^2 x^2} g\left(u^2, v^2\right) s \geq 0, \tag{3.8}$$

$$\left(u^2\right)^T \left[\nabla_{x^2} g\left(u^2, v^2\right) + w^2 + \nabla_{x^2 x^2} g\left(u^2, v^2\right) s\right] \leq 0, \tag{3.9}$$

$$w^1 \in C_1, \quad w^2 \in C_2, \tag{3.10}$$

where

(i) $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ are differentiable functions,

(ii) C_1, C_2, D_1 and D_2 are compact convex sets in $R^{|J_1|}$, $R^{|J_2|}$, $R^{|K_1|}$ and $R^{|K_2|}$, respectively,

(iii) $p \in R^{|K_1|}$, $r \in R^{|K_2|}$, $q \in R^{|J_1|}$ and $s \in R^{|J_2|}$.

Theorem 3.1 (Weak duality). *Let* $(x^1, y^1, x^2, y^2, z^1, z^2, p, r)$ *be feasible for (SMNP) and* $(u^1, v^1, u^2, v^2, w^1, w^2, q, s)$ *be feasible for (SMND). Let the sublinear functionals* $F_1 : R^{|J_1|} \times R^{|J_1|} \times R^{|J_1|} \mapsto R$, $F_2 : R^{|K_1|} \times R^{|K_1|} \times R^{|K_1|} \mapsto R$, $G_1 : R^{|J_2|} \times R^{|J_2|} \times R^{|J_2|} \mapsto R$ *and* $G_2 : R^{|K_2|} \times R^{|K_2|} \times R^{|K_2|} \mapsto R$ *satisfy the following conditions:*

$$F_1\left(x^1, u^1; a^1\right) + \left(a^1\right)^T u^1 \geq 0, \quad \forall a^1 \in R_+^{|J_1|}, \tag{A}$$

$$F_2\left(v^1, y^1; a^2\right) + \left(a^2\right)^T y^1 \geq 0, \quad \forall a^2 \in R_+^{|K_1|}, \tag{B}$$

$$G_1\left(x^2, u^2; b^1\right) + \left(b^1\right)^T u^2 \geq 0, \quad \forall b^1 \in R_+^{|J_2|}, \tag{C}$$

$$G_2\left(v^2, y^2; b^2\right) + \left(b^2\right)^T y^2 \geq 0, \quad \forall b^2 \in R_+^{|K_2|}. \tag{D}$$

Suppose that

(i) $f(\cdot, v^1) + (\cdot)^T w^1$ *is second-order* F_1*-convex at* u^1*, and* $f(x^1, \cdot) - (\cdot)^T z^1$ *is second-order* F_2*-concave at* y^1*,*

(ii) $g(\cdot, v^2) + (\cdot)^T w^2$ *is second-order* G_1*-pseudoconvex at* u^2*, and* $g(x^2, \cdot) - (\cdot)^T z^2$ *is second-order* G_2*-pseudoconcave at* y^2*.*

Then,

$$G\left(x^1, y^1, x^2, y^2, z^2, p, r\right) \geq H\left(u^1, v^1, u^2, v^2, w^2, q, s\right). \tag{3.11}$$

Proof. By the second-order F_1-convexity of $f(\cdot, v^1) + (\cdot)^T w^1$ at u^1 and the second-order F_2-concavity of $f(x^1, \cdot) - (\cdot)^T z^1$ at y^1, we have

$$\begin{aligned} &f\left(x^1, v^1\right) + \left(x^1\right)^T w^1 - f\left(u^1, v^1\right) - \left(u^1\right)^T w^1 + \frac{1}{2} q^T \nabla_{x^1 x^1} f\left(u^1, v^1\right) q \\ &\quad \geq F_1\left(x^1, u^1; \nabla_{x^1} f\left(u^1, v^1\right) + w^1 + \nabla_{x^1 x^1} f\left(u^1, v^1\right) q\right), \end{aligned} \tag{3.12}$$

$$\begin{aligned} &f\left(x^1, y^1\right) - \left(y^1\right)^T z_1 - f\left(x^1, v^1\right) + \left(v^1\right)^T z_1 - \frac{1}{2} p^T \nabla_{y^1 y^1} f\left(x^1, y^1\right) p \\ &\quad \geq F_2\left(v^1, y^1; -\left(\nabla_{y^1} f\left(x^1, y^1\right) - z^1 + \nabla_{y^1 y^1} f\left(x^1, y^1\right) p\right)\right). \end{aligned} \tag{3.13}$$

Since $(x^1, y^1, x^2, y^2, z^1, z^2, p, r)$ is feasible for primal problem (SMNP) and $(u^1, v^1, u^2, v^2, w^1, w^2, q, s)$ is feasible for dual problem (SMND), by the dual constraint (3.7), the vector $a^1 = \nabla_{x^1} f(u^1, v^1) + w^1 + \nabla_{x^1x^1} f(u^1, v^1)q \in R_+^{|J_1|}$, and so from the hypothesis (A), we obtain

$$F_1\left(x^1, u^1; a^1\right) + \left(a^1\right)^T u^1 \geq 0. \tag{3.14}$$

Similarly,

$$F_2\left(v^1, y^1; a^2\right) + \left(a^2\right)^T y^1 \geq 0, \tag{3.15}$$

for the vector $a^2 = -[\nabla_{y^1} f(x^1, y^1) - z^1 + \nabla_{y^1y^1} f(x^1, y^1)p] \in R_+^{|K_1|}$.

Using (3.14) in (3.12) and (3.15) in (3.13), we have

$$\begin{aligned} &f\left(x^1, v^1\right) + \left(x^1\right)^T w^1 - f\left(u^1, v^1\right) - \left(u^1\right)^T w^1 + \frac{1}{2} q^T \nabla_{x^1x^1} f\left(u^1, v^1\right) q \geq -\left(u^1\right)^T a^1, \\ &f\left(x^1, y^1\right) - \left(y^1\right)^T z_1 - f\left(x^1, v^1\right) + \left(v^1\right)^T z_1 - \frac{1}{2} p^T \nabla_{y^1y^1} f\left(x^1, y^1\right) p \geq -\left(y^1\right)^T a^2. \end{aligned} \tag{3.16}$$

Adding the above two inequalities, we obtain

$$\begin{aligned} &f\left(x^1, y^1\right) + \left(x^1\right)^T w^1 - \left(y^1\right)^T z_1 + \left(y^1\right)^T a^2 - \frac{1}{2} p^T \nabla_{y^1y^1} f\left(x^1, y^1\right) p \\ &\quad \geq f\left(u^1, v^1\right) + \left(u^1\right)^T w^1 - \left(v^1\right)^T z_1 - \left(u^1\right)^T a^1 - \frac{1}{2} q^T \nabla_{x^1x^1} f\left(u^1, v^1\right) q. \end{aligned} \tag{3.17}$$

Substituting the values of a^1 and a^2 in (3.17), we get

$$\begin{aligned} &f\left(x^1, y^1\right) + \left(x^1\right)^T w^1 - \left(y^1\right)^T \left[\nabla_{y^1} f\left(x^1, y^1\right) + \nabla_{y^1y^1} f\left(x^1, y^1\right) p\right] - \frac{1}{2} p^T \nabla_{y^1y^1} f\left(x^1, y^1\right) p \\ &\quad \geq f\left(u^1, v^1\right) - \left(v^1\right)^T z_1 - \left(u^1\right)^T \left[\nabla_{x^1} f\left(u^1, v^1\right) + \nabla_{x^1x^1} f\left(u^1, v^1\right) q\right] \\ &\quad - \frac{1}{2} q^T \nabla_{x^1x^1} f\left(u^1, v^1\right) q. \end{aligned} \tag{3.18}$$

Using $(x^1)^T w^1 \leq S(x^1 \mid C_1)$ and $(v^1)^T z^1 \leq S(v^1 \mid D_1)$, we have

$$\begin{aligned} &f\left(x^1, y^1\right) + S\left(x^1 \mid C_1\right) - \left(y^1\right)^T \left[\nabla_{y^1} f\left(x^1, y^1\right) + \nabla_{y^1y^1} f\left(x^1, y^1\right) p\right] - \frac{1}{2} p^T \nabla_{y^1y^1} f\left(x^1, y^1\right) p \\ &\quad \geq f\left(u^1, v^1\right) - S\left(v^1 \mid D_1\right) - \left(u^1\right)^T \left[\nabla_{x^1} f\left(u^1, v^1\right) + \nabla_{x^1x^1} f\left(u^1, v^1\right) q\right] \\ &\quad - \frac{1}{2} q^T \nabla_{x^1x^1} f\left(u^1, v^1\right) q. \end{aligned} \tag{3.19}$$

By hypothesis (C) and the dual constraint (3.8), we obtain

$$\begin{aligned} &G_1\left(x^2, u^2; \nabla_{x^2} g\left(u^2, v^2\right) + w^2 + \nabla_{x^2x^2} g\left(u^2, v^2\right) s\right) \\ &\quad \geq -\left(u^2\right)^T \left[\nabla_{x^2} g\left(u^2, v^2\right) + w^2 + \nabla_{x^2x^2} g\left(u^2, v^2\right) s\right], \end{aligned} \tag{3.20}$$

which on using the dual constraint (3.9) yields

$$G_1\left(x^2, u^2; \nabla_{x^2} g\left(u^2, v^2\right) + w^2 + \nabla_{x^2x^2} g\left(u^2, v^2\right) s\right) \geq 0. \tag{3.21}$$

Since $g(\cdot, v^2) + (\cdot)^T w^2$ is second-order G_1-pseudoconvex at u^2, we have

$$g\left(x^2, v^2\right) + \left(x^2\right)^T w^2 \geq g\left(u^2, v^2\right) + \left(u^2\right)^T w^2 - \frac{1}{2} s^T \nabla_{x^2x^2} g\left(u^2, v^2\right) s. \tag{3.22}$$

Similarly, from (3.3) and (3.4) and hypothesis (D) along with second-order G_2-pseudoconcavity of $g(x^2, \cdot) - (\cdot)^T z^2$ at y^2, we get

$$g\left(x^2, y^2\right) - \left(y^2\right)^T z^2 \geq g\left(x^2, v^2\right) - \left(v^2\right)^T z^2 + \frac{1}{2} r^T \nabla_{y^2y^2} g\left(x^2, y^2\right) r. \tag{3.23}$$

Adding (3.22) and (3.23), we obtain

$$\begin{aligned} &g\left(x^2, y^2\right) + \left(x^2\right)^T w^2 - \left(y^2\right)^T z^2 - \frac{1}{2} r^T \nabla_{y^2y^2} g\left(x^2, y^2\right) r \\ &\quad \geq g\left(u^2, v^2\right) + \left(u^2\right)^T w^2 - \left(v^2\right)^T z^2 - \frac{1}{2} s^T \nabla_{x^2x^2} g\left(u^2, v^2\right) s. \end{aligned} \tag{3.24}$$

Using $(x^2)^T w^2 \leq S(x^2|C_2)$ and $(v^2)^T z^2 \leq S(v^2|D_2)$, we have

$$\begin{aligned} &g\left(x^2, y^2\right) + S\left(x^2 \mid C_2\right) - \left(y^2\right)^T z^2 - \frac{1}{2} r^T \nabla_{y^2y^2} g\left(x^2, y^2\right) r \\ &\quad \geq g\left(u^2, v^2\right) + \left(u^2\right)^T w^2 - S\left(v^2 \mid D_2\right) - \frac{1}{2} s^T \nabla_{x^2x^2} g\left(u^2, v^2\right) s. \end{aligned} \tag{3.25}$$

Inequalities (3.19) and (3.25) together yield

$$
\begin{aligned}
& f\left(x^1,y^1\right)+S\left(x^1\mid C_1\right)+g\left(x^2,y^2\right)+S\left(x^2\mid C_2\right)-\left(y^2\right)^T z^2 \\
&\quad -\left(y^1\right)^T\left[\nabla_{y^1} f\left(x^1,y^1\right)+\nabla_{y^1y^1} f\left(x^1,y^1\right)p\right]-\frac{1}{2}p^T\nabla_{y^1y^1} f\left(x^1,y^1\right)p-\frac{1}{2}r^T\nabla_{y^2y^2} g\left(x^2,y^2\right)r \\
&\geq f\left(u^1,v^1\right)-S\left(v^1\mid D_1\right)+g\left(u^2,v^2\right)-S\left(v^2\mid D_2\right)+\left(u^2\right)^T w^2 \\
&\quad -\left(u^1\right)^T\left[\nabla_{x^1} f\left(u^1,v^1\right)+\nabla_{x^1x^1} f\left(u^1,v^1\right)q\right]-\frac{1}{2}q^T\nabla_{x^1x^1} f\left(u^1,v^1\right)q \\
&\quad -\frac{1}{2}s^T\nabla_{x^2x^2} g\left(u^2,v^2\right)s,
\end{aligned}
\tag{3.26}
$$

that is,$G(x^1,y^1,x^2,y^2,z^2,p,r)\geq H(u^1,v^1,u^2,v^2,w^2,q,s)$. □

Theorem 3.2 (Strong duality). *Let $f: R^{|J_1|}\times R^{|K_1|}\to R$ and $g: R^{|J_2|}\times R^{|K_2|}\to R$ be differentiable functions, and let $(\overline{x}^1,\overline{y}^1,\overline{x}^2,\overline{y}^2,\overline{z}^1,\overline{z}^2,\overline{p},\overline{r})$ be a local optimal solution of (SMNP). Suppose that*

(i) *the matrix $\nabla_{y^1y^1} f(\overline{x}^1,\overline{y}^1)$ is non singular,*

(ii) $\nabla_{y^2y^2} g(\overline{x}^2,\overline{y}^2)$ *is positive definite,* $\overline{r}^T(\nabla_{y^2} g(\overline{x}^2,\overline{y}^2)-\overline{z}^2)\geq 0$ *or* $\nabla_{y^2y^2} g(\overline{x}^2,\overline{y}^2)$ *is negative definite, and* $\overline{r}^T(\nabla_{y^2} g(\overline{x}^2,\overline{y}^2)-\overline{z}^2)\leq 0$,

(iii) $\nabla_{y^2} g(\overline{x}^2,\overline{y}^2)-\overline{z}^2+\nabla_{y^2y^2} g(\overline{x}^2,\overline{y}^2)\overline{r}\neq 0$,

(iv) *one of the matrices $(\partial/\partial y_i^1)(\nabla_{y^1y^1} f(\overline{x}^1,\overline{y}^1))$, $i=1,2,\ldots,|K_1|$, is positive or negative definite.*

Then, there exist $\overline{w}^1\in C_1$ and $\overline{w}^2\in C_2$ such that $(\overline{x}^1,\overline{y}^1,\overline{x}^2,\overline{y}^2,\overline{w}^1,\overline{w}^2,\overline{q}=0,\overline{s}=0)$ is feasible for (SMND), and the objective function values of (SMNP) and (SMND) are equal. Furthermore, if the assumptions of weak duality (Theorem 3.1) are satisfied for all feasible solutions of (SMNP) and (SMND), then $(\overline{x}^1,\overline{y}^1,\overline{x}^2,\overline{y}^2,\overline{z}^1,\overline{z}^2,\overline{p},\overline{r})$ and $(\overline{x}^1,\overline{y}^1,\overline{x}^2,\overline{y}^2,\overline{w}^1,\overline{w}^2,\overline{q},\overline{s})$ are global optimal solutions for (SMNP) and (SMND), respectively.

Proof. Since $(\overline{x}^1,\overline{y}^1,\overline{x}^2,\overline{y}^2,\overline{z}^1,\overline{z}^2,\overline{p},\overline{r})$ is a local optimal solution of (SMNP), there exist $\alpha\in R$, $\beta\in R^{|K_1|}$, $\gamma\in R^{|K_2|}$, $\delta\in R$, $\eta_1\in R^{|J_1|}$, and $\eta_2\in R^{|J_2|}$ such that the following by Fritz John optimality conditions [18] are satisfied at $(\overline{x}^1,\overline{y}^1,\overline{x}^2,\overline{y}^2,\overline{z}^1,\overline{z}^2,\overline{p},\overline{r})$

$$\alpha^T\left(\nabla_{x^1} f\left(\overline{x}^1,\overline{y}^1\right)+\eta_1\right)+\nabla_{y^1x^1} f\left(\overline{x}^1,\overline{y}^1\right)\left[\beta-\alpha\overline{y}^1\right] + \nabla_{x^1}\left(\nabla_{y^1y^1} f\left(\overline{x}^1,\overline{y}^1\right)\overline{p}\right)\left[\beta-\alpha\left(\overline{y}^1+\frac{1}{2}\overline{p}\right)\right]=0, \tag{3.27}$$

$$\alpha^T\left(\nabla_{x^2} g\left(\overline{x}^2,\overline{y}^2\right)+\eta_2\right)+\nabla_{y^2x^2} g\left(\overline{x}^2,\overline{y}^2\right)\left[\gamma-\delta\overline{y}^2\right] + \nabla_{x^2}\left(\nabla_{y^2y^2} g\left(\overline{x}^2,\overline{y}^2\right)\overline{r}\right)\left[\gamma-\delta\overline{y}^2-\frac{1}{2}\alpha\overline{r}\right]=0, \tag{3.28}$$

$$\nabla_{y^1y^1} f\left(\overline{x}^1,\overline{y}^1\right)\left[\beta-\alpha\left(\overline{y}^1+\overline{p}\right)\right] + \nabla_{y^1}\left(\nabla_{y^1y^1} f\left(\overline{x}^1,\overline{y}^1\right)\overline{p}\right)\left[\beta-\alpha\left(\overline{y}^1+\frac{1}{2}\overline{p}\right)\right]=0, \tag{3.29}$$

$$\left[\nabla_{y^2} g\left(\overline{x}^2,\overline{y}^2\right)-\overline{z}^2\right][\alpha-\delta]+\nabla_{y^2y^2} g\left(\overline{x}^2,\overline{y}^2\right)\left[\gamma-\delta\left(\overline{y}^2+\overline{r}\right)\right] + \nabla_{y^2}\left(\nabla_{y^2y^2} g\left(\overline{x}^2,\overline{y}^2\right)\overline{r}\right)\left[\gamma-\delta\overline{y}^2-\frac{1}{2}\alpha\overline{r}\right]=0, \tag{3.30}$$

$$\nabla_{y^1y^1} f\left(\overline{x}^1,\overline{y}^1\right)\left[\beta-\alpha\left(\overline{y}^1+\overline{p}\right)\right]=0, \tag{3.31}$$

$$\nabla_{y^2y^2} g\left(\overline{x}^2,\overline{y}^2\right)\left[\gamma-\delta\overline{y}^2-\alpha\overline{r}\right]=0, \tag{3.32}$$

$$\beta^T\left[\nabla_{y^1} f\left(\overline{x}^1,\overline{y}^1\right)-\overline{z}^1+\nabla_{y^1y^1} f\left(\overline{x}^1,\overline{y}^1\right)\overline{p}\right]=0, \tag{3.33}$$

$$\gamma^T\left[\nabla_{y^2} g\left(\overline{x}^2,\overline{y}^2\right)-\overline{z}^2+\nabla_{y^2y^2} g\left(\overline{x}^2,\overline{y}^2\right)\overline{r}\right]=0, \tag{3.34}$$

$$\delta\left(\overline{y}^2\right)^T\left[\nabla_{y^2} g\left(\overline{x}^2,\overline{y}^2\right)-\overline{z}^2+\nabla_{y^2y^2} g\left(\overline{x}^2,\overline{y}^2\right)\overline{r}\right]=0, \tag{3.35}$$

$$\beta\in N_{D_1}\left(\overline{z}^1\right), \tag{3.36}$$

$$(\alpha-\delta)\overline{y}^2+\gamma\in N_{D_2}\left(\overline{z}^2\right), \tag{3.37}$$

$$\eta_1\in C_1,\quad \eta_1^T\overline{x}^1=S\left(\overline{x}^1\mid C_1\right), \tag{3.38}$$

$$\eta_2\in C_2,\quad \eta_2^T\overline{x}^2=S\left(\overline{x}^2\mid C_2\right), \tag{3.39}$$

$$(\alpha,\beta,\gamma,\delta)\geq 0,\qquad (\alpha,\beta,\gamma,\delta)\neq 0. \tag{3.40}$$

By hypothesis (i), (3.31) gives

$$\beta=\alpha\left(\overline{y}^1+\overline{p}\right). \tag{3.41}$$

Since $\nabla_{y^2y^2} g(\overline{x}^2,\overline{y}^2)$ is positive or negative definite, (3.32) yields

$$\gamma=\delta\overline{y}^2+\alpha\overline{r}. \tag{3.42}$$

Suppose that $\alpha = 0$, then (3.42) implies

$$\gamma = \delta \overline{y}^2. \tag{3.43}$$

Using (3.42) in (3.30), we get

$$(\alpha - \delta)\left[\nabla_{y^2} g\left(\overline{x}^2, \overline{y}^2\right) - \overline{z}^2 + \nabla_{y^2 y^2} g\left(\overline{x}^2, \overline{y}^2\right)\overline{r}\right] + \frac{1}{2}\nabla_{y^2}\left(\nabla_{y^2 y^2} g\left(\overline{x}^2, \overline{y}^2\right)\overline{r}\right)\left[\gamma - \delta \overline{y}^2\right] = 0, \tag{3.44}$$

which on using hypothesis (iii) and $\gamma = \delta \overline{y}^2$ yields

$$\alpha = \delta. \tag{3.45}$$

As $\alpha = 0$, therefore the equations $\alpha = \delta$ and $\gamma = \delta \overline{y}^2$ give $\delta = 0$ and $\gamma = 0$, respectively. Further, (3.41) implies $\beta = 0$. Consequently, $(\alpha, \beta, \gamma, \delta) = 0$, contradicting (3.40). Hence, we have

$$\alpha > 0. \tag{3.46}$$

Subtracting (3.35) from (3.34) yields

$$\left[\gamma - \delta\left(\overline{y}^2\right)\right]^T \left[\nabla_{y^2} g\left(\overline{x}^2, \overline{y}^2\right) - \overline{z}^2 + \nabla_{y^2 y^2} g\left(\overline{x}^2, \overline{y}^2\right)\overline{r}\right] = 0. \tag{3.47}$$

Using (3.42) and (3.46) in the above equation, we get

$$\overline{r}^T \left(\nabla_{y^2} g\left(\overline{x}^2, \overline{y}^2\right) - \overline{z}^2\right) + \overline{r}^T \nabla_{y^2 y^2} g\left(\overline{x}^2, \overline{y}^2\right)\overline{r} = 0, \tag{3.48}$$

which contradicts hypothesis (ii) unless

$$\overline{r} = 0. \tag{3.49}$$

Equation (3.42) yields

$$\gamma = \delta \overline{y}^2. \tag{3.50}$$

Using (3.49) and (3.50) in (3.30), we obtain

$$(\alpha - \delta)\left(\nabla_{y^2} g\left(\overline{x}^2, \overline{y}^2\right) - \overline{z}^2\right) = 0, \tag{3.51}$$

which on using hypothesis (iii) and (3.49) gives

$$\alpha = \delta. \tag{3.52}$$

Since $\alpha > 0$, therefore

$$\delta > 0. \tag{3.53}$$

Now, using (3.41) and (3.46) in (3.29), we get

$$\left(\nabla_{y^1}\left(\nabla_{y^1y^1} f\left(\overline{x}^1,\overline{y}^1\right)\overline{p}\right)\right)\overline{p} = 0, \tag{3.54}$$

which by hypothesis (iv) implies

$$\overline{p} = 0. \tag{3.55}$$

By (3.41) and (3.55), we have

$$\beta = \alpha\overline{y}^1. \tag{3.56}$$

Using (3.46), (3.55), and (3.56) in (3.27), we get

$$\nabla_{x^1} f\left(\overline{x}^1,\overline{y}^1\right) + \eta_1 = 0. \tag{3.57}$$

Equations (3.28), (3.46), (3.49), and (3.50) give

$$\nabla_{x^2} g\left(\overline{x}^2,\overline{y}^2\right) + \eta_2 = 0, \tag{3.58}$$

and hence, we also have

$$\left(\overline{x}^2\right)^T\left(\nabla_{x^2} g\left(\overline{x}^2,\overline{y}^2\right) + \eta_2\right) = 0. \tag{3.59}$$

Thus, $(\overline{x}^1,\overline{y}^1,\overline{x}^2,\overline{y}^2,\overline{w}^1 = \eta_1, \overline{w}^2 = \eta_2, \overline{q} = 0, \overline{s} = 0)$ satisfies the dual constraints from (3.7) to (3.10), and so it is a feasible solution for the dual problem (SMND).

Further, using (3.46), (3.55), and (3.56) in (3.33), we obtain

$$\left(\overline{y}^1\right)^T \nabla_{y^1} f\left(\overline{x}^1,\overline{y}^1\right) = \left(\overline{y}^1\right)^T \overline{z}^1. \tag{3.60}$$

Moreover, since $\beta = \alpha\overline{y}^1$ and $\alpha > 0$, (3.36) implies $\overline{y}^1 \in N_{D_1}(\overline{z}^1)$ so that

$$\left(\overline{y}^1\right)^T \overline{z}^1 = S\left(\overline{y}^1 \mid D_1\right). \tag{3.61}$$

From (3.37), (3.50), (3.52), and (3.53), we get

$$\overline{y}^2 \in N_{D_2}\left(\overline{z}^2\right). \tag{3.62}$$

Since D_2 is a compact convex set in $R^{|K_2|}$,

$$\left(\overline{y}^2\right)^T \overline{z}^2 = S\left(\overline{y}^2 \mid D_2\right). \tag{3.63}$$

Therefore, using (3.38), (3.39), (3.49), (3.55), (3.57), and (3.60)–(3.63), we obtain

$$\begin{aligned}
&f\left(\overline{x}^1,\overline{y}^1\right) + S\left(\overline{x}^1 \mid C_1\right) + g\left(\overline{x}^2,\overline{y}^2\right) + S\left(\overline{x}^2 \mid C_2\right) - \left(\overline{y}^2\right)^T \overline{z}^2 \\
&\quad - \left(\overline{y}^1\right)^T \left[\nabla_{y^1} f\left(\overline{x}^1,\overline{y}^1\right) + \nabla_{y^1y^1} f\left(\overline{x}^1,\overline{y}^1\right)\overline{p}\right] - \frac{1}{2}\overline{p}^T \nabla_{y^1y^1} f\left(\overline{x}^1,\overline{y}^1\right)\overline{p} \\
&\quad - \frac{1}{2}\overline{r}^T \nabla_{y^2y^2} g\left(\overline{x}^2,\overline{y}^2\right)\overline{r} \\
&\quad\quad = f\left(\overline{x}^1,\overline{y}^1\right) - S\left(\overline{y}^1 \mid D_1\right) + g\left(\overline{x}^2,\overline{y}^2\right) - S\left(\overline{y}^2 \mid D_2\right) + \left(\overline{x}^2\right)^T \overline{w}^2 \\
&\quad\quad\quad - \left(\overline{x}^1\right)^T \left[\nabla_{x^1} f\left(\overline{x}^1,\overline{y}^1\right) + \nabla_{x^1x^1} f\left(\overline{x}^1,\overline{y}^1\right)\overline{q}\right] - \frac{1}{2}\overline{q}^T \nabla_{x^1x^1} f\left(\overline{x}^1,\overline{y}^1\right)\overline{q} \\
&\quad\quad\quad - \frac{1}{2}\overline{s}^T \nabla_{x^2x^2} g\left(\overline{x}^2,\overline{y}^2\right)\overline{s},
\end{aligned} \tag{3.64}$$

that is, the two objective function values are equal.

Finally, from Theorem 3.1, we get that $(\overline{x}^1,\overline{y}^1,\overline{x}^2,\overline{y}^2,\overline{z}^1,\overline{z}^2,\overline{p},\overline{r})$ and $(\overline{x}^1,\overline{y}^1,\overline{x}^2,\overline{y}^2,\overline{w}^1,\overline{w}^2,\overline{q},\overline{s})$ are global optimal solutions for (SMNP) and (SMND), respectively. □

Theorem 3.3 (Converse duality). *Let $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ be differentiable functions, and let $(\overline{u}^1,\overline{v}^1,\overline{u}^2,\overline{v}^2,\overline{w}^1,\overline{w}^2,\overline{q},\overline{s})$ be a local optimal solution of (SMND). Suppose that*

(i) *the matrix $\nabla_{x^1x^1} f(\overline{u}^1,\overline{v}^1)$ is non singular,*

(ii) $\nabla_{x^2x^2} g(\overline{u}^2,\overline{v}^2)$ *is positive definite and* $\overline{s}^T(\nabla_{x^2} g(\overline{u}^2,\overline{v}^2) + \overline{w}^2) \geq 0$ *or* $\nabla_{x^2x^2} g(\overline{u}^2,\overline{v}^2)$ *is negative definite and* $\overline{s}^T(\nabla_{x^2} g(\overline{u}^2,\overline{v}^2) + \overline{w}^2) \leq 0$,

(iii) $\nabla_{x^2} g(\overline{u}^2,\overline{v}^2) + \overline{w}^2 + \nabla_{x^2x^2} g(\overline{u}^2,\overline{v}^2)\overline{s} \neq 0$,

(iv) *one of the matrices* $(\partial/\partial x_i^1)(\nabla_{x^1x^1} f(\overline{u}^1,\overline{v}^1))$, $i = 1,2,\ldots,|J_1|$, *is positive or negative definite.*

Then, there exist $\overline{z}^1 \in D_1$ and $\overline{z}^2 \in D_2$ such that $(\overline{u}^1,\overline{v}^1,\overline{u}^2,\overline{v}^2,\overline{z}^1,\overline{z}^2,\overline{p} = 0,\overline{r} = 0)$ is feasible for (SMNP) and the objective function values of (SMNP), and (SMND) are equal. Furthermore, if the assumptions of weak duality (Theorem 3.1) are satisfied for all feasible solutions of (SMNP) and (SMND), then $(\overline{u}^1,\overline{v}^1,\overline{u}^2,\overline{v}^2,\overline{w}^1,\overline{w}^2,\overline{q},\overline{s})$ and $(\overline{u}^1,\overline{v}^1,\overline{u}^2,\overline{v}^2,\overline{z}^1,\overline{z}^2,\overline{p},\overline{r})$ are global optimal solutions for (SMND) and (SMNP), respectively.

Proof. It follows on the lines of Theorem 3.2. □

4. Special Cases

In this section, we consider some of the special cases of the problems studied in Section 3.

(i) If $J_2 = \emptyset$ and $K_2 = \emptyset$, then our problems (SMNP) and (SMND) reduce to the programs (PP) and (DP) studied in Gulati and Gupta [17].

(ii) If $J_2 = \emptyset$, $K_2 = \emptyset$, $C_1 = \{0\}$ and $D_1 = \{0\}$, then (SMNP) and (SMND) are reduced to the programs (SP) and (SD) studied in Gulati et al. [16] with the omission of nonnegativity constraints from (SP) and (SD).

(iii) If $J_2 = \emptyset$, $K_2 = \emptyset$, $p = 0$, and $q = 0$, then (SMNP) and (SMND) become a pair of symmetric nondifferentiable dual programs considered in Mond and Schechter [6] with the omission of nonnegativity constraints from the programs (P) and (D) studied in Mond and Schechter.

(iv) If $J_2 = \emptyset$, $K_2 = \emptyset$, $p = 0$, $q = 0$, $C_1 = \{0\}$, and $D_1 = \{0\}$, then the programs (WP) and (WD) of [15] are obtained with the omission of nonnegativity constraints from (WP) and (WD).

(v) If $J_1 = \emptyset$ and $K_1 = \emptyset$ in (SMNP) and (SMND), then the programs studied in [10] are obtained.

(vi) If $J_1 = \emptyset$, $K_1 = \emptyset$, $C_2 = \{0\}$, and $D_2 = \{0\}$ in (SMNP) and (SMND), then the programs (SP1) and (SD1) of [16] are obtained with the omission of nonnegativity constraints from (SP1) and (SD1).

(vii) If $J_1 = \emptyset$, $K_1 = \emptyset$, $r = 0$, and $s = 0$, then (SMNP) and (SMND) become a pair of symmetric nondifferentiable dual programs considered in [6] with the omission of nonnegativity constraints from the programs (P1) and (D1) studied in Mond and Schechter.

(viii) If $J_1 = \emptyset$, $K_1 = \emptyset$, $r = 0$, $s = 0$, $C_2 = \{0\}$, and $D_2 = \{0\}$, then (SMNP) and (SMND) become a pair of single objective symmetric differentiable dual programs considered in [15] with the omission of nonnegativity constraints from (MP) and (MD).

(ix) By eliminating the second-order and nondifferentiable terms, our problems (SMNP) and (SMND) reduce to the mixed symmetric dual programs studied by Chandra et al. [11] with the omission of $x^1 > 0$, $x^2 > 0$, $v^1 > 0$, and $v^2 > 0$ from the programs studied in Chandra et al. [11].

(x) By eliminating the second-order terms, our problems are reduced to the programs (MP) and (MD) studied in [12] with the omission of nonnegativity constraints from (MP) and (MD).

5. Concluding Remarks

It is to be noted that previously known results [6, 10–12, 15–17] are special cases of our study. It is not clear whether the second-order mixed symmetric duality in mathematical programming can be further extended to higher-order multiobjective symmetric dual programs formulated in [19].

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