Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2010, Article ID 509286, 9 pages
doi:10.1155/2010/509286

Research Article

The Periodic Solutions of the Compound Singular Fractional Differential System with Delay

XuTing Wei and XuanZhu Lu

School of Mathematics and Computer Science, Fuzhou Universiy, Fuzhou 350108, China

Correspondence should be addressed to XuTing Wei, wxt1985wan@163.com

Received 31 July 2009; Revised 16 November 2009; Accepted 1 December 2009

Academic Editor: Fawang Liu

The paper gives sufficient conditions on the existence of periodic solution for a class of compound singular fractional differential systems with delay, involving Nishimoto fractional derivative. Furthermore, for the particular functions, the necessary conditions on the existence of periodic solution are also derived. Especially, for two-dimensional compound singular fractional differential equation with delay, the criteria of existence of periodic solution are obtained. Finally, two examples are presented to verify the validity of criteria.

1. Introduction

In real life, there are many phenomena with time delay. The mathematical model derived from engineering, physics, mechanics, control theory, chemical reactions, biology, and medicine was made with a significant amount of delay, such as the limited signal transmission speed human reaction time to the outside world. Therefore, the delay is widespread in nature and society, in the introduction of time-delay differential equations there can be a more accurate description and explanation of various phenomena and processes.

Fractional calculus is the promotion of classical calculus. The study found that fractional calculus was very suitable to describe long memory and hereditary properties of various materials and processes [1, 2]. In the recent years, fractional calculus becomes a research hotspot, its field of concern has become wide, such as the numerical method of the equation in [3], the existence and uniqueness of equations in [4], fractional Brownian motion, fractional reaction-diffusion equation and random walk [5, 6], fractional wavelet transform [7], and fractional control [8].

Most of the above-mentioned studies, utilize the Riemann-liouville fractional derivative definition, which due to its nature of its definition is simple and relatively good. But Nishimoto definition of fractional calculus [9, 10], has not received a lot of attention, this may be part of the naturalization due to the complexity of its definition, but compared to Riemann-liouville fractional calculus, it has a better nature, relevant results more concise useful.

The existence of periodic solutions of differential equations is one of the important research directions of biomathematics [11–15], which has a wide range of applications, such as the existence of periodic orbits of celestial movement and its stability.

In [12], the author discussed the following system:

$$E\dot{x}(t) = Ax(t) + Bx(t - \tau_1) + Cx(t - \tau_2), \tag{1.1}$$

and obtained sufficient and necessary conditions for the existence of periodic solutions for the system. Taking into account the periodic solutions of the fractional time-delay system will be a very important practical significance; we are tried to generalize the corresponding results to the case of fractional order.

For the above reasons we consider the following compound singular fractional differential system with delay:

$$HD^{\alpha}x(t) = Ax(t) + Bx(t - \tau_1) + Cx(t + \tau_2), \tag{1.2}$$

where D^{α} denotes Nishimoto fractional derivative of order α, $\alpha > 0$. H, A, B, and C are constant system matrices of appropriate dimensions, and τ_1 and τ_2 are constants with $\tau_1 > 0$, $\tau_2 > 0$, $|H| = 0$.

2. Definitions and Notations

In this section we introduce the definitions of fractional derivative/integral and related basic properties used in the paper; more information can be obtained from [9, 10].

Definition 2.1 (see [9]). If the function $f(z)$ is analytic (regular) inside and on C, here $C := \{C^-, C^+\}$, C^- is a contour along the cut joining the points z and $-\infty + i\Im(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, C^+ is a contour along the cut joining the points z and $+\infty + i\Im(z)$, which starts from the point at $+\infty$, encircles the point z once counter-clockwise, and returns to the point at $+\infty$,

$$\begin{aligned} f_\upsilon(z) &= \left(f(z)\right)_\upsilon := \frac{\Gamma(\upsilon+1)}{2\pi i}\int_c \frac{f(\varsigma)}{(\varsigma - z)^{\upsilon+1}}d\varsigma \quad \left(\upsilon \in \frac{R}{Z^-}; Z^- := \{-1,-2,-3,\dots\}\right), \\ f_{-n}(z) &:= \lim_{\upsilon\to -n}\left\{f_\upsilon(z)\right\} \quad (n \in N := \{1,2,3,\dots\}), \end{aligned} \tag{2.1}$$

where ς, $-\pi \le \arg(\varsigma - z) \le \pi$, for C^-, and $0 \le \arg(\varsigma - z) \le 2\pi$, for C^+.

Then $f_\upsilon(z)$ $(\upsilon > 0)$ is said to be the fractional derivative of $f(z)$ of order υ and $f_\upsilon(z)$ $(\upsilon < 0)$ is said to be the fractional integral of $f(z)$ of order $-\upsilon$, provided that $|f_\upsilon(z)| < \infty$ $(\upsilon \in R)$.

Let us recall the following useful properties associated with the definition introduced above [9].

Property 1. For a constant λ,

$$\left(e^{\lambda z}\right)_{\upsilon} = \lambda^{\upsilon} e^{\lambda z} \quad (\lambda \neq 0; \upsilon \in R; z \in C). \tag{2.2}$$

Property 2. For a constant λ,

$$\left(e^{-\lambda z}\right)_{\upsilon} = e^{-i\pi\upsilon} \lambda^{\upsilon} e^{-\lambda z} \quad (\lambda \neq 0; \upsilon \in R; z \in C). \tag{2.3}$$

Property 3. If the function $f(z)$ is singlevalued and analytic in some domain $\Omega \subseteq C$, then

$$\left(f_{\mu}(z)\right)_{\upsilon} = f_{\mu+\upsilon}(z) = \left(f_{\upsilon}(z)\right)_{\mu}. \tag{2.4}$$

Property 4. For a constant λ,

$$\left(z^{\lambda}\right)_{\upsilon} = e^{-i\pi\upsilon} \frac{\Gamma(\upsilon-\lambda)}{\Gamma(-\lambda)} z^{\lambda-\upsilon} \quad \left(\upsilon \in R;\ z \in C;\ \left|\frac{\Gamma(\upsilon-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right). \tag{2.5}$$

In the following section of this paper, we let D^{α} denote the α order Nishimoto derivative.

3. The Main Results

In this section, we discuss some problems to the system of the system (1.2).

Theorem 3.1. *The sufficient condition for the existence of the nonconstant periodic solutions of system* (1.2) *is that the following equation exists pure imaginary roots*

$$\det\left(\lambda^{\alpha} H - A - Be^{-\tau_1 \lambda} - Ce^{\tau_2 \lambda}\right) = 0. \tag{3.1}$$

Proof. Assume that ηi is pure imaginary root of (3.1), let $x(t) = Ke^{\eta i t} (K \in R^n)$, substituting $x(t)$ in (1.2), then

$$\left((\eta i)^{\alpha} H - A - Be^{-\tau_1(\eta i)} - Ce^{\tau_2(\eta i)}\right) K = 0. \tag{3.2}$$

As ηi is pure imaginary roots of (3.1), note that

$$\det\left((\eta i)^{\alpha} H - A - Be^{-\tau_1(\eta i)} - Ce^{\tau_2(\eta i)}\right) = 0. \tag{3.3}$$

So (3.1) exists nonzero solution K, then, $x(t) = Ke^{\eta i t}$ is the nonconstant periodic solution of (1.2).

If the system have nonconstant periodic solutions, then we may wonder whether the solution satisfy (3.1), in fact, as you will see it holds when the function satisfy some conditions. We know that if the function $f(t)$ is a continuous smooth periodic function with period 2l, then it can be expressed as its fourier series form

$$f(t)=\sum_{k=-\infty}^{+\infty} C_k e^{ik\pi t/l}, \tag{3.4}$$

where $C_k=(1/2l)\int_{-l}^{l} f(\xi)e^{-ik\pi t/l}d\xi$.

And as we also know that its fourier series expansion has the following properties:

$$f'(t)=\sum_{k=-\infty}^{+\infty} C_k \frac{ik\pi}{l} e^{ik\pi t/l}, \tag{3.5}$$

we can even get the following relation if $f(t)$ satisfy some more strictly condition:

$$f^{(k)}(t)=\sum_{k=-\infty}^{+\infty} C_k \left(\frac{ik\pi}{l}\right)^k e^{ik\pi t/l}. \tag{3.6}$$

To obtain the similar property of our fractional derivative, what conditions the function should satisfy? We give the following function space. □

Definition 3.2. If the periodic function $f(t)$ is continuous and smooth on R, its α $(\alpha>1)$ order Nishimoto derivative exists, then we let $\Omega(t)$ denote the corresponding function space whose elements have the following property:

$$D^\alpha f(t)=D^\alpha\left(\sum_{k=-\infty}^{+\infty} C_k e^{ik\pi t/l}\right)=\sum_{k=-\infty}^{+\infty} C_k D^\alpha\left(e^{ik\pi t/l}\right)=\sum_{k=-\infty}^{+\infty} C_k\left(\frac{ik\pi t}{l}\right)^\alpha e^{ik\pi t/l}, \tag{3.7}$$

it is easy to know from the definition that $e^{i\lambda t}\in\Omega(t)$ $(\lambda\in R)$, and so $\Omega(t)$ is nonempty.

Theorem 3.3. *If $x(t)$ is the non-constant periodic solution of* (1.2)*, and further $x(t)\in\Omega(t)$, one can obtain the necessity of Theorem 3.1.*

Proof. Suppose the period of $x(t)$ is 2l, $x(t)$ is continuous and differentiable because of $\alpha>1$, then we can denote it in the form of its fourier series:

$$x(t)=\sum_{k=-\infty}^{+\infty} C_k e^{ik\pi t/l}. \tag{3.8}$$

Since $x(t)\in\Omega(t)$, we have

$$D^\alpha x(t)=\sum_{k=-\infty}^{+\infty} C_k\left(\frac{ik\pi t}{l}\right)^\alpha e^{ik\pi t/l}, \tag{3.9}$$

where $C_k=(1/2l)\int_{-l}^{l} x(\xi)e^{-ik\pi t/l}d\xi$.

We put (3.8) and (3.9) into (1.2), and obtain

$$\sum_{k=-\infty}^{\infty}\left[\left(\frac{ik\pi}{l}\right)^{\alpha}H - A - Be^{-ik\pi\tau_1/l} - Ce^{ik\pi\tau_2/l}\right]C_k e^{ik\pi t/l} = 0, \tag{3.10}$$

then multiply $(1/2l)e^{-im\pi t/l}$ $(m = 0, \pm 1, \pm 2, \ldots)$ on both sides of (3.10) and integrate it from $-l$ to l, hence

$$\sum_{k=-\infty}^{\infty}\left[\left(\frac{ik\pi}{l}\right)^{\alpha}H - A - Be^{-ik\pi\tau_1/l} - Ce^{ik\pi\tau_2/l}\right]C_k \frac{1}{2l}\int_{-l}^{l} e^{i(k-m)\pi t/l}dt = 0. \tag{3.11}$$

It is easy to deduce that

$$\frac{1}{2l}\int_{-l}^{l} e^{i(k-m)\pi t/l}dt = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases} \tag{3.12}$$

recalling (3.10), it reduces to

$$\left[\left(\frac{ik\pi}{l}\right)^{\alpha}H - A - Be^{-ik\pi\tau_1/l} - Ce^{ik\pi\tau_2/l}\right]C_m = 0 \quad (m = 0, \pm 1, \pm 2, \ldots). \tag{3.13}$$

Thus, if there are no pure imaginary roots in (3.1), then for every k we have $C_k = 0$, according to (3.8), we conclude that $x(t)\ = \text{cons tan } t$ vector which conflicts the suppose that $x(t)$ is the non-constant periodic solution of (1.2). □

4. Two-Dimensional Case

For the case of the two-dimensional compound singular fractional differential system with delay, there is

$$HD^{\alpha}x(t) = Ax(t) + Bx(t - \tau_1) + Cx(t + \tau_2), \tag{4.1}$$

where $\alpha > 0$, $H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$, $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, and $x_1(t)$, $x_2(t)$ is scalar function.

Using Theorem 3.1, we obtained the following theorem.

Theorem 4.1. *If one of the following equations exists non-zero real root, then system*(4.1) *exists non-constant periodic solution, further more, if $x(t) \in \Omega(t)$, then the conclusion is sufficient and necessary*

$$\begin{aligned}
&|A| + |B|\cos(2\tau_1 y) + |C|\cos(2\tau_2 y) + E\cos(\tau_1 y) + F\cos(\tau_2 y) + G\cos((\tau_2 - \tau_1)y) \\
&\quad -b_{22}|y|^{\alpha}\cos\left(-\tau_1 y + \frac{\pi}{2}\alpha\right) - c_{22}|y|^{\alpha}\cos\left(\tau_2 y + \frac{\pi}{2}\alpha\right) - a_{22}|y|^{\alpha}\cos\left(\frac{\pi}{2}\alpha\right) = 0 \\
&\quad -|B|\sin(2\tau_1 y) + |C|\sin(2\tau_2 y) - E\sin(\tau_1 y) + F\sin(\tau_2 y) + G\sin((\tau_2 - \tau_1)y) \\
&\quad -b_{22}|y|^{\alpha}\sin\left(-\tau_1 y + \frac{\pi}{2}\alpha\right) - c_{22}|y|^{\alpha}\sin\left(\tau_2 y + \frac{\pi}{2}\alpha\right) - a_{22}|y|^{\alpha}\sin\left(\frac{\pi}{2}\alpha\right) = 0,
\end{aligned} \tag{4.2}$$

or

$$\begin{aligned}
&|A| + |B|\cos(2\tau_1 y) + |C|\cos(2\tau_2 y) + E\cos(\tau_1 y) + F\cos(\tau_2 y) + G\cos((\tau_2 - \tau_1)y) \\
&\quad -b_{22}|y|^{\alpha}\cos\left(-\tau_1 y - \frac{\pi}{2}\alpha\right) - c_{22}|y|^{\alpha}\cos\left(\tau_2 y - \frac{\pi}{2}\alpha\right) - a_{22}|y|^{\alpha}\cos\left(-\frac{\pi}{2}\alpha\right) = 0 \\
&\quad -|B|\sin(2\tau_1 y) + |C|\sin(2\tau_2 y) - E\sin(\tau_1 y) + F\sin(\tau_2 y) + G\sin((\tau_2 - \tau_1)y) \\
&\quad -b_{22}|y|^{\alpha}\sin\left(-\tau_1 y - \frac{\pi}{2}\alpha\right) - c_{22}|y|^{\alpha}\sin\left(\tau_2 y - \frac{\pi}{2}\alpha\right) - a_{22}|y|^{\alpha}\sin\left(-\frac{\pi}{2}\alpha\right) = 0,
\end{aligned} \tag{4.3}$$

where $E = \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $F = \begin{vmatrix} a_{11} & a_{12} \\ c_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $G = \begin{vmatrix} b_{11} & b_{12} \\ c_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} \\ b_{21} & b_{22} \end{vmatrix}$.

Proof. First of all we know that

$$(yi)^{\alpha} = \begin{cases} y^{\alpha} e^{\pi\alpha i/2}, & y \geq 0, \\ |y|^{\alpha} e^{-\pi\alpha i/2}, & y < 0, \end{cases} \tag{4.4}$$

according to Theorem 3.1, we have

$$\begin{aligned}
h(yi) &= \left| yiI - A - Be^{-\tau_1 yi} - Ce^{\tau_2 yi} \right| \\
&= \begin{vmatrix} (yi)^{\alpha} - a_{11} - b_{11}e^{-\tau_1 yi} - c_{11}e^{\tau_2 yi} & -a_{12} - b_{12}e^{-\tau_1 yi} - c_{12}e^{\tau_2 yi} \\ -a_{21} - b_{21}e^{-\tau_1 yi} - c_{21}e^{\tau_2 yi} & -a_{22} - b_{22}e^{-\tau_1 yi} - c_{22}e^{\tau_2 yi} \end{vmatrix} \\
&= |A| + |B|e^{-\tau_1 yi} + |C|e^{2\tau_2 yi} + |E|e^{-\tau_1 yi} + Fe^{\tau_2 yi} + Ge^{(\tau_2 - \tau_1)yi} \\
&\quad - b_{22}(yi)^{\alpha}e^{-\tau_1 yi} - c_{22}(yi)^{\alpha}e^{\tau_2 yi} - a_{22}(yi)^{\alpha}.
\end{aligned} \tag{4.5}$$

then

$$\begin{aligned}
\Re[h(yi)] &= |A| + |B|\cos(2\tau_1 y) + |C|\cos(2\tau_2 y) + E\cos(\tau_1 y) + F\cos(\tau_2 y) \\
&\quad + G\cos((\tau_2 - \tau_1)y) - b_{22}|y|^{\alpha}\cos\left(-\tau_1 y \pm \frac{\pi}{2}\alpha\right) \\
&\quad - c_{22}|y|^{\alpha}\cos\left(\tau_2 y \pm \frac{\pi}{2}\alpha\right) - a_{22}|y|^{\alpha}\cos\left(\pm\frac{\pi}{2}\alpha\right) = 0, \\
\Im[h(yi)] &= -|B|\sin(2\tau_1 y) + |C|\sin(2\tau_2 y) - E\sin(\tau_1 y) + F\sin(\tau_2 y) \\
&\quad + G\sin((\tau_2 - \tau_1)y) - b_{22}|y|^{\alpha}\sin\left(-\tau_1 y \pm \frac{\pi}{2}\alpha\right) \\
&\quad - c_{22}|y|^{\alpha}\sin\left(\tau_2 y \pm \frac{\pi}{2}\alpha\right) - a_{22}|y|^{\alpha}\sin\left(\pm\frac{\pi}{2}\alpha\right) = 0,
\end{aligned} \tag{4.6}$$

where $\Re\ [z]$ and $\Im\ [z]$ denote the real and imaginary parts of z, respectively. Then using Theorem 3.1, if there exists $y \in R,\ y \neq 0,\ h(yi) = 0$, we obtain that

$$\begin{aligned}
\Re[h(yi)] &= 0, \\
\Im[h(yi)] &= 0,
\end{aligned} \tag{4.7}$$

hence system (4.1) exists non-constant periodic solution. This proved the theorem. □

5. Examples

In this section we give some concrete examples to illustrate our conclusions.

Example 5.1. We consider the following two-dimensional compound singular fractional differential system with delay:

$$\begin{aligned}
D^{\alpha}x_1(t) &= -x_2(t) + x_1(t-1) + x_2(t-1) + x_2(t+1), \\
0 &= x_1(t) + x_1(t-1) + x_1(t+1),
\end{aligned} \tag{5.1}$$

where $H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tau_1 = 1,\ \tau_2 = 1,\ \alpha > 1$. so we have $|A| = 1,\ |B| = -1,\ |C| = -1,\ E = 0,\ F = 0,\ G = -2$.

Using the discriminant of Theorem 4.1, we have

$$\begin{aligned}
2\cos(2y) &= -1, \\
0 &= 0,
\end{aligned} \tag{5.2}$$

the solution is $y = k\pi + \pi/3\ (k = 0, \pm 1, \pm 2, \dots)$.

According to Theorem 4.1 , system (5.1) has non-constant periodic solution. We suppose that $y = \pi/3$, as

$$\left((yi)^{\alpha} I - A - Be^{-yi} - Ce^{yi}\right) K = 0 \tag{5.3}$$

exists non-zero solution, it means that

$$\begin{pmatrix} \left[\left(\frac{\pi}{3}\right)^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right) - \frac{\sqrt{3}}{2}\right] + \left[\left(\frac{\pi}{3}\right)^{\alpha} \sin\left(\frac{\pi\alpha}{2}\right) + \frac{\sqrt{3}}{2}\right] i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.4}$$

So we have $k_1 = 0$, k_2 for any real number. Supposed that $k_2 = 1$, then we obtained a non-constant periodic solution of system (5.1):

$$x(t) = e^{(\pi/3)it} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.5}$$

We can verify that $x(t)$ is a non-constant periodic solution of system (5.1).

Example 5.2. Consider the following two-dimension compounded with singular fractional differential equation delay system:

$$\begin{gathered} D^{1/2} x_1(t) = -x_2(t) + x_1(t-1) + x_2(t+1), \\ 0 = x_1(t) + x_2(t) + x_1(t-1) + x_2(t-1) + x_1(t+1) + x_2(t+1). \end{gathered} \tag{5.6}$$

where $H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\tau_1 = 1$, $\tau_2 = 1$, $\alpha = 1/2$.

So we have $|A| = 1$, $|B| = -1$, $|C| = -1$, $E = 0$, $F = 0$, $G = -2$.

Using the discriminant of Theorem 4.1, we obtained

$$\begin{gathered} (2\cos y + 1)|y|^{1/2} \cos\frac{\pi}{4} + 2\cos 2y + 1 = 0, \\ (2\cos y + 1)|y|^{1/2} \sin\frac{\pi}{4} = 2\sin 2y. \end{gathered} \tag{5.7}$$

Through the simplication of this equation, we have

$$\sqrt{2}\sin(2y + (\pi/4)) = -1/2, \tag{5.8}$$

and the solution is $y = k\pi + (1/2)\text{arc}\ \sin(-1/2\sqrt{2}) - \pi/8$, $k = (0, \pm1, \pm2, \ldots)$.

According to Theorem 4.1, there exists non-constant periodic solution in the system.

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