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Research Article

A Numerical Method for a Singularly Perturbed Three-Point Boundary Value Problem

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The purpose of this paper is to present a uniform finite difference method for numerical solution of nonlinear singularly perturbed convection-diffusion problem with nonlocal and third type boundary conditions. The numerical method is constructed on piecewise uniform Shishkin type mesh. The method is shown to be convergent, uniformly in the diffusion parameter ε, of first order in the discrete maximum norm. Some numerical experiments illustrate in practice the result of convergence proved theoretically.

1. Introduction

This paper is concerned with ε-uniform numerical method for the singularly perturbed semilinear boundary-value problem (BVP):

$$Lu := \varepsilon^2 u'' + \varepsilon a(x) u' - f(x,u) = 0, \quad 0 < x < \ell, \tag{1.1}$$

$$L_0 u := -\varepsilon u'(0) + \psi(u(0)) = 0, \tag{1.2}$$

$$u(\ell) - \varphi(u(\ell_1)) = 0, \quad 0 < \ell_1 < \ell, \tag{1.3}$$

where ε is a small positive parameter, the functions $a(x) \geq 0$, $f(x,u)$, and $\psi(u)$, $\varphi(u)$ are sufficiently smooth on $[0,\ell]$, $[0,\ell] \times \mathbb{R}$, and $\mathbb{R}$, respectively, and furthermore

$$\begin{gathered} 0 < \beta \leq \frac{\partial f}{\partial u} \leq \beta^* < \infty, \\ \frac{d\psi}{du} \geq \delta > 0, \qquad \left|\frac{d\varphi}{du}\right| \leq \kappa < 1. \end{gathered} \tag{1.4}$$

The solution u generally has boundary layers near $x = 0$ and $x = \ell$.

Singularly perturbed differential equations are characterized by the presence of a small parameter ε multiplying the highest-order derivatives. Such problems arise in many areas of applied mathematics. Among these are the Navier-Stokes equations of fluid flow at high Reynolds number, mathematical models of liquid crystal materials and chemical reactions, control theory, reaction-diffusion processes, quantum mechanics, and electrical networks. The solutions of singularly perturbed differential equations typically have steep gradients, in thin regions of the domain, whose magnitude depends inversely on some positive power of ε. Such regions are called either interior or boundary layers, depending on whether their location is in the interior or at the boundary of the domain. An overview of some existence and uniqueness results and applications of singularly perturbed equations can be found in [1–6].

It is known that these problems depend on a small positive parameter ε in such away that the solution exhibits a multiscale character; that is, there are thin transition layers where the solution varies rapidly, while away from layers it behaves regularly and varies slowly. The treatment of singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions. Therefore it is important to develop suitable numerical methods for solving these problems, whose accuracy does not depend on the value of parameter ε, that is, methods that are convergent ε-uniformly. These include fitted finite-difference methods, finite element methods using special elements such as exponential elements, and methods which use a priori refined or special piecewise uniform grids which condense in the boundary layers in a special manner. One of the simplest ways to derive parameter-uniform methods consists of using a class of special piecewise uniform meshes, such as Shishkin meshes (see [3, 6–15] for the motivation for this type of mesh), which are constructed a priori and depend on the parameter ε, the problem data, and the number of corresponding mesh points. The various approaches to numerical solution of differential equations with stepwise continuous solutions can be found in [2, 3, 6].

There is also an increasing interest in the application of Shishkin meshes to singularly perturbed convection-diffusion problems (see [16, 17] and references cited therein). However, much of the Shishkin mesh literature is concerned with linear or quasilinear singularly perturbed two-point problems with first-order reduced equations. In [18] has been obtained a result ε-uniform for the two-point boundary value problem of (1.1), by using a fitted operator method on uniform meshes.

In the present paper, we analyse a fitted finite-difference scheme on a Shishkin type mesh for the numerical solution of the semilinear nonlocal boundary value problem (1.1)–(1.3). The origin of the fitted finite difference method can be traced to [19]; for subsequent work on fitted operator method and its applications, see [2, 3]. Nonlocal boundary value problems have also been studied extensively in the literature. For a discussion of existence and uniqueness results and for applications of nonlocal problems see [20–26] and the references cited therein. Some approaches to approximating this type of problem have also been considered [20, 21, 26–28]. However, the algorithms developed in the papers cited above are mainly concerned with regular cases (i.e., when boundary layers are absent). In [27] has been studied the fitted difference schemes on an equidistant mesh for the numerical solution of the linear three-point reaction-diffusion problem.

The numerical method presented here comprises a fitted difference scheme on a piecewise uniform mesh. We have derived this approach on the basis of the method of integral identities using interpolating quadrature rules with the weight and remainder terms in integral form. This results in a local truncation error containing only first-order derivatives of exact solution and hence facilitates examination of the convergence. A summary of paper is as follows. Section 2 contains results concerning the exact solutions of problem (1.1)–(1.3). In Section 3, we describe the finite-difference discretization and construct a piecewise uniform mesh, which is fitted to the boundary layers. In Section 4, we present the error analysis for the approximate solution. Uniform convergence is proved in the discrete maximum norm. In the following section numerical results are presented, which are in agreement with the theoretical results. The approach to the construction of the discrete problem and the error analysis for the approximate solution are similar to those in [18, 27–29].

2. Continuous Solution

In this section, we give uniform bounds for the solution of the BVP (1.1)–(1.3), which will be used to analyze properties of appropriate difference problem.

Lemma 2.1. *Let* $a, f \in C^1[0,\ell]$. *Then the solution* $u(x)$ *of problem* (1.1)–(1.3) *satisfies the inequalities*

$$\|u\|_\infty \le C_0, \tag{2.1}$$

where

$$\begin{gathered} C_0 = (1-k)^{-1}\left\{|B| + k\left(\delta^{-1}|A| + \beta^{-1}\|F\|_\infty\right)\right\} \\ F(x) = f(x,0), \qquad A = -\varphi(0), \qquad B = \varphi(0), \\ \|u\|_\infty = \max_{[0,\ell]}|u(x)|, \end{gathered} \tag{2.2}$$

$$\left|u'(x)\right| \le C\left\{1 + \frac{1}{\varepsilon}\left(e^{-\mu_1 x/\varepsilon} + e^{-\mu_2(\ell-x)/\varepsilon}\right)\right\}, \quad 0 \le x \le \ell, \tag{2.3}$$

with

$$\begin{aligned} \mu_1 &= \frac{1}{2}\left(\sqrt{a^2(0)+4\beta} + a(0)\right), \\ \mu_2 &= \frac{1}{2}\left(\sqrt{a^2(\ell)+4\beta} - a(\ell)\right), \end{aligned} \tag{2.4}$$

providing that $(\partial f/\partial x)(x,u)$ *is bounded for* $x \in [0,\ell]$ *and* $|u| \le C_0$.

Proof. We rewrite the problem (1.1)–(1.3) as

$$Lu \equiv \varepsilon^2 u''^{(x)} + \varepsilon a(x)u' - b(x)u(x) = F(x), \quad 0 < x < \ell, \tag{2.5}$$

$$L_0 u \equiv -\varepsilon u'^{(0)} + \rho u(0) = A, \tag{2.6}$$

$$u(\ell) - \gamma u(\ell_1) = B, \tag{2.7}$$

where

$$b(x) = \frac{\partial f}{\partial u}(x, \xi u), \quad 0 < \xi < 1,$$
$$\rho = \frac{d\psi}{du}(\eta_1 u(0)), \quad 0 < \eta_1 < 1, \tag{2.8}$$
$$\gamma = \frac{d\varphi}{du}(\eta_2 u(\ell_1)), \quad 0 < \eta_2 < 1.$$

Here we use the Maximum Principle: *let* L *and* L_0 *be the differential operators in* (2.5)-(2.6) *and* $v \in C^2[0, \ell]$. *If* $L_0 v \geq 0$, $v(\ell) \geq 0$, *and* $Lv \leq 0$ *for all, then* $v(x) \geq 0$ *for all* $x < 0 < \ell$. Then, from (2.5)-(2.6) we have the following inequality:

$$|u(x)| \leq \delta^{-1}|A| + |u(\ell)| + \beta^{-1}\|F\|_\infty, \quad x \in [0, \ell]. \tag{2.9}$$

Next, from boundary condition (2.7), we get

$$|u(\ell)| \leq |B| + k|u(\ell_1)|. \tag{2.10}$$

By setting the value $x = \ell_1$ in the inequality (2.9), we obtain

$$|u(\ell_1)| \leq \delta^{-1}|A| + |u(\ell)| + \beta^{-1}\|F\|_\infty. \tag{2.11}$$

From (2.10) and (2.11), then we have

$$|u(\ell)| \leq (1-k)^{-1}\left\{|B| + k\left(\delta^{-1}|A| + \beta^{-1}\|F\|_\infty\right)\right\}, \tag{2.12}$$

which along with (2.9) leads to (2.1).

After establishing (2.1) the further part of the proof is almost identical to that of [28]. □

3. Discretization and Mesh

Let ω_N be any nonuniform mesh on $[0, \ell]$:

$$\omega_N = \{0 < x_1 < \cdots < x_{N-1} < \ell\}, \tag{3.1}$$

and $\overline{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = \ell\}$. For each $i \geq 1$ we set the stepsize $h_i = x_i - x_{i-1}$.

Before describing our numerical method, we introduce some notation for the mesh functions. For any mesh function $g(x)$ defined on $\overline{\omega}_N$ we use

$$
\begin{gathered}
g_i = g(x_i), \qquad g_{\bar{x},i} = \frac{(g_i - g_{i-1})}{h_i}, \qquad g_{x,i} = \frac{(g_{i+1} - g_i)}{h_{i+1}}, \\
g_{\overset{0}{x},i} = \frac{(g_{\bar{x},i} + g_{x,i})}{2}, \qquad g_{\hat{x},i} = \frac{(g_{i+1} - g_i)}{\hbar_i}, \qquad \hbar_i = \frac{(h_i + h_{i+1})}{2}, \\
g_{\bar{x}\hat{x},i} = \frac{(g_{x,i} - g_{\bar{x},i})}{\hbar}, \\
\|w\|_\infty \equiv \|w\|_{\infty,\overline{\omega}_N} := \max_{0 \le i \le N} |w_i|.
\end{gathered}
\tag{3.2}
$$

The difference scheme we will construct follows from the identity

$$
\chi_i^{-1} \hbar_i^{-1} \int_0^\ell L u \varphi_i(x) dx = 0, \quad i = 1, 2, \ldots, N-1 \tag{3.3}
$$

with the basis functions $\{\varphi_i(x)\}_{i=1}^{N-1}$ having the form

$$
\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x), & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x), & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases} \tag{3.4}
$$

where $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$, respectively, are the solutions of the following problems:

$$
\begin{gathered}
\varepsilon \varphi'' - a_i \varphi' = 0, \quad x_{i-1} < x < x_i, \\
\varphi(x_{i-1}) = 0, \qquad \varphi(x_i) = 1, \\
\varepsilon \varphi'' - a_i \varphi' = 0, \quad x_i < x < x_{i+1}, \\
\varphi(x_i) = 1, \qquad \varphi(x_{i+1}) = 0.
\end{gathered}
\tag{3.5}
$$

The functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ can be explicitly expressed as follows:

$$
\begin{gathered}
\varphi_i^{(1)}(x) = \frac{e^{a_i(x - x_{i-1})/\varepsilon} - 1}{e^{a_i h_i/\varepsilon} - 1}, \quad \varphi_i^{(2)}(x) = \frac{1 - e^{-a_i(x_{i+1} - x)/\varepsilon}}{1 - e^{-a_i h_{i+1}/\varepsilon}}, \quad \text{for } a_i \neq 0, \\
\varphi_i^{(1)}(x) = \frac{x - x_{i-1}}{h_i}, \quad \varphi_i^{(2)}(x) = \frac{x_{i+1} - x}{h_{i+1}}, \quad \text{for } a_i = 0.
\end{gathered}
\tag{3.6}
$$

The coefficient χ_i in (3.3) is given by

$$\chi_i = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)dx = \begin{cases} \hbar_i^{-1}\left(\dfrac{h_i}{1-e^{a_ih_i/\varepsilon}} + \dfrac{h_{i+1}}{1-e^{-a_ih_{i+1}/\varepsilon}}\right), & a_i \neq 0, \\ 1, & a_i = 0. \end{cases} \tag{3.7}$$

Rearranging (3.3) gives

$$-\varepsilon^2\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x)u'(x)dx + \varepsilon a_i\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)u'(x)dx - f(x_i,u_i) + R_i = 0, \quad i = \overline{1, N-1} \tag{3.8}$$

with

$$\begin{gathered} R_i = \varepsilon\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [a(x)-a(x_i)]u'(x)\varphi_i(x)dx - \hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\,\varphi_i(x) \\ \times \int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx}f(\xi,u(\xi))K_{0,i}^*(x,\xi)d\xi, \\ K_{0,i}^*(x,\xi) = T_0(x-\xi) - T_0(x_i-\xi), \quad 1 \le i \le N-1, \\ T_0(\lambda) = \begin{cases} 1, & \lambda \ge 0, \\ 0, & \lambda < 0. \end{cases} \end{gathered} \tag{3.9}$$

As consistent with [26, 27], we obtain the precise relation:

$$\begin{aligned} &-\varepsilon^2\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x)u'(x)dx + \varepsilon a_i\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)u'(x)dx \\ &\quad = \varepsilon^2\left\{\chi_i^{-1}\left(1+0.5\varepsilon^{-1}\hbar_ia_i\left(\chi_{2,i}-\chi_{1,i}\right)\right)\right\}u_{\bar{x}\hat{x},i} + \varepsilon a_iu_{\overset{0}{x},i}, \end{aligned} \tag{3.10}$$

where

$$\begin{gathered} \chi_{1,i} = \hbar_i^{-1}\int_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x)dx = \begin{cases} \hbar_i^{-1}\left(\dfrac{\varepsilon}{a_i} + \dfrac{h_i}{1-e^{a_ih_i/\varepsilon}}\right), & a_i \neq 0, \\ \dfrac{h_i\hbar^{-1}}{2}, & a_i = 0, \end{cases} \\ \chi_{2,i} = \hbar_i^{-1}\int_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x)dx = \begin{cases} \hbar_i^{-1}\left(\dfrac{h_{i+1}}{1-e^{-a_ih_{i+1}/\varepsilon}} - \dfrac{\varepsilon}{a_i}\right), & a_i \neq 0, \\ \dfrac{h_{i+1}\hbar^{-1}}{2}, & a_i = 0. \end{cases} \end{gathered} \tag{3.11}$$

It then follows from (3.8) that

$$\ell u_i + R_i \equiv \varepsilon^2\theta_i u_{\bar{x}\hat{x},i} + \varepsilon a_i u_{\overset{0}{x},i} - f(x_i, u_i) + R_i = 0, \quad i = \overline{1, N-1}, \tag{3.12}$$

where

$$\theta_i = \chi_i^{-1}\left[1 + 0.5\varepsilon^{-1} a_i \hbar_i (\chi_{2,i} - \chi_{1,i})\right] \tag{3.13}$$

and after a simple calculation

$$\theta_i = \begin{cases} \dfrac{a_i\hbar_i}{2\varepsilon}\left(\dfrac{h_{i+1}(e^{a_i h_i/\varepsilon} - 1) + h_i(1 - e^{-a_i h_{i+1}/\varepsilon})}{h_{i+1}(e^{a_i h_i/\varepsilon} - 1) - h_i(1 - e^{-a_i h_{i+1}/\varepsilon})}\right), & a_i \neq 0, \\ 1, & a_i = 0. \end{cases} \tag{3.14}$$

To define an approximation for the boundary condition (1.2), we proceed our discretization process by

$$\int_0^{x_1} Lu\varphi_0(x)dx = 0, \tag{3.15}$$

with

$$\varphi_0(x) = \begin{cases} \dfrac{1 - e^{-a_0(x_1 - x)/\varepsilon}}{1 - e^{-a_0 h_1/\varepsilon}}, & x_0 < x < x_1,\ a_0 \neq 0, \\ \dfrac{x_1 - x}{h_1}, & x_0 < x < x_1,\ a_0 = 0, \\ 0, & x \notin (x_0, x_1). \end{cases} \tag{3.16}$$

Here the function $\varphi_0(x)$ is the solution of the following problem:

$$\begin{gathered} \varepsilon\varphi_0'' - a_0\varphi_0' = 0, \quad x_0 < x < x_1, \\ \varphi_0(x_0) = 1, \qquad \varphi_0(x_1) = 0. \end{gathered} \tag{3.17}$$

In the analogous way, as in construction of (3.12), we obtain

$$-\varepsilon\theta_0^{(0)} u_{x,0} + \psi(u_0) + \theta_0^{(1)} f(x_0, u_0) - r^{(0)} = 0, \tag{3.18}$$

where

$$\theta_0^{(0)} = \begin{cases} \dfrac{a_0\rho}{1-e^{-a_0\rho}}, & a_0 \neq 0, \\ 1, & a_0 = 0, \end{cases} \qquad \rho = \frac{h_1}{\varepsilon}, \tag{3.19}$$

$$\theta_0^{(1)} = \begin{cases} \dfrac{\rho}{1-e^{-a_0\rho}} - a_0^{-1}, & a_0 \neq 0, \\ \dfrac{h_1}{2\varepsilon}, & a_0 = 0, \end{cases} \tag{3.20}$$

$$r^{(0)} = \int_{x_0}^{x_1} [a(x) - a_0] u'(x) \varphi_0(x) dx - \varepsilon^{-1} \int_{x_0}^{x_1} dx\, \varphi_0(x) \int_{x_0}^{x_1} \frac{d}{d\xi} f(\xi, u(\xi)) T_0(x-\xi) d\xi, \quad \xi \in (x_0, x_1). \tag{3.21}$$

Now, it remains to define an approximation for the second boundary condition (1.3). Let x_{N_0} be the mesh point nearest to ℓ_1. Then

$$u_N - \varphi(u_{N_0}) + r^{(1)} = 0, \tag{3.22}$$

where

$$r^{(1)} = (u(\ell_1) - u(x_{N_0}))\varphi'^{(\xi)}, \tag{3.23}$$

being ξ-intermediate point between $u(x_{N_0})$ and $u(\ell_1)$.

Based on (3.12), (3.18), and (3.22), we propose the following difference scheme for approximating (1.1) and (1.3):

$$\varepsilon^2 \theta y_{\bar{x}\hat{x}} + \varepsilon a y_{\overset{0}{x}} - f(x, y) = 0, \quad x \in \omega_N, \tag{3.24}$$

$$-\varepsilon \theta_0^{(0)} y_{x,0} + \psi(y_0) + \theta_0^{(1)} f(x_0, y_0) = 0, \tag{3.25}$$

$$y(\ell) - \varphi(y_{N_0}) = 0, \tag{3.26}$$

where θ, $\theta_0^{(0)}$, and $\theta_0^{(1)}$ are defined by (3.14), (3.19), and (3.20), respectively.

The difference scheme (3.24)–(3.26) in order to be ε-uniform convergent, we will use the Shishkin mesh. For a divisible by 4 positive integer N, we divide each of the intervals $[0, \sigma_1]$ and $[\ell - \sigma_2, \ell]$ into $N/4$ equidistant subintervals and also $[\sigma_1, \ell - \sigma_2]$ into $N/2$ equidistant subintervals, where the transition points σ_1 and σ_2, which separate the fine and coarse portions of the mesh, are obtained by taking

$$\sigma_1 = \min\left\{\frac{\ell}{4}, \mu_1^{-1}\varepsilon \ln N\right\}, \qquad \sigma_2 = \min\left\{\frac{\ell}{4}, \mu_2^{-1}\varepsilon \ln N\right\}, \tag{3.27}$$

where μ_1 and μ_2 are given in Lemma 2.1. In practice one usually has $\sigma_i \ll \ell$ $(i = 1,2)$; so the mesh is fine on $[0, \sigma_1]$, $[\ell - \sigma_2, \ell]$, and coarse on $[\sigma_1, \ell - \sigma_2]$. Hence, if we denote by $h^{(1)}$, $h^{(2)}$, and $h^{(3)}$ the step-size in $[0, \sigma_1]$, $[\sigma_1, \ell - \sigma_2]$, and $[\ell - \sigma_2, \ell]$, respectively, we have

$$\begin{gathered} h^{(1)} = \frac{4\sigma_1}{N}, \qquad h^{(2)} = \frac{2(\ell - \sigma_2 - \sigma_1)}{N}, \qquad h^{(3)} = \frac{4\sigma_2}{N}, \\ \frac{1}{2}\left(h^{(1)} + h^{(3)}\right) = \frac{2\ell}{N}, \qquad h^{(k)} \le \ell N^{-1}, \quad k = 1,3, \qquad \ell N^{-1} \le h^{(2)} < 2\ell N^{-1}, \end{gathered} \tag{3.28}$$

and so

$$\begin{aligned} \overline{\omega}_N = \bigg\{ & x_i = ih^{(1)},\ i = 0,1,\dots,\frac{N}{4};\ x_i = \sigma_1 + \left(i - \frac{N}{4}\right)h^{(2)},\ i = \frac{N}{4}+1,\dots,\frac{3N}{4}; \\ & x_i = \ell - \sigma_2 + \left(i - \frac{3N}{4}\right)h^{(3)},\ i = \frac{3N}{4}+1,\dots,N,\ h^{(1)} = \frac{4\sigma_1}{N}, \\ & h^{(2)} = \frac{2(\ell - \sigma_2 - \sigma_1)}{N},\ h^{(3)} = \frac{4\sigma_2}{N} \bigg\}. \end{aligned} \tag{3.29}$$

In the rest of the paper we only consider this mesh.

We note that on this mesh the coefficient θ_i which is defined by (3.14) simplifies to

$$\theta_i = \begin{cases} \dfrac{a_i h^{(1)}}{2\varepsilon} \coth \dfrac{a_i h^{(1)}}{2\varepsilon}, & \text{for } 1 \le i \le \dfrac{N}{4} - 1, \\[2ex] \dfrac{a_i h^{(2)}}{2\varepsilon} \coth \dfrac{a_i h^{(2)}}{2\varepsilon}, & \text{for } \dfrac{N}{4} + 1 \le i \le \dfrac{3N}{4} - 1, \\[2ex] \dfrac{a_i h^{(3)}}{2\varepsilon} \coth \dfrac{a_i h^{(3)}}{2\varepsilon}, & \text{for } \dfrac{3N}{4} + 1 \le i \le N. \end{cases} \tag{3.30}$$

For the evaluation of the rest values $\theta_{N/4}$ and $\theta_{3N/4}$ we will use the form (3.14).

4. Error Analysis

Let $z = y - u$, $x \in \overline{\omega}_N$. Then for the error of the difference scheme (3.24)–(3.26) we get

$$\varepsilon^2 \theta_i z_{\bar{x}\hat{x},i} + \varepsilon a_i z_{\overset{0}{x},i} - \left[f(x_i, y_i) - f(x_i, u_i)\right] = R_i, \quad 1 \le i \le N-1, \tag{4.1}$$

$$-\varepsilon \theta_0^{(0)} z_{x,0} + \left[\varphi(y_0) - \varphi(u_0)\right] + \theta_0^{(1)}\left[f(x_0, y_0) - f(x_0, u_0)\right] + r^{(0)} = 0, \tag{4.2}$$

$$z_N - \left[\varphi(y_{N_0}) - \varphi(u_{N_0})\right] = r^{(1)}, \tag{4.3}$$

where the truncation errors R_i, $r^{(0)}$, and $r^{(1)}$ are defined by (3.9), (3.21), and (3.23), respectively.

Lemma 4.1. *The solution z_i of problem* (4.1)–(4.3) *satisfies*

$$\|z\|_{\infty,\overline{\omega}_N} \leq (1-\kappa)^{-1}\left(\left(\delta+\beta\theta_0^{(1)}\right)^{-1}\left|r^{(0)}\right| + \left|r^{(1)}\right| + \beta^{-1}\|R\|_{\infty,\omega_N}\right). \tag{4.4}$$

Proof. The problem (4.1)–(4.3) can be rewritten as

$$\begin{gathered}
\varepsilon^2\theta_i z_{\overline{x}\hat{x},i} + \varepsilon a_i z_{\overset{0}{x},i} - \widetilde{b}_i z_i = R_i, \quad 1 \leq i \leq N-1, \\
-\varepsilon\theta_0^{(0)} z_{x,0} + \left(\widetilde{\delta} + \widetilde{b}\theta_0^{(1)}\right) z_0 = -r^{(0)}, \\
z_N - \widetilde{\gamma} z_{N_0} = r^{(1)},
\end{gathered} \tag{4.5}$$

where

$$\widetilde{b}_i = \frac{\partial f}{\partial u}(x_i, \widetilde{y}_i), \qquad \widetilde{\delta} = \varphi^{'(\widetilde{y}_0)}, \qquad \widetilde{\gamma} = \varphi^{'(\widetilde{y}_{N_0})}, \tag{4.6}$$

$\widetilde{y}_0$, $\widetilde{y}_{N_0}$, $\widetilde{y}_i$-intermediate points called for by the mean value theorem.

Since the discrete maximum principle is valid here, we have the proof of (4.4) by analogy with the proof of Lemma 2.1. □

Lemma 4.2. *Under the above assumptions of Section 1 and Lemma 2.1, for the error functions R_i, $r^{(0)}$, and $r^{(1)}$, the following estimates hold:*

$$\begin{gathered}
\|R\|_{\infty,\omega_N} \leq CN^{-1}\ln N, \\
\left|r^{(0)}\right| \leq CN^{-1}\ln N, \\
\left|r^{(1)}\right| \leq CN^{-1}\ln N.
\end{gathered} \tag{4.7}$$

Proof. From explicit expression (3.9) for R_i, on an arbitrary mesh we have

$$\begin{aligned}
|R_i| &\leq C\left\{\varepsilon \max_{[x_{i-1},x_{i+1}]}\left|u'(x)\right| \max_{[x_{i-1},x_{i+1}]}|x_i - x| + \int_{x_{i-1}}^{x_{i+1}}\left|\frac{\partial f(\xi, u(\xi))}{\partial \xi} + \frac{\partial f}{\partial u}\frac{du(\xi)}{d\xi}\right|d\xi\right\} \\
&\leq C\left\{h_i + h_{i+1} + \int_{x_{i-1}}^{x_{i+1}}\left(1+\left|u'(\xi)\right|\right)d\xi\right\}, \quad 1 \leq i \leq N.
\end{aligned} \tag{4.8}$$

This inequality together with (2.3) enables us to write

$$|R_i| \leq C\left\{h_i + h_{i+1} + \frac{1}{\varepsilon}\int_{x_{i-1}}^{x_{i+1}}\left(e^{-\mu_1 x/\varepsilon} + e^{-\mu_2(\ell-x)/\varepsilon}\right)dx\right\}, \tag{4.9}$$

in which

$$h_i = \begin{cases} h^{(1)}, & 1 \le i \le \dfrac{N}{4}, \\ h^{(2)}, & \dfrac{N}{4} + 1 \le i \le \dfrac{3N}{4}, \\ h^{(3)}, & \dfrac{3N}{4} + 1 \le i \le N. \end{cases} \tag{4.10}$$

We consider first the case $\sigma_1 = \sigma_2 = \ell/4$, and so $\ell/4 < \mu_k^{-1}\varepsilon \ln N$, $k = 1,2$, and $h^{(1)} = h^{(2)} = h^{(3)} = h = \ell N^{-1}$. Hereby, from (4.9) we can write

$$\begin{aligned} |R_i| &\le C\left\{N^{-1} + \varepsilon^{-1}h\right\} \le C\left\{N^{-1} + \frac{\ell}{N}\frac{4\mu_1^{-1}}{\ell}\ln N\right\} \\ &= C\left\{N^{-1} + 4\mu_1^{-1}N^{-1}\ln N\right\}. \end{aligned} \tag{4.11}$$

Hence

$$|R_i| \le CN^{-1}\ln N, \quad 1 \le i \le N. \tag{4.12}$$

We now consider the case $\sigma_1 = \mu_1^{-1}\varepsilon \ln N$ and $\sigma_2 = \mu_2^{-1}\varepsilon \ln N$, and so $\mu_k^{-1}\varepsilon \ln N < \ell/4$, $k = 1,2$ and estimate R_i on $[0,\sigma_1]$, $[\sigma_1, \ell - \sigma_2]$, and $[\ell - \sigma_2, \ell]$ separately. In the layer region $[0,\sigma_1]$, the inequality (4.9) reduces to

$$|R_i| \le C\left(1 + \varepsilon^{-1}\right)h^{(1)} \le C\left(1 + \varepsilon^{-1}\right)\frac{4\mu_1^{-1}\varepsilon \ln N}{N}, \quad 1 \le i \le \frac{N}{4} - 1. \tag{4.13}$$

Hence

$$|R_i| \le CN^{-1}\ln N, \quad 1 \le i \le \frac{N}{4} - 1. \tag{4.14}$$

The same estimate is obtained in the layer region $[\ell - \sigma_2, \ell]$ in the similar manner.

It remains to estimate R_i for $N/4 + 1 \le i \le 3N/4 - 1$. In this case we are able to write (4.9) as

$$\begin{aligned} |R_i| \le C\Big\{h^{(2)} + \mu_1^{-1}\left(e^{-\mu_1 x_{i-1}/\varepsilon} - e^{-\mu_1 x_{i+1}/\varepsilon}\right) + \mu_2^{-1}\left(e^{-\mu_2(\ell - x_{i+1})/\varepsilon} - e^{-\mu_2(\ell - x_{i-1})/\varepsilon}\right)\Big\}&, \\ \frac{N}{4} + 1 \le i \le \frac{3N}{4} - 1&. \end{aligned} \tag{4.15}$$

Since $x_i = \mu_1^{-1}\varepsilon \ln N + (i - N/4)h^{(2)}$, it follows that

$$e^{-\mu_1 x_{i-1}/\varepsilon} - e^{-\mu_1 x_{i+1}/\varepsilon} = \frac{1}{N}e^{-\mu_1(i-1-N/4)h^{(2)}/\varepsilon}\left(1 - e^{-2\mu_1 h^{(2)}/\varepsilon}\right) < N^{-1}. \tag{4.16}$$

Also, if we rewrite the mesh points in the form $x_i = \ell - \sigma_2 - (3N/4 - i)h^{(2)}$, evidently

$$e^{-\mu_2(\ell - x_{i+1})/\varepsilon} - e^{-\mu_2(\ell - x_{i-1})/\varepsilon} = \frac{1}{N} e^{-\mu_2(3N/4-i-1)h^{(2)}/\varepsilon}\left(1 - e^{-2\mu_2 h^{(2)}/\varepsilon}\right) < N^{-1}. \tag{4.17}$$

The last two inequalities together with (4.15) give the bound

$$|R_i| \le CN^{-1}. \tag{4.18}$$

It remains to estimate R_i for the mesh points $x_{N/4}$ and $x_{3N/4}$. For the mesh point $x_{N/4}$, inequality (4.9) reduces to

$$|R_{N/4}| \le C\left\{ \left(1 + \varepsilon^{-1}\right)h^{(1)} + h^{(2)} + \frac{1}{\varepsilon}\int_{x_{N/4}}^{x_{N/4+1}} \left(e^{-\mu_1 x/\varepsilon} + e^{-\mu_2(\ell - x)/\varepsilon}\right)dx \right\}. \tag{4.19}$$

Since

$$\begin{gathered} e^{-\mu_1 x_{N/4}/\varepsilon} - e^{-\mu_1 x_{N/4+1}/\varepsilon} = \frac{1}{N}\left(1 - e^{-\mu_1 h^{(2)}/\varepsilon}\right) < N^{-1}, \\ e^{-\mu_2(\ell - x_{N/4+1})/\varepsilon} - e^{-\mu_2(\ell - x_{N/4})/\varepsilon} = \frac{1}{N} e^{-\mu_2 h^{(1)}/\varepsilon}\left(1 - e^{-\mu_2 h^{(1)}/\varepsilon}\right) < N^{-1}, \end{gathered} \tag{4.20}$$

it then follows that

$$|R_{N/4}| \le CN^{-1}\ln N. \tag{4.21}$$

The same estimate is obtained for $i = 3N/4$ in the similar manner.

The same estimate is valid when only one of the values σ_1 and σ_2 is equal to $\ell/4$.

Next, we estimate the remainder term $r^{(0)}$. From the explicit expression (3.21), taking into consideration that $\left(\delta + \beta\theta_0^{(1)}\right)^{-1} \le \delta^{-1}$ and $|\varphi_0(x)| \le 1$, we obtain

$$\begin{aligned} \left|r^{(0)}\right| &= \frac{1}{\delta}\left\{ h_1 \int_{x_0}^{x_1} \left|u'(x)\right| dx + \frac{h_1}{\varepsilon}\int_{x_0}^{x_1} \left| \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} u'(x) \right| dx \right\} \\ &\le C\left\{ \frac{h_1^2}{\varepsilon} + \left(h_1 + \frac{h_1}{\varepsilon}\right)\int_{x_0}^{x_1} \left|u'(x)\right| dx \right\} \\ &\le C\left\{ \frac{h_1^2}{\varepsilon} + \left(h_1 + \frac{h_1}{\varepsilon}\right)\int_{x_0}^{x_1} \left(1 + \frac{1}{\varepsilon} e^{-\mu_1 x/\varepsilon}\right) dx \right\} \\ &\le C\left\{ \frac{h_1^2 + \varepsilon h_1^2 + h_1}{\varepsilon} \right\}. \end{aligned} \tag{4.22}$$

Hence

$$\left|r^{(0)}\right| \leq CN^{-1} \ln N. \tag{4.23}$$

Finally, we estimate the remainder term $r^{(1)}$. From the expression (3.23) we obtain

$$\begin{aligned}
\left|r^{(1)}\right| &\leq \left|\varphi'(\xi)\right| \int_{x_{N_0}}^{\ell_1} \left|u'(\tau)\right| d\tau \\
&\leq C\left\{\ell_1 - x_{N_0} + \frac{1}{\varepsilon}\int_{x_{N_0}}^{\ell_1}\left(e^{-\mu_1\tau/\varepsilon} + e^{-\mu_2(\ell-\tau)/\varepsilon}\right)d\tau\right\} \\
&\leq C\left\{N^{-1} + \frac{1}{\varepsilon}\int_{x_{N_0}}^{x_{N_0+1}}\left(e^{-\mu_1\tau/\varepsilon} + e^{-\mu_2(\ell-\tau)/\varepsilon}\right)d\tau\right\},
\end{aligned} \tag{4.24}$$

where we have assumed that $x_{N_0}$is on the left-hand side of ℓ_1 (if x_{N_0} is on right side of ℓ_1, the integral will be over (x_{N_0-1}, x_{N_0})). In the same manner as above we therefore obtain from here that

$$\left|r^{(1)}\right| \leq CN^{-1} \ln N. \tag{4.25}$$

Thus Lemma 4.2 is proved. □

Combining the two previous lemmas gives us the following convergence result.

Theorem 4.3. *Let $u(x)$ be the solution of* (1.1)–(1.3) *and y the solution* (3.24)–(3.26). *Then*

$$\|y - u\|_{\infty,\overline{\omega}_N} \leq CN^{-1} \ln N. \tag{4.26}$$

5. Numerical Results

In this section, we present some numerical results which illustrate the present method.

Example 5.1. Consider the test problem:

$$\begin{aligned}
&a(x) = 1 + x, \quad f(x,u) = u + \tan^{-1}(x+u), \quad 0 < x < 1, \\
&\psi(u) = \sin u + 2u, \qquad \varphi(u) = 1 + \cos\frac{\pi u}{4}, \qquad \ell_1 = \frac{1}{2}.
\end{aligned} \tag{5.1}$$

The exact solution of our test problem is unknown. Therefore, we use a double-mesh method [2] to estimate the errors and compute the experimental rates of convergence in our computed solutions; that is, we compare the computed solution with the solution on a mesh that is twice as fine (for details see [13, 28]). The error estimates obtained in this way are denoted by

$$e_\varepsilon^N = \max_i \left| y_i^{\varepsilon,N} - \tilde{y}_i^{\varepsilon,2N} \right|, \tag{5.2}$$

where $\tilde{y}_i^{\varepsilon,2N}$ is the approximate solution of the respective method on the mesh

$$\tilde{\omega}_{2N} = \{ x_{i/2} : i = 0, 1, \ldots, 2N \} \tag{5.3}$$

with

$$x_{i+1/2} = \frac{x_i + x_{i+1}}{2} \quad \text{for } i = 0, 1, \ldots, N-1. \tag{5.4}$$

The convergence rates are

$$p_\varepsilon^N = \frac{\ln\left(e_\varepsilon^N / e_\varepsilon^{2N}\right)}{\ln 2}. \tag{5.5}$$

Approximations to the ε-uniform rates of convergence are estimated from

$$e^N = \max_\varepsilon e_\varepsilon^N. \tag{5.6}$$

The corresponding ε-uniform convergence rates are computed using the formula

$$p^N = \frac{\ln\left(e^N / e^{2N}\right)}{\ln 2}. \tag{5.7}$$

To solve the nonlinear problem (3.24)–(3.26) we use the following iteration technique:

$$\varepsilon^2 \theta_i y_{\bar{x}\hat{x},i}^{(n)} + \varepsilon a_i y_{\overset{0}{x},i}^{(n)} - f(x_i, y_i) - \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right)\left(y_i^{(n)} - y_i^{(n-1)}\right) = 0, \quad 1 \le i \le N, \tag{5.8}$$

$$\begin{aligned} &-\varepsilon\theta_0^{(0)} y_{x,0}^{(n)} + \varphi\left(y_0^{(n-1)}\right) + \frac{\partial \varphi}{\partial y}\left(y_0^{(n-1)}\right)\left(y_0^{(n)} - y_0^{(n-1)}\right) \\ &+ \theta_0^{(1)}\left(f\left(x_0, y_0^{(n-1)}\right) + \frac{\partial f}{\partial y}\left(x_0, y_0^{(n-1)}\right)\left(y_0^{(n)} - y_0^{(n-1)}\right)\right) = 0, \end{aligned} \tag{5.9}$$

$$\begin{aligned} &y_N^{(n)} = \varphi\left(y_{N_0}^{(n-1)}\right) + \frac{\partial \varphi}{\partial y}\left(y_{N_0}^{(n-1)}\right)\left(y_{N_0}^{(n)} - y_{N_0}^{(n-1)}\right), \\ &n = 1, 2, \ldots; \quad y_i^{(0)} \quad \text{given} \quad 1 \le i \le N. \end{aligned} \tag{5.10}$$

Table 1: For the case of $a(x) \neq 0$, approximate errors e_ε^N and e^N and the computed orders of convergence p_ε^N on the piecewise uniform mesh ω_N for various values of ε and N.

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-2}	0.02233220	0.01350608	0.00954256	0.00428337	0.00310038	0.00176533
	0.77	0.82	0.86	0.93	0.98	0.99
10^{-4}	0.02233215	0.01376835	0.00850615	0.00538213	0.00315446	0.00160520
	0.75	0.80	0.85	0.92	0.97	0.99
10^{-6}	0.02233210	0.01376065	0.00851454	0.00540203	0.00314356	0.00160544
	0.75	0.80	0.85	0.92	0.97	0.99
10^{-8}	0.02233205	0.01376074	0.00851405	0.00540203	0.00314356	0.00160522
	0.75	0.80	0.85	0.92	0.97	0.99
10^{-10}	0.02233209	0.01376065	0.00851405	0.00540203	0.00314356	0.00160522
	0.75	0.80	0.85	0.92	0.97	0.99
10^{-12}	0.02233209	0.01376065	0.00851405	0.00540203	0.00314356	0.00160522
	0.75	0.80	0.85	0.92	0.97	0.99
10^{-14}	0.02233209	0.01376065	0.00851405	0.00540203	0.00314356	0.00160522
	0.75	0.80	0.85	0.92	0.97	0.99
10^{-16}	0.02233213	0.01376074	0.00851405	0.00540203	0.00314356	0.00160522
	0.75	0.80	0.85	0.92	0.97	0.99
10^{-18}	0.02233213	0.01376074	0.00851405	0.00540203	0.00314356	0.00160522
	0.75	0.80	0.85	0.92	0.97	0.99
10^{-20}	0.02233213	0.01376074	0.00851405	0.00540203	0.00314356	0.00160522
	0.75	0.80	0.85	0.92	0.97	0.99
e^N	0.02233220	0.01376835	0.00954256	0.00540203	0.00315446	0.00176533
p^N	0.75	0.80	0.85	0.92	0.97	0.99

To solve (5.8)–(5.10), we take the initial approximation as $y_i^{(0)} = x_i^2$ and the stopping criterion is

$$\max_i \left| y_i^{(n)} - y_i^{(n-1)} \right| \leq 10^{-5}. \tag{5.11}$$

The computed maximum pointwise errors e_ε^N and e_ε^{2N}, and the orders of uniform convergence p_ε^N for different values of ε and N, based on the double-mesh principle are presented in Tables 1 and 2. The results established here are that the discrete solution is uniformly convergent with respect to the perturbation parameter p^N and the errors are uniformly convergent with rates of almost unity as predicted by our theoretical analysis.

Example 5.2. Consider the test problem:

$$\begin{gathered} a(x) = \sin\left(\frac{\pi x}{2}\right), \quad f(x,u) = 1 + x^2 + u + \tanh u, \quad 0 < x < 1, \\ \psi(u) = u - 1, \qquad \varphi(u) = \frac{1}{2}u + 2, \qquad \ell_1 = \frac{1}{2}. \end{gathered} \tag{5.12}$$

The exact solution of our test problem is unknown. In the same manner as above we solve this problem.

Table 2: For the case $a(0) = 0$ of approximate errors e_ε^N and e^N and the computed orders of convergence p_ε^N on the piecewise uniform mesh ω_N for various values of ε and N.

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-2}	0.02453225	0.01721110	0.01123085	0.00652485	0.00439674	0.00189543
	0.68	0.79	0.83	0.88	0.96	0.99
10^{-4}	0.02453220	0.01716585	0.01120973	0.00651194	0.00439686	0.00188515
	0.67	0.76	0.80	0.85	0.94	0.99
10^{-6}	0.02453221	0.01706582	0.01120403	0.00651089	0.0043 9225	0.00188523
	0.67	0.76	0.80	0.85	0.94	0.99
10^{-8}	0.02453215	0.01706553	0.01120282	0.00651046	0.00439214	0.00188517
	0.67	0.76	0.80	0.85	0.94	0.99
10^{-10}	0.02453215	0.01706593	0.01120275	0.00651093	0.00439214	0.00188517
	0.67	0.76	0.80	0.85	0.94	0.99
10^{-12}	0.02453216	0.01706452	0.01120270	0.00651025	0.00439214	0.00188517
	0.67	0.76	0.80	0.85	0.94	0.99
10^{-14}	0.02453209	0.01706425	0.01120256	0.00651025	0.00439214	0.00188517
	0.67	0.76	0.80	0.85	0.94	0.99
10^{-16}	0.02453209	0.01706425	0.01120256	0.00651025	0.00439214	0.00188517
	0.67	0.76	0.80	0.85	0.94	0.99
10^{-18}	0.02453209	0.01706425	0.01120256	0.00651025	0.00439214	0.00188517
	0.67	0.76	0.80	0.85	0.94	0.99
10^{-20}	0.02453209	0.01706425	0.01120256	0.00651025	0.00439214	0.00188517
	0.67	0.76	0.80	0.85	0.94	0.99
e^N	0.02453225	0.01716585	0.01123085	0.00652485	0.00439686	0.00189543
p^N	0.68	0.79	0.83	0.88	0.96	0.99

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