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Research Article

On the Critical Case in Oscillation for Differential Equations with a Single Delay and with Several Delays

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New nonoscillation and oscillation criteria are derived for scalar delay differential equations $\dot{x}(t) + a(t)x(h(t)) = 0, a(t) \geq 0, h(t) \leq t, t \geq t_0$, and $\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, a_k(t) \geq 0, h_k(t) \leq t$, and $t \geq t_0$, in the critical case including equations with several unbounded delays, without the usual assumption that the parameters a, h, a_k, and h_k of the equations are continuous functions. These conditions improve and extend some known oscillation results in the critical case for delay differential equations.

1. Introduction

It is well known that a scalar linear equation with delay

$$\dot{x}(t) + \frac{1}{e}x(t-1) = 0 \tag{1.1}$$

has a nonoscillatory solution as $t \to \infty$. This means that there exists an eventually positive solution. The coefficient $1/e$ is called critical with the following meaning: for any $\alpha > 1/e$, all solutions of the equation

$$\dot{x}(t) + \alpha x(t-1) = 0 \tag{1.2}$$

are oscillatory while, for $\alpha \leq 1/e$, there exists an eventually positive solution.

In [1] the third author considered the equation

$$\dot{x}(t) + a(t)x(t-\tau) = 0, \tag{1.3}$$

where $a : [t_0, \infty) \to (0, \infty)$, $t_0 \in \mathbb{R}$ (throughout this paper we assume that $t_0 \geq 0$ is sufficiently large), is a continuous function and the delay $\tau > 0$ is a constant. For the critical case, he obtained the following result.

Theorem 1.1. (a) *Let an integer $k \geq 0$ exists such that $a(t) \leq a_k(t)$ if $t \to \infty$ where*

$$a_k(t) := \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2}. \tag{1.4}$$

Then there exists an eventually positive solution x of (1.3).

(b) *Let an integer $k \geq 2$ and $\theta > 1$, $\theta \in \mathbb{R}$, exist such that*

$$a(t) > a_{k-2}(t) + \frac{\theta\tau}{8e(t\ln t\ln_2 t\cdots\ln_{k-1} t)^2} \tag{1.5}$$

if $t \to \infty$. Then all solutions of (1.3) *oscillate.*

In this theorem for $k \geq 1$, $\ln_k t = \ln(\ln_{k-1} t)$, $\ln_0 t = t$, $t > \exp_{k-2} 1$ where $\exp_k t = \exp(\exp_{k-1} t)$, $\exp_0 t = t$, and $\exp_{-1} t = 0$.

Further results on the critical case for (1.3) can be found in [2–6]. Theorem 1.1 was generalized in [7] for the following equation with a variable delay

$$\dot{x}(t) + a(t)x(t-\tau(t)) = 0, \tag{1.6}$$

where $a : [t_0, \infty) \to (0, \infty)$, $t_0 \in \mathbb{R}$, and $\tau : [t_0, \infty) \to (0, \infty)$, $t_0 \in \mathbb{R}$, are continuous functions.

The main results of this paper include the following.

Theorem 1.2 (see [7]). *Let $t - \tau(t) \geq t_0 - \tau(t_0)$ if $t \geq t_0$. Let an integer $k \geq 0$ exists such that $a(t) \leq a_{k\tau}(t)$ for $t \to \infty$, where*

$$a_{k\tau}(t) := \frac{1}{e\tau(t)} + \frac{\tau(t)}{8et^2} + \frac{\tau(t)}{8e(t\ln t)^2} + \cdots + \frac{\tau(t)}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2}. \tag{1.7}$$

If moreover

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)}\,d\xi \leq 1, \quad \text{when } t \longrightarrow \infty, \tag{1.8}$$

$$\lim_{t\to\infty} \tau(t) \cdot \left(\frac{1}{t}\ln t\ln_2 t\cdots\ln_k t\right) = 0, \tag{1.9}$$

then there exists an eventually positive solution x of (1.6) *for $t \to \infty$.*

Theorem 1.3 (see [7]). *Let one assume that* $t-\tau(t) \geq t_0-\tau(t_0)$ *if* $t \geq t_0$ *and*

$$a(t) \leq \frac{1}{\tau(t)} \exp\left[-\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi\right] \tag{1.10}$$

as $t \to \infty$. *Then there exists an eventually positive solution* x *of* (1.6).

In this paper we obtain new nonoscillation and oscillation sufficient conditions for (1.6) in the critical case, independent of Theorems 1.1–1.3. We also obtain nonoscillation and oscillation conditions for equations with several delays, including equations with unbounded delays. To the best of our knowledge, we are the first to investigate the critical case of such equations.

2. Preliminaries

We consider a scalar delay differential equation

$$\dot{x}(t) + \sum_{i=1}^{m} b_i(t) x(h_i(t)) = 0, \quad t \geq t_0, \tag{2.1}$$

subject to the following conditions:

(a1) $b_i : [t_0, \infty) \to [0, \infty)$, $i = 1, \ldots, m$, are Lebesgue measurable functions essentially bounded in each finite interval $[t_0, b]$ with $b > t_0$.

(a2) $h_i : [t_0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_i(t) \leq t$, $t \in [t_0, \infty)$, and $\limsup_{t\to\infty} h_i(t) = +\infty$, $i = 1, \ldots, m$.

Along with (2.1) we consider an initial value problem

$$\dot{x}(t) + \sum_{i=1}^{m} b_i(t) x(h_i(t)) = f(t), \quad t \geq t_0, \tag{2.2}$$

$$x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0. \tag{2.3}$$

We also assume that the following hypothesis holds:

(a3) $f : [t_0, \infty) \to \mathbb{R}$ is a Lebesgue measurable function essentially bounded in each finite interval $[t_0, b]$ with $b > t_0$, and $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

Definition 2.1. A function absolutely continuous on each interval $[t_0, b]$ with $b > t_0$, $x : \mathbb{R} \to \mathbb{R}$, is called a solution of problem (2.2), (2.3) if it satisfies (2.2) for almost all $t \in [t_0, \infty)$ and equalities (2.3) for $t \leq t_0$.

Lemma 2.2 (see [8]). *Let* (*a*1)-(*a*3) *hold. Then there exists exactly one solution of problem* (2.2), (2.3).

Definition 2.3. One will say that (2.1) has a nonoscillatory solution if, for some problem (2.2), (2.3) with $f(t) \equiv 0$, $t \geq t_0$, there exists an eventually positive solution. Otherwise, all solutions of (2.1) oscillate.

To formulate a comparison result, consider the following equation:

$$\dot{x}(t) + \sum_{i=1}^{m} c_i(t) x\bigl(g_i(t)\bigr) = 0. \tag{2.4}$$

Let (a1) holds with $b_i(t) := c_i(t)$ and (a2) holds with $h_i(t) := g_i(t)$, $i = 1, 2, \ldots, m$.

Lemma 2.4 (see [9, 10]). *Let* (2.4) *have a nonoscillatory solution. If*

$$b_i(t) \le c_i(t), \quad g_i(t) \le h_i(t), \quad t \ge t_0, \tag{2.5}$$

then (2.1) *has a nonoscillatory solution as well.*

Suppose that all solutions of (2.4) *are oscillatory. If*

$$b_i(t) \ge c_i(t), \quad g_i(t) \ge h_i(t), \quad t \ge t_0, \tag{2.6}$$

then all solutions of (2.1) *are oscillatory as well.*

Lemma 2.5 (see [9, 10]). *Let exist t_0 such that*

$$\int_{\min_{i=1,\ldots,m}\{\max\{t_0, h_i(t)\}\}}^{t} \sum_{j=1}^{m} b_j(s)\,ds \le \frac{1}{e}, \quad t \ge t_0. \tag{2.7}$$

Then there exists a positive solution of (2.1) *for $t \ge t_0$.*

Lemma 2.6 (see [9, 10]). *A nonoscillatory solution of* (2.1) *exists if and only if, for some t_0, there exists a nonnegative locally integrable function $u(t) \ge 0$, $t \in \mathbb{R}$, such that*

$$\begin{gathered} u(t) \ge \sum_{i=1}^{m} b_i(t) e^{\int_{h_i(t)}^{t} u(s)\,ds}, \quad t \ge t_0, \\ u(t) = 0, \quad t < t_0. \end{gathered} \tag{2.8}$$

3. Differential Equation with a Single Delay

Equation (1.6) is a special case of (2.1) for $m = 1$, $b_1(t) = a(t)$, and $h_1(t) = t - \tau(t)$.

Our first result is a simple consequence of Theorem 1.1 and Lemma 2.4. Theorem 1.1 was obtained under the assumption that $a(t)$ and $\tau(t)$ are continuous functions. But the proof of this theorem remains valid even for more general conditions (a1)-(a2).

Theorem 3.1. (A) *Let $\tau > 0$, $0 \le \tau(t) \le \tau$, for $t \to \infty$, and let condition (a) of Theorem 1.1 holds. Then* (1.6) *has a nonoscillatory solution.*

(B) *Let $\tau(t) \ge \tau > 0$ for $t \to \infty$, and let condition (b) of Theorem 1.1 holds. Then all solutions of* (1.6) *oscillate.*

Proof. (A) We set $h_1(t) := t - \tau(t)$, $g_1(t) := t - \tau$. Obviously $h_1(t) \geq g_1(t)$ for $t \to \infty$. By Theorem 1.1, (1.3) has a nonoscillatory solution. By Lemma 2.4 (with $m = 1$, $b_1(t) = c_1(t) = a(t)$), (1.6) also has a nonoscillatory solution.

(B) The proof of this part is much the same (using Theorem 1.1 and Lemma 2.4) as the proof of part (A). □

Theorems 1.2 and 1.3 can be applied to equations with one unbounded delay. Here, we want to give some new nonoscillation and oscillation conditions for equations with one delay, also including equations with unbounded delays. We remove some conditions of Theorems 1.2 and 1.3, in particular conditions (1.8) and (1.9). Moreover, the delay function $\tau(t)$ used in Theorems 1.2 and 1.3 as a coefficient appears in our conditions in both integral and nonintegral expressions.

For every integer $k \geq 0$, $\delta > 0$, and $t \to \infty$ we define

$$A_k(t) := \frac{1}{e\delta\tau(t)} + \frac{\delta}{8e\tau(t)s^2} + \frac{\delta}{8e\tau(t)(s\ln s)^2} + \cdots + \frac{\delta}{8e\tau(t)(s\ln s\ln_2 s\cdots\ln_k s)^2}, \tag{3.1}$$

where

$$s = p(t) := \int_{t_0}^{t} \frac{1}{\tau(\xi)} d\xi. \tag{3.2}$$

Theorem 3.2. *Let for t_0 sufficiently large and $t \geq t_0$: $\tau(t) > 0$ a.e. $1/\tau(t)$ be a locally integrable function,*

$$\lim_{t\to\infty} (t - \tau(t)) = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{\tau(\xi)} d\xi = \infty, \tag{3.3}$$

and let there exists $t_1 > t_0$ such that $t - \tau(t) \geq t_0$, $t \geq t_1$.

(a) *If there exists a $\delta \in (0,\infty)$ such that*

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \leq \delta, \quad t \geq t_1 \tag{3.4}$$

and, for a fixed integer, $k \geq 0$,

$$a(t) \leq A_k(t), \quad t \geq t_1, \tag{3.5}$$

then there exists an eventually positive solution of (1.6)*.*

(b) *If there exists a $\delta \in (0,\infty)$ such that*

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \geq \delta, \quad t \geq t_1, \tag{3.6}$$

and, for a fixed integer $k \geq 2$ *and* $\theta > 1$, $\theta \in \mathbb{R}$,

$$a(t) > A_{k-2}(t) + \frac{\theta\delta}{8e\tau(t)(s \ln s \ln_2 s \cdots \ln_{k-1} s)^2} \tag{3.7}$$

if $t \geq t_1$, *then all solutions of* (1.6) *oscillate.*

Proof. (a) For the proof we will use a transformation applied to delay equations for the first time in [11]. Consider (1.6) for $t \geq t_1$. Denote

$$h(t) := t - \tau(t). \tag{3.8}$$

Since, by (3.2), $s > 0$, p is a strictly increasing function and, hence, there exists an inverse function $t = p^{-1}(s)$. Denote

$$y(s) := x(t) = x\left(p^{-1}(s)\right), \qquad l(s) := p(h(t)) = p\left(h\left(p^{-1}(s)\right)\right). \tag{3.9}$$

Since $h(t) \leq t$, we have $l(s) \leq s$ (by (3.9) and (3.2)). From (3.9) we also have

$$l(s) = \int_{t_0}^{h(t)} \frac{1}{\tau(\xi)} d\xi. \tag{3.10}$$

Substituting $x(t) = y(s)$ in (1.6), we have $\dot{x}(t) = \dot{y}(s)/\tau(t)$ and (using (3.9))

$$\begin{aligned} x(t - \tau(t)) = x(h(t)) &= x\left(h\left(p^{-1}(s)\right)\right) = x\left(p^{-1}\left[p\left(h\left(p^{-1}(s)\right)\right)\right]\right) \\ &= x\left(p^{-1}(l(s))\right) = y(l(s)). \end{aligned} \tag{3.11}$$

Hence, (1.6) takes the form

$$\dot{y}(s) + \tau(t)a(t)y(l(s)) = 0, \tag{3.12}$$

where $\tau(t)a(t) = \tau(p^{-1}(s))a(p^{-1}(s))$. Equality

$$y(s) = x(t) \tag{3.13}$$

implies that the oscillation properties of (1.6) and (3.12) are equivalent. We have (by (3.2), (3.10), (3.8), and (3.4))

$$s - l(s) = \int_{t_0}^{t} \frac{1}{\tau(s)} ds - \int_{t_0}^{h(t)} \frac{1}{\tau(s)} ds = \int_{h(t)}^{t} \frac{1}{\tau(s)} ds = \int_{t-\tau(t)}^{t} \frac{1}{\tau(s)} ds \leq \delta. \tag{3.14}$$

Hence

$$l(s) \geq s - \delta. \tag{3.15}$$

Consider an equation

$$\dot{y}(s) + a_k^*(s)y(s-\delta) = 0, \tag{3.16}$$

where $a_k^*(s)$ is defined similar to $a_k(t)$ by (1.4), where τ is replaced by δ and t by s, that is,

$$a_k^*(s) := \frac{1}{e\delta} + \frac{\delta}{8es^2} + \frac{\delta}{8e(s\ln s)^2} + \cdots + \frac{\delta}{8e(s\ln s\ln_2 s\cdots\ln_k s)^2}. \tag{3.17}$$

By Theorem 1.1, (3.16) has a positive solution. Equation (3.12) is of type (2.1) with

$$m = 1, \qquad b_1(s) = \tau\left(p^{-1}(s)\right)a\left(p^{-1}(s)\right), \qquad h_1(s) = l(s). \tag{3.18}$$

Now we use comparison of Lemma 2.4 where (2.4) is replaced by (3.16), that is,

$$m = 1, \qquad c_1(s) = a_k^*(s), \qquad g_1(s) = s - \delta. \tag{3.19}$$

Since, by (3.5) and (3.1),

$$b_1(s) = \tau\left(p^{-1}(s)\right)a\left(p^{-1}(s)\right) \leq \tau\left(p^{-1}(s)\right)A_k\left(p^{-1}(s)\right) = a_k^*(s) = c_1(s) \tag{3.20}$$

and, by (3.15),

$$g_1(s) = s - \delta \leq l(s) = h_1(s), \tag{3.21}$$

equation (3.12) and, due to (3.13), equation (1.6) also has a positive (i.e., nonoscillatory) solution. Part (b) can be proved in much the same way. □

Now we want to compare Theorem 3.2 and Theorems 1.2 and 1.3 for equations with unbounded delays. Note that Theorem 1.1 is not valid for such equations and Theorem 1.2 contains additional restrictions (1.8), (1.9). Theorem 1.3 is not explicitly valid for the critical case.

Example 3.3. Let (1.6) be of the form

$$\dot{x}(t) + a(t)x\left(\frac{t}{2}\right) = 0, \quad t \geq t_0 = 1, \tag{3.22}$$

where $a : [1, \infty) \to (0, \infty)$. Here

$$\tau(t) = \frac{t}{2}, \quad s = p(t) = \int_{t_0}^{t} \frac{1}{\tau(\xi)}\, d\xi = \int_{1}^{t} \frac{2}{\xi}\, d\xi = 2\ln t. \tag{3.23}$$

We set

$$\delta := \int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)}\, d\xi = \int_{t/2}^{t} \frac{2}{\xi}\, d\xi = 2\ln 2. \tag{3.24}$$

In accordance with Theorem 3.2 (case (a) where $k = 0$), (3.22) has a nonoscillatory solution if

$$a(t) \le A_0(t) = \frac{1}{e\delta\tau(t)} + \frac{\delta}{8e\tau(t)s^2} = \frac{1}{(e\ln 2)t} + \frac{\ln 2}{8et(\ln t)^2}. \tag{3.25}$$

Since (by Theorem 3.2, case (b) with $k = 2$) all solutions of (3.22) oscillate if

$$a(t) > A_0(t) + \frac{\theta\delta}{8e\tau(t)\ (s\ln s)^2} = \frac{1}{(e\ln 2)t} + \frac{\ln 2}{8et(\ln t)^2} + \frac{\theta\ln 2}{8et(\ln t)^2(\ln(2\ln t))^2}, \tag{3.26}$$

we conclude that the value

$$a(t) = a^*(t) := \frac{1}{(e\ln 2)t} + \frac{\ln 2}{8et(\ln t)^2} \tag{3.27}$$

is a critical value for the nonoscillation of (3.22).

The above statement is corroborated by Lemma 2.5 since, for $m = 1$, $h_1(t) = t - \tau(t) = t/2$, $b_1(t) = a^*(t)$, and $t \in [1, 2]$,

$$\int_{\max\{t_0, h_1(t)\}}^{t} b_1(s)ds = \int_{t_0}^{t} a^*(s)ds = \int_{1}^{t} a^*(s)ds = \frac{\ln t}{e\ln 2} \le \frac{1}{e} \tag{3.28}$$

and for $t \ge 2$

$$\begin{aligned} \int_{\max\{t_0, h_1(t)\}}^{t} b_1(s)ds &= \int_{h_1(t)}^{t} a^*(s)ds = \int_{t/2}^{t} a^*(s)ds \\ &= \int_{t/2}^{t} \frac{1}{(e\ln 2)s}\, ds = \frac{1}{e}, \end{aligned} \tag{3.29}$$

and (2.7) turns into an equality for $t \ge 2$.

To apply Theorem 1.2, we verify condition (1.8). But, unfortunately, for (3.22) we have

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi = \int_{t/2}^{t} \frac{2}{\xi}\, d\xi = 2\ln 2 \doteq 1.386 > 1. \tag{3.30}$$

Thus, this theorem is not applicable to (3.22).

To compare Theorem 1.3 with Theorem 3.2 we set $a(t) := a^*(t)$ (where a^* is defined by (3.27)) in (3.22). By Theorem 3.2, (3.22) has a nonoscillatory solution. By Theorem 1.3, (3.22) has a nonoscillatory solution if (we set $\tau(t) := t/2$ in (1.10))

$$a(t) \le \frac{1}{\tau(t)} \exp\left[-\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi\right] = \frac{2}{t} \exp\left[-\int_{t/2}^{t} \frac{2}{\xi} d\xi\right] = \frac{1}{2t}. \tag{3.31}$$

But in our case

$$a(t) = a^*(t) = \frac{1}{(e \ln 2)t} + \frac{\ln 2}{8et(\ln t)^2} \ge \frac{1}{1.885t} > \frac{1}{2t}, \tag{3.32}$$

and Theorem 1.3 fails for this equation.

4. Differential Equation with Several Delays

We start with the following question: for what functions $b(t) \ge 0$ and delay $\sigma > 0$ the equation

$$\dot{x}(t) + b(t)x(t-\sigma) + \frac{1}{e\tau}x(t-\tau) = 0 \tag{4.1}$$

can have a nonoscillatory solution? It is easy to see that b should be vanishing.

Theorem 4.1. *Let* $\liminf_{t\to\infty} b(t) = b > 0, \tau > 0$, *and* $\sigma \ge 0$. *Then all solutions of* (4.1) *are oscillatory.*

Proof. Consider first the equation

$$\dot{x}(t) + bx(t-\sigma) + \frac{1}{e\tau}x(t-\tau) = 0. \tag{4.2}$$

Suppose that (4.2) has a nonoscillatory solution. We set

$$\begin{gathered} m = 2, \\ b_1(t) = c_1(t) := b, \qquad b_2(t) = c_2(t) := \frac{1}{e\tau}, \\ g_1(t) := t-\sigma, \qquad h_2(t) = g_2(t) := t-\tau, \\ h_1(t) := t. \end{gathered} \tag{4.3}$$

Since $g_1(t) \le h_1(t)$, by Lemma 2.4, the equation

$$\dot{x}(t) + bx(t) + \frac{1}{e\tau}x(t-\tau) = 0 \tag{4.4}$$

has a nonoscillatory solution. After the substitution $x(t) = e^{-bt}y(t)$, (4.4) takes a form

$$\dot{y}(t) + \frac{e^{b\tau}}{e\tau}y(t-\tau) = 0. \tag{4.5}$$

Since $e^{b\tau} > 1$, all solutions of (4.5) are oscillatory by Theorem 1.1 (b). This is a contradiction. Hence, all solutions of (4.2) are oscillatory.

For sufficiently large t_0, we have $b(t) \geq b$, $t \geq t_0$. We set

$$\begin{gathered} m = 2, \\ b_1(t) := b(t) \geq c_1(t) := b, \\ b_2(t) = c_2(t) := \frac{1}{e\tau}, \\ h_1(t) = g_1(t) := t - \sigma, \qquad h_2(t) = g_2(t) := t - \tau. \end{gathered} \tag{4.6}$$

Now, Lemma 2.4 implies the statement of the theorem. □

We consider general equation (2.1) with delays subject to restrictions (a1), (a2).

Theorem 4.2. (a) *Let an integer $k \geq 0$ and $\tau > 0$ exist such that, for all sufficiently large t, inequalities*

$$t - h_i(t) \leq \tau, \quad i = 1, 2, \ldots, m, \tag{4.7}$$

$$\sum_{i=1}^{m} b_i(t) \leq a_k(t), \tag{4.8}$$

where a_k is defined by (1.4)*, are valid. Then there exists an eventually positive solution x of* (2.1)*.*

(b) *Let an integer $k \geq 2$, $\tau > 0$, and $\theta > 1$ exist such that, for all sufficiently large t, inequalities*

$$t - h_i(t) \geq \tau, \quad i = 1, 2, \ldots, m, \tag{4.9}$$

$$\sum_{i=1}^{m} b_i(t) \geq a_{k-2}(t) + \frac{\theta\tau}{8e(t \ln t \ln_2 t \cdots \ln_{k-1} t)^2}, \tag{4.10}$$

where a_{k-2} is defined by (1.4)*, are valid. Then all solutions of* (2.1) *oscillate.*

Proof. Let the assumptions of case (a) be valid. Then, by Theorem 1.1, the equation

$$\dot{x}(t) + a_k(t)x(t-\tau) = 0 \tag{4.11}$$

has a nonoscillatory solution. By Lemma 2.6 (with $m = 1$, $b_1(t) = a_k(t)$, and $h_1(t) = t - \tau$), there exist a t_0 and a locally integrable function $u : \mathbb{R} \to [0, \infty)$ such that

$$u(t) \geq a_k(t) e^{\int_{t-\tau}^{t} u(s)ds}, \quad t \geq t_0, \qquad u(t) = 0, \quad t < t_0. \tag{4.12}$$

We have

$$\sum_{i=1}^{m} b_i(t) e^{\int_{h_i(t)}^{t} u(s)ds} \leq \left(\sum_{i=1}^{m} b_i(t) \right) e^{\int_{t-\tau}^{t} u(s)ds} \leq a_k(t) e^{\int_{t-\tau}^{t} u(s)ds}. \tag{4.13}$$

Hence

$$u(t) \geq \sum_{i=1}^{m} b_i(t) e^{\int_{h_i(t)}^{t} u(s)ds}, \quad t \geq t_0, \qquad u(t) = 0, \quad t < t_0. \tag{4.14}$$

Now using Lemma 2.6 again, we conclude that there exists an eventually positive solution x of (2.1).

Let the assumptions of case (b) be valid. Suppose, on the contrary, that (2.1) has a nonoscillatory solution. Using calculations similar to those of the previous part of the proof, one can deduce that (by Lemma 2.6) there exist a t_0 and a locally integrable function $u(t) \geq 0$ such that

$$u(t) \geq \sum_{i=1}^{m} b_i(t) e^{\int_{h_i(t)}^{t} u(s)ds} \geq \left(a_{k-2}(t) + \frac{\theta\tau}{8e(t \ln t \ln_2 t \cdots \ln_{k-1} t)^2} \right) e^{\int_{t-\tau}^{t} u(s)ds}, \quad t \geq t_0, \qquad u(t) = 0, \quad t < t_0. \tag{4.15}$$

Hence (using Lemma 2.6 again), an equation

$$\dot{x}(t) + \left(a_{k-2}(t) + \frac{\theta\tau}{8e(t \ln t \ln_2 t \cdots \ln_{k-1} t)^2} \right) x(t - \tau) = 0 \tag{4.16}$$

should have a nonoscillatory solution. Due to $\theta > 1$ being arbitrary, we easily get a contradiction to statement (b) of Theorem 1.1. □

Example 4.3. We show that equation of type (2.1)

$$\dot{x}(t) + \frac{1}{8et^2} x(t - \sigma) + \frac{1}{e} x(t - 1) = 0 \tag{4.17}$$

has a nonoscillatory solution for any positive $\sigma \leq 1$. Indeed, set $m = 2$, $\tau = 1$, $h_1(t) = t - \sigma$, $h_2(t) = t - \tau = t - 1$, $b_1(t) = 1/(8et^2)$, and $b_2(t) = 1/e$. Then (4.8), where $k = 0$, holds and Part (a) of Theorem 4.2 is valid.

Now we consider (2.1) with unbounded delays.

Theorem 4.4. *Let t_0 be sufficiently large, for $t \geq t_0$, $\tau(t) > 0$ a.e., $1/\tau(t)$ let locally integrable function,*

$$\lim_{t\to\infty}(t-\tau(t)) = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{\tau(\xi)} d\xi = \infty, \tag{4.18}$$

$$\lim_{t\to\infty} h_i(t) = +\infty, \quad t - h_i(t) \leq \tau(t), \quad i = 1,2,\ldots,m, \tag{4.19}$$

and let there exists $t_1 > t_0$ such that $t - \tau(t) \geq t_0$ if $t \geq t_1$.

(a) *If there exists a $\sigma \in (0,\infty)$ such that*

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \leq \sigma, \quad t \geq t_1, \tag{4.20}$$

and, for a fixed integer $k \geq 0$,

$$\sum_{i=1}^{m} b_i(t) \leq A_k(t), \quad t \geq t_1, \tag{4.21}$$

where $A_k(t)$ is defined by (3.1), (3.2), *then there exists an eventually positive solution of* (2.1).

(b) *If there exists a $\sigma \in (0,\infty)$ such that*

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \geq \sigma, \quad t \geq t_1, \tag{4.22}$$

and, for a fixed integer $k \geq 2$ and $\theta > 1$, $\theta \in \mathbb{R}$,

$$\sum_{i=1}^{m} b_i(t) \geq A_{k-2}(t) + \frac{\theta\sigma}{8e\tau(t)(s\ln s\ln_2 s\cdots\ln_{k-1}s)^2}, \quad t \geq t_1, \tag{4.23}$$

where $A_{k-2}(t)$ is defined by (3.1), (3.2), *then all solutions of* (2.1) *oscillate.*

Proof. Let the assumptions of case (a) be valid. By Theorem 3.2, the equation

$$\dot{x}(t) + \left(\sum_{i=1}^{m} b_i(t)\right) x(t-\tau(t)) = 0 \tag{4.24}$$

has a nonoscillatory solution. Equation (4.24) is of a form of (2.4) with $c_i(t) = b_i(t)$, $g_i(t) = t - \tau(t)$, $i = 1,2,\ldots,m$. This means that we see (4.24) as an equation with m delayed terms.

Compare (4.24) with (2.1). We have $b_i(t) \leq c_i(t)$ and, due to (4.19), $g_i(t) \leq h_i(t)$, $i = 1,2,\ldots,m$. By Lemma 2.4, (2.1) has a nonoscillatory solution.

The proof of part (b) can be carried out in a way similar to that of the proof of part (a) and, therefore, it is omitted. □

Example 4.5. Consider the equation of the type of (2.1):

$$\dot{x}(t)+\frac{\alpha}{te\ln 3}x\left(\frac{t}{3}\right)+\frac{\ln 3}{8et(\ln t)^2}x\left(\frac{t}{2}\right)=0,\quad t\geq t_0=1. \tag{4.25}$$

First let $0<\alpha\leq 1$. We set $m=2$, $t_1:=3$, $\tau(t)=:2t/3$, and

$$\sigma:=\int_{t-\tau(t)}^{t}\frac{1}{\tau(\xi)}d\xi=\int_{t-(2t)/3}^{t}\frac{3}{2\xi}d\xi=\int_{t/3}^{t}\frac{3}{2\xi}d\xi=\frac{3}{2}\ln 3. \tag{4.26}$$

Moreover, we put

$$\begin{gathered}
h_1(t)=\frac{t}{3},\qquad h_2(t)=\frac{t}{2},\qquad b_1(t)=\frac{\alpha}{te\ln 3},\qquad b_2(t)=\frac{\ln 3}{8et(\ln t)^2},\\
t-h_k(t)\leq\tau(t)=\frac{2}{3}t,\quad k=1,2,\\
s=\int_{t_0}^{t}\frac{1}{\tau(\xi)}d\xi=\int_{1}^{t}\frac{3}{2\xi}d\xi=\frac{3}{2}\ln t.
\end{gathered} \tag{4.27}$$

By (3.1),

$$A_0(t)=\frac{1}{e\sigma\tau(t)}+\frac{\sigma}{8e\tau(t)s^2}=\frac{1}{te\ln 3}+\frac{\ln 3}{8et(\ln t)^2}\geq b_1(t)+b_2(t). \tag{4.28}$$

All conditions of Theorem 4.4 part (a) hold, hence (4.25) has a nonoscillatory solution.

Similarly, one can show (by Theorem 4.4 part (b)) that, for $\alpha>1$, all solutions of (4.25) are oscillatory.

5. Differential Equation with Two Delays

In [12] the authors consider a differential equation with two delays

$$\dot{x}(t)+b_1(t)x(h_1(t))+b_2(t)x(h_2(t))=0,\quad t\geq t_0, \tag{5.1}$$

where $b_i:[t_0,\infty)\to[0,\infty)$, $i=1,2$,

$$h_1(t)=t-\tau,\qquad h_2(t)=t-\sigma, \tag{5.2}$$

and τ, σ are positive constants. Let

$$\liminf_{t\to\infty}b_1(t)=p,\qquad \liminf_{t\to\infty}b_2(t)=q. \tag{5.3}$$

In accordance with [12] we say that (5.1) is in a critical state if there exists $\lambda_0 \geq 0$ such that

$$\lambda_0 = pe^{\lambda_0 \tau} + qe^{\lambda_0 \sigma} \tag{5.4}$$

and, for any $\lambda > 0$, $\lambda \neq \lambda_0$, we have

$$\lambda < pe^{\lambda \tau} + qe^{\lambda \sigma}. \tag{5.5}$$

Theorem 5.1 (see [12]). *Let* (5.1) *be in a critical case,* $p > 0$, $q > 0$, *and*

$$\begin{aligned} \liminf_{t \to \infty} \left[b_1(t+\tau) - p\right] t &= \alpha, \\ \liminf_{t \to \infty} \left[b_2(t+\sigma) - q\right] t &= \beta, \end{aligned} \tag{5.6}$$

where $-\infty < \alpha, \beta \leq +\infty$. *If* $\alpha e^{\lambda_0 \tau} + \beta e^{\lambda_0 \sigma} > 0$, *then all solutions of* (5.1) *oscillate.*

Theorem 5.2 (see [12]). *Let*

$$\begin{aligned} \liminf_{t \to \infty} b_1(t) = \frac{1}{e\tau}, \qquad & \liminf_{t \to \infty} b_2(t) = 0, \\ \liminf_{t \to \infty} \left[b_1(t+\tau) - \frac{1}{e\tau}\right] t = \alpha, \qquad & \liminf_{t \to \infty} t b_2(t+\sigma) = \beta, \end{aligned} \tag{5.7}$$

where $-\infty < \alpha \leq +\infty$, $\beta > 0$. *If* $e\alpha + e^{\sigma/\tau}\beta > 0$, *then all solutions of* (5.1) *oscillate.*

The aim of the following theorems is to obtain nonoscillation conditions for (5.1) in the above-mentioned critical case. This will complete the oscillation results given by Theorems 5.1 and 5.2. Note that, in Theorems 5.3 and 5.4 below, delays h_1 and h_2 being not defined by (5.2) are arbitrary and subject only to the restrictions indicated.

Theorem 5.3. *Let* $t - h_1(t) \leq \tau$ *and* $t - h_2(t) \leq \sigma$ *for* $t \geq t_0$,

$$\begin{aligned} \limsup_{t \to \infty} \left[b_1(t) - p\right] t &= \alpha, \\ \limsup_{t \to \infty} \left[b_2(t) - q\right] t &= \beta, \end{aligned} \tag{5.8}$$

where $-\infty < \alpha, \beta < +\infty$, *and let there exists a* $\lambda_0 > 0$ *such that*

$$\lambda_0 \geq pe^{\lambda_0 \tau} + qe^{\lambda_0 \sigma}, \tag{5.9}$$

$$\alpha e^{\lambda_0 \tau} + \beta e^{\lambda_0 \sigma} < 0. \tag{5.10}$$

Then (5.1) *has a nonoscillatory solution.*

Proof. There exist $t_1 > t_0 + \max(\tau, \sigma)$ and $\epsilon > 0$ such that, owing to (5.8) and (5.10),

$$b_1(t) \le p + \frac{\alpha + \epsilon}{t},$$
$$b_2(t) \le q + \frac{\beta + \epsilon}{t}, \tag{5.11}$$
$$(\alpha + \epsilon)e^{\lambda_0 \tau} + (\beta + \epsilon)e^{\lambda_0 \sigma} \le 0$$

if $t \ge t_1$. By Lemma 2.4, with $m = 2$,

$$c_1(t) := p + \frac{\alpha + \epsilon}{t},$$
$$c_2(t) := q + \frac{\beta + \epsilon}{t}, \tag{5.12}$$
$$g_1(t) := t - \tau,$$
$$g_2(t) := t - \sigma$$

in (2.4), the existence of a nonoscillatory solution of the equation

$$\dot{x}(t) + \left(p + \frac{\alpha + \epsilon}{t}\right)x(t - \tau) + \left(q + \frac{\beta + \epsilon}{t}\right)x(t - \sigma) = 0 \tag{5.13}$$

implies the existence of a nonoscillatory solution of (5.1).

By Lemma 2.6, for the existence of a positive solution of (5.13), it is sufficient to find a nonnegative solution of the inequality

$$u(t) \ge \left(p + \frac{\alpha + \epsilon}{t}\right)e^{\int_{t-\tau}^{t} u(s)ds} + \left(q + \frac{\beta + \epsilon}{t}\right)e^{\int_{t-\sigma}^{t} u(s)ds}, \tag{5.14}$$

where $t \ge t_1$. Substituting $u(t)$ for λ_0 in inequality (5.14), we have

$$\lambda_0 \ge pe^{\lambda_0 \tau} + qe^{\lambda_0 \sigma} + \frac{(\alpha + \epsilon)e^{\lambda_0 \tau} + (\beta + \epsilon)e^{\lambda_0 \sigma}}{t}. \tag{5.15}$$

Due to (5.9) and (5.11), we conclude that the last inequality holds, and, consequently, $u(t) = \lambda_0$ is a solution of inequality (5.14). By Lemma 2.6, (5.13) has a nonoscillatory solution. Hence, (5.1) has a nonoscillatory solution, too. □

Theorem 5.4. *Let $t - h_1(t) \le \tau$, $t - h_2(t) \le \sigma$ for $t \ge t_0$,*

$$\limsup_{t \to \infty} \left[b_1(t) - \frac{1}{e\tau}\right]t = \alpha,$$
$$\limsup_{t \to \infty} tb_2(t) = \beta, \tag{5.16}$$

where $\alpha, \beta \in \mathbb{R}$. *If*

$$e\alpha + e^{\sigma/\tau}\beta < 0, \tag{5.17}$$

then (5.1) *has a nonoscillatory solution.*

Proof. There exist $t_1 > t_0 + \max(\tau, \sigma)$ and $\epsilon > 0$ such that, owing to (5.16) and (5.17),

$$\begin{gathered} b_1(t) \leq \frac{1}{e\tau} + \frac{\alpha + \epsilon}{t}, \\ b_2(t) \leq \frac{\beta + \epsilon}{t}, \\ e(\alpha + \epsilon) + e^{\sigma/\tau}(\beta + \epsilon) \leq 0 \end{gathered} \tag{5.18}$$

if $t \geq t_1$. By Lemma 2.4, with $m = 2$,

$$\begin{gathered} c_1(t) = \frac{1}{e\tau} + \frac{\alpha + \epsilon}{t}, \\ c_1(t) = \frac{\beta + \epsilon}{t}, \\ g_1(t) = t - \tau, \\ g_2(t) = t - \sigma \end{gathered} \tag{5.19}$$

in (2.4), the existence of a nonoscillatory solution of the equation

$$\dot{x}(t) + \left(\frac{1}{e\tau} + \frac{\alpha + \epsilon}{t}\right)x(t - \tau) + \left(\frac{\beta + \epsilon}{t}\right)x(t - \sigma) = 0 \tag{5.20}$$

implies the existence of a nonoscillatory solution of (5.1). By Lemma 2.6, it is sufficient to find a nonnegative solution of the inequality

$$u(t) \geq \left(\frac{1}{e\tau} + \frac{\alpha + \epsilon}{t}\right)e^{\int_{t-\tau}^{t} u(s)ds} + \left(\frac{\beta + \epsilon}{t}\right)e^{\int_{t-\sigma}^{t} u(s)ds}, \tag{5.21}$$

where $t \geq t_1$. Put $u(t) = 1/\tau$, $t \geq t_1$ in inequality (5.21). We have

$$\frac{1}{\tau} \geq \frac{1}{e\tau}e + \frac{e(\alpha + \epsilon) + e^{\sigma/\tau}(\beta + \epsilon)}{t}. \tag{5.22}$$

Due to (5.18), we conclude that the last inequality holds, and, consequently, $u(t) = 1/\tau$ is a solution of inequality (5.21). Let $u(t) = 0$ for $t < t_1$. By Lemma 2.6, (5.20) has a nonoscillatory solution. Hence, (5.1) also has a nonoscillatory solution. □

Example 5.5. Consider (5.1) with

$$b_1(t) = \frac{1}{e\tau} + \frac{|\sin t| - 2}{t}, \qquad b_2(t) = \frac{|\cos t| + 1}{t} \tag{5.23}$$

and with $h_1(t)$, $h_2(t)$ defined by (5.2), that is,

$$\dot{x}(t) + \left(\frac{1}{e\tau} + \frac{|\sin t| - 2}{t}\right)x(t-\tau) + \frac{|\cos t| + 1}{t}x(t-\sigma) = 0, \tag{5.24}$$

where $t \geq t_0$. Since

$$\begin{gathered} \liminf_{t\to\infty} b_1(t) = \frac{1}{e\tau}, \qquad \liminf_{t\to\infty} b_2(t) = 0, \\ \liminf_{t\to\infty} \left[b_1(t+\tau) - \frac{1}{e\tau}\right]t = -2, \\ \liminf_{t\to\infty} tb_2(t+\sigma) = 1, \end{gathered} \tag{5.25}$$

then, by Theorem 5.2 (with $\alpha = -2$ and $\beta = 1$), all solutions of (5.24) oscillate if

$$\frac{\sigma}{\tau} > 1 + \ln 2. \tag{5.26}$$

Since

$$\begin{gathered} \limsup_{t\to\infty} \left[b_1(t) - \frac{1}{e\tau}\right]t = -1, \\ \limsup_{t\to\infty} tb_2(t) = 2, \end{gathered} \tag{5.27}$$

by Theorem 5.4 (with $\alpha = -1$ and $\beta = 2$), (5.24) has a nonoscillatory solution if

$$\frac{\sigma}{\tau} < 1 - \ln 2. \tag{5.28}$$

5.1. Generalization for Equations with Several Delays

It is easy to generalize Theorem 5.3 for a general equation (2.1) with several delays. Denote

$$p_i := \liminf_{t\to\infty} b_i(t), \tag{5.29}$$

where $i = 1, 2, \ldots, m$. We omit the proof of this generalization since it is similar to that of Theorem 5.3.

Theorem 5.6. *Let τ_i, $i = 1, 2, \ldots, m$, be positive constants such that*

$$t - h_i(t) \le \tau_i \tag{5.30}$$

for $t \ge t_0$,

$$\limsup_{t \to \infty} \left[b_i(t) - p_i\right] t = \alpha_i, \tag{5.31}$$

where $\alpha_i \in \mathbb{R}$, and let there exists a $\lambda_0 > 0$ such that

$$\begin{gathered} \lambda_0 \ge \sum_{i=1}^{m} p_i e^{\lambda_0 \tau_i}, \\ \sum_{i=1}^{m} \alpha_i e^{\lambda_0 \tau_i} < 0. \end{gathered} \tag{5.32}$$

Then (2.1) *has a nonoscillatory solution.*

The following statement generalizes Theorem 5.4. We will formulate this result for (2.4).

Theorem 5.7. *Let $I \subset \{1, \ldots, m\}$ be a set of indices such that*

$$g_k(t) \le \begin{cases} h_1(t), & \text{if } k \in I, \\ h_2(t), & \text{if } k \notin I, \end{cases} \tag{5.33}$$

where $h_i : [t_0, \infty) \to \mathbb{R}$ and $h_i(t) \le t$. Let

$$b_1(t) := \sum_{k \in I} c_k(t), \qquad b_2(t) := \sum_{k \notin I} c_k(t). \tag{5.34}$$

If, for functions b_i, h_i, $i = 1, 2$, all assumptions of Theorem 5.4 are true, then (2.4) *has a nonoscillatory solution.*

The proof of Theorem 5.7 is omitted as it can be done easily using Lemma 2.4 and Theorem 5.4.

6. Concluding Remarks

In conclusion we note that there exist numerous results on nonoscillation for various classes of delay differential equations in a noncritical case. We refer, for example, to monographs [6, 9, 13, 14], recent papers [15–22], and references therein. Some of the books and papers mentioned discuss the critical case from various points of view different from our approach, and we mentioned them above. In the paper we investigate the critical case for delayed differential equations. It will be interesting as a motivation for further investigation along these lines to consider cases, critical is a sense to other classes of equations, in particular, for integrodifferential equations, differential equations with distributed delay, or differential equations of a neutral type. Finally, for nonoscillation results for difference equations we refer to [23–28].

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