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Research Article

Convergence of Iterative Methods Applied to Generalized Fisher Equation

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A generalized Fisher's equation is solved by using the modified Adomian decomposition method (MADM), variational iteration method (VIM), homotopy analysis method (HAM), and modified homotopy perturbation method (MHPM). The approximation solution of this equation is calculated in the form of series whose components are computed easily. The existence, uniqueness, and convergence of the proposed methods are proved. Numerical example is studied to demonstrate the accuracy of the present methods.

1. Introduction

Fisher proposed equation $\partial u/\partial t = \partial^2 u/\partial x^2 + u(1-u)$ as a model for the propagation of a mutant gene, with u denoting the density of an advantageous. This equation is encountered in chemical kinetics [1] and population dynamics which includes problems such as nonlinear evolution of a population in a nuclear reaction and branching. Moreover, the same equation occurs in logistic population growth models [2], flame propagation, neurophysiology, autocatalytic chemical reaction, and branching Brownian motion processes. A lot of works have been done in order to find the numerical solution of this equation, for example, variational iteration method and modified variational iteration method for solving the generalized Fisher equation [3–5], an analytical study of Fisher equation by using Adomian decomposition method [6], numerical solution for solving Burger-Fisher equation [7–10], a novel approach for solving the Fisher equation using Exp-function method [11]. In this paper, we develop the MADM, VIM, HAM, and MHPM to solve the generalized Fisher equation as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u^s), \tag{1.1}$$

with the initial conditions given by

$$u(x,0) = f(x). \tag{1.2}$$

The paper is organized as follows. In Section 2, the iteration methods MADM, VIM, HAM and MHPM are introduced for solving (1.1). Also, the existance, uniqueness and convergence of the proposed in Section 3. Finally, the numerical example is presented in Section 4 to illustrate the accuracy of these methods.

To obtain the approximation solution of (1.1), by integrating one time from (1.1) with respect to t and using the initial conditions, we obtain

$$u(x,t) = f(x) + \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2} d\tau + \int_0^t u(x,\tau)(1 - u^s(x,\tau)) d\tau. \tag{1.3}$$

We set

$$F(u) = u(1 - u^s). \tag{1.4}$$

In (1.3), we assume $f(x)$ is bounded for all x in $J = [0,T] (T \in \mathbb{R})$ and

$$|t - \tau| \leq M', \quad \forall 0 \leq t, \ \tau \leq T. \tag{1.5}$$

The terms $D^2(u)$ and $F(u)$ are Lipschitz continuous with $|D^2(u) - D^2(u^*)| \leq L_1 |u - u^*|$, $|F(u) - F(u^*)| \leq L_2 |u - u^*|$.

We set

$$\begin{gathered} \alpha = T\left(M' L_1 + M' L_2\right), \\ \beta = 1 - T(1 - \alpha). \end{gathered} \tag{1.6}$$

Now we decompose the unknown function $u(x,t)$ by a sum of components defined by the following decomposition series with u_0 identified as $u(x,0)$:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{1.7}$$

2. Iterative Methods

2.1. Preliminaries of the MADM

The Adomian decomposition method is applied to the following general nonlinear equation:

$$Lu + Ru + Nu = g(x), \tag{2.1}$$

where u is the unknown function, L is the highest-order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of (2.1), and using the given conditions, we obtain

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.2}$$

where the function $f(x)$ represents the terms arising from integrating the source term $g(x)$. The nonlinear operator $Nu = G(u)$ is decomposed as

$$G(u) = \sum_{n=0}^{\infty} A_n, \tag{2.3}$$

where A_n, $n \geq 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]\right]_{\lambda=0}. \tag{2.4}$$

Adomian polynomials were introduced in [12–15] as

$$\begin{aligned}
A_0 &= G(u_0), \\
A_1 &= u_1 G'(u_0), \\
A_2 &= u_2 G'(u_0) + \frac{1}{2!}u_1^2 G''(u_0), \\
A_3 &= u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3!}u_1^3 G'''(u_0), \ldots.
\end{aligned} \tag{2.5}$$

2.1.1. *Adomian Decomposition Method*

The standard decomposition technique represents the solution of u in (2.1) as the following series,

$$u = \sum_{n=0}^{\infty} u_i, \tag{2.6}$$

where, the components $u_0, u_1, \ldots$ are usually determined recursively by

$$\begin{aligned}
u_0 &= f(x), \\
u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 0.
\end{aligned} \tag{2.7}$$

Substituting (2.5) into (2.7) leads to the determination of the components of u. Having determined the components $u_0, u_1, \ldots$ the solution u in a series form defined by (2.6) follows immediately.

2.1.2. The Modified Adomian Decomposition Method

The modified decomposition method was introduced by Wazwaz in [16]. The modified forms was established based on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x). \tag{2.8}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part f_1 be assigned to the zeroth component u_0, whereas the remaining part f_2 be combined with the other terms given in (2.7) to define u_1. Consequently, the modified recursive relation

$$\begin{aligned} u_0 &= f_1(x), \\ u_1 &= f_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ &\vdots \\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1, \end{aligned} \tag{2.9}$$

was developed.

2.2. Description of the MADM

To obtain the approximation solution of (1.1), according to the MADM, we can write the iterative formula (2.9) as follows:

$$\begin{aligned} u_0(x,t) &= f_1(x), \\ u_1(x,t) &= f_2(x) + \int_0^t D^2(u_0(x,\tau)) + \int_0^t F(u_0(x,\tau))d\tau, \\ &\vdots \\ u_{n+1}(x,t) &= \int_0^t D^2(u_n(x,\tau)) + \int_0^t F(u_n(x,\tau))d\tau. \end{aligned} \tag{2.10}$$

The operators $D^2(u(x,\tau)) = (d^2/dx^2)u(x,t)$ and $F(u(x,\tau))$ are usually represented by an infinite series of the so-called Adomian polynomials as follows:

$$F(u) = \sum_{i=0}^{\infty} A_i, \qquad D^2(u) = \sum_{i=0}^{\infty} L_i. \tag{2.11}$$

where A_i and $L_i (i \geq 0)$ are the Adomian polynomials were introduced in [12].

From [12], we can write another formula for the Adomian polynomials:

$$\begin{aligned} L_n &= D^2(s_n) - \sum_{i=0}^{n-1} L_i, \\ A_n &= F(s_n) - \sum_{i=0}^{n-1} A_i, \end{aligned} \tag{2.12}$$

where the partial sum is $s_n = \sum_{i=0}^{n} u_i(x,t)$.

2.3. Preliminaries of the VIM

In the VIM [17–20], we consider the following nonlinear differential equation:

$$L(u) + N(u) = g(t), \tag{2.13}$$

where L is a linear operator,N is a nonlinear operator and $g(x,t)$ is a known analytical function. In this case, a correction functional can be constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\tau)\{L(u_n(x,\tau)) + N(u_n(x,\tau)) - g(x,\tau)\}d\tau, \quad n \geq 0, \tag{2.14}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here the function $u_n(x,\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t)$, $n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $u_n(x,t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{2.15}$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

2.4. Description of the VIM

To obtain the approximation solution of (1.1), according to the VIM, we can write iteration formula (2.14) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + L_t^{-1}\left(\lambda\left[u(x,t) - f(x) - \int_0^t D^2(u_n(x,\tau))d\tau - \int_0^t F(u_n(x,\tau))d\tau\right]\right), \tag{2.16}$$

where,

$$L_t^{-1}(\cdot) = \int_0^t (\cdot)d\tau. \tag{2.17}$$

To find the optimal λ, we proceed as

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta L_t^{-1}\left(\lambda\left[u_n(x,t) - f(x) - \int_0^t D^2(u_n(x,\tau))d\tau + \int_0^t F(u_n(x,\tau))d\tau\right]\right)\\ &= \delta u_n(x,t) + \lambda(x)\delta u_n(x,t) - L_t^{-1}\left[\delta u_n(x,t)\lambda'(x)\right].\end{aligned} \tag{2.18}$$

From (2.18), the stationary conditions can be obtained as follows:

$$\lambda' = 0, \qquad 1 + \lambda = 0. \tag{2.19}$$

Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in (2.16), the following iteration formula is obtained.

$$\begin{gathered}u_0(x,t) = f(x),\\ u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}\left(\left[u_n(x,t) - f(x) - \int_0^t D^2(u_n(x,\tau))d\tau - \int_0^t F(u_n(x,\tau))d\tau\right]\right), \quad n \geq 0.\end{gathered} \tag{2.20}$$

Relation (2.20) will enable us to determine the components $u_n(x,t)$ recursively for $n \geq 0$.

2.5. Preliminaries of the HAM

Consider

$$N[u] = 0, \tag{2.21}$$

where N is a nonlinear operator, $u(x,t)$ is unknown function and x is an independent variable. let $u_0(x,t)$ denote an initial guess of the exact solution $u(x,t)$, $h \neq 0$ an auxiliary parameter, $H(x,t) \neq 0$ an auxiliary function, and L an auxiliary nonlinear operator with the property $L[r(x,t)] = 0$ when $r(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] - qhH(x,t)N[\phi(x,t;q)] = \widehat{H}[\phi(x,t;q); u_0(x,t), H(x,t), h, q]. \tag{2.22}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary nonlinear operator L, the nonzero auxiliary parameter h, and the auxiliary function $H(x,t)$.

Enforcing the homotopy (2.22) to be zero, that is,

$$\widehat{H}\left[\phi(x,t;q);u_0(x,t),H(x,t),h,q\right]=0, \tag{2.23}$$

we have the so-called zero-order deformation equation

$$(1-q)L\left[\phi(x,t;q)-u_0(x,t)\right]=qhH(x,t)N\left[\phi(x,t;q)\right]. \tag{2.24}$$

When $q=0$, the zero-order deformation (2.24) becomes

$$\phi(x,t;0)=u_0(x,t), \tag{2.25}$$

and when $q=1$, since $h\neq 0$ and $H(x,t)\neq 0$, the zero-order deformation (2.24) is equivalent to

$$\phi(x,t;1)=u(x,t). \tag{2.26}$$

Thus, according to (2.25) and (2.26), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy [21, 22].

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows:

$$\phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{2.27}$$

where

$$u_m(x,t)=\frac{1}{m!}\left.\frac{\partial^m\phi(x,t;q)}{\partial q^m}\right|_{q=0}. \tag{2.28}$$

Let the initial guess $u_0(x,t)$, the auxiliary nonlinear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(x,t)$ be properly chosen so that the power series (2.27) of $\phi(x,t;q)$ converges at $q=1$, then, we have under these assumptions the solution series

$$u(x,t)=\phi(x,t;1)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t). \tag{2.29}$$

From (2.27), we can write (2.24) as follows:

$$\begin{aligned}
&(1-q)L[\phi(x,t;q)-u_0(x,t)] \\
&\quad = (1-q)L\left[\sum_{m=1}^{\infty}u_m(x,t)q^m\right] = qhH(x,t)N[\phi(x,t;q)] \\
&\quad \Longrightarrow L\left[\sum_{m=1}^{\infty}u_m(x,t)\ q^m\right] - qL\left[\sum_{m=1}^{\infty}u_m(x,t)q^m\right] = qhH(x,t)N[\phi(x,t;q)].
\end{aligned} \tag{2.30}$$

By differentiating (2.30) m times with respect to q, we obtain

$$\begin{aligned}
&\left\{L\left[\sum_{m=1}^{\infty}u_m(x,t)q^m\right] - qL\left[\sum_{m=1}^{\infty}u_m(x,t)q^m\right]\right\}^{(m)} \\
&\quad = \{qhH(x,t)N[\phi(x,t;q)]\}^{(m)} \\
&\quad = m!L[u_m(x,t)-u_{m-1}(x,t)] \\
&\quad = h\,H(x,t)m\left.\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\right|_{q=0}.
\end{aligned} \tag{2.31}$$

Therefore,

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = hH(x,t)\Re_m(y_{m-1}(x)), \tag{2.32}$$

where,

$$\Re_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\right|_{q=0}, \tag{2.33}$$

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{2.34}$$

Note that the high-order deformation (2.32) is governing the nonlinear operator L, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by (2.33) for any nonlinear operator N.

2.6. Description of the HAM

To obtain the approximation solution of (1.1), according to HAM, let

$$N[u] = u(x,t) - f(x) - \int_0^t D^2(u(x,\tau))d\tau - \int_0^t F(u(x,\tau))d\tau, \tag{2.35}$$

so

$$\Re_m(u_{m-1}(x,t)) = u_{m-1}(x,t) - \int_0^t D^2(u_{m-1}(x,\tau))d\tau - \int_0^t F(u_{m-1}(x,\tau))d\tau - (1-\chi_m)f(x). \tag{2.36}$$

Substituting (2.36) into(2.32)

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)\left[u_{m-1}(x,t) - \int_0^t D^2(u_{m-1}(x,\tau))d\tau - \int_0^t F(u_{m-1}(x,\tau))d\tau - (1-\chi_m)f(x)\right]. \tag{2.37}$$

We take an initial guess $u_0(x,t) = f(x)$, an auxiliary nonlinear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H(x,t) = 1$. This is substituted into (2.37) to give the recurrence relation

$$\begin{gathered} u_0(x,t) = f(x), \\ u_n(x,t) = \int_0^t D^2(u_{n-1}(x,\tau))d\tau + \int_0^t F(u_{n-1}(x,\tau))d\tau, \quad n \geq 1. \end{gathered} \tag{2.38}$$

Therefore, the solution $u(x,t)$ becomes

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = f(x) + \sum_{n=0}^{\infty}\left(\int_0^t D^2(u_{n-1}(x,\tau))d\tau + \int_0^t F(u_{n-1}(x,\tau))d\tau\right), \tag{2.39}$$

which is the method of successive approximations. If

$$|u_n(x,t)| < 1 \tag{2.40}$$

then the series solution (2.39) convergence uniformly.

2.7. Description of the MHPM

To explain MHPM, we consider (1.1) as

$$L(u) = u(x,t) - f(x) - \int_0^t D^2(u_{n-1}(x,\tau))d\tau - \int_0^t F(u_{n-1}(x,\tau))d\tau, \tag{2.41}$$

where $D^2(u(x,\tau)) = g_1(x)h_1(\tau)$ and $F(u(x,\tau)) = g_2(x)h_2(\tau)$. We can define homotopy $H(u,p,m)$ by

$$H(u,o,m) = f(u), \qquad H(u,1,m) = L(u), \tag{2.42}$$

where m is an unknown real number and

$$f(u(x,t)) = u(x,t) - G(x,t). \tag{2.43}$$

Typically we may choose a convex homotopy by

$$H(u,p,m) = (1-p)f(u) + pL(u) + p(1-p)\left[m\left(g_1(x) + g_2(x)\right)\right] = 0, \quad 0 \le p \le 1. \tag{2.44}$$

where m is called the accelerating parameters, and for $m = 0$ we define $H(u,p,0) = H(u,p)$, which is the standard HPM. The convex homotopy (2.44) continuously trace an implicity defined curve from a starting point $H(u(x,t) - f(u), 0, m)$ to a solution function $H(u(x,t), 1, m)$. The embedding parameter p monotonically increase from o to 1 as trivial problem $f(u) = 0$ is continuously deformed to original problem $L(u) = 0$. [23, 24]

The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n; \tag{2.45}$$

when $p \to 1$ (2.44) corresponds to the original one, (2.45) becomes the approximate solution of (1.1), that is,

$$u = \lim_{p\to 1} v = \sum_{m=0}^{\infty} u_m, \tag{2.46}$$

where,

$$u_m(x,t) = f(x) + \int_0^t D^2(u_{m-1}(x,\tau))d\tau + \int_0^t F(u_{m-1}(x,\tau))d\tau. \tag{2.47}$$

3. Existence and Convergency of Iterative Methods

Theorem 3.1. *Let* $0 < \alpha < 1$, *then Fisher equation* (1.1), *has a unique solution.*

Proof. Let u and u^* be two different solutions of (1.3) then

$$\begin{aligned} |u-u^*| &= \left|\int_0^t \left[D^2(u(x,\tau)) - D^2(u^*(x,\tau))\right]d\tau + \int_0^t [F(u(x,\tau)) - F(u^*(x,\tau))]d\tau\right| \\ &\le \int_0^t \left|D^2(u(x,\tau)) - D^2(u^*(x,\tau))\right|d\tau + \int_0^t |F(u(x,\tau)) - F(u^*(x,\tau))|d\tau \\ &\le T \quad (M'L_1 + M'L_2)|u-u^*| = \alpha|u-u^*|. \end{aligned} \tag{3.1}$$

From which we get $(1-\alpha)|u-u^*| \le 0$. Since $0 < \alpha < 1$. then $|u-u^*| = 0$. Implies $u = u^*$ and completes the proof. □

Theorem 3.2. *The series solution* $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ *of problem* (1.1) *using MADM convergence when* $0 < \alpha < 1$, $|u_1(x,t)| < \infty$.

Proof. Denote as $(C[J], \|\cdot\|)$ the Banach space of all continuous functions on J with the norm $\|f(t)\| = \max|f(t)|$, for all t in J. Define the sequence of partial sums s_n, and let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\begin{aligned}
\|s_n - s_m\| &= \max_{\forall t\in J}|s_n - s_m| = \max_{\forall t\in J}\left|\sum_{i=m+1}^{n} u_i(x,t)\right| \\
&= \max_{\forall t\in J}\left|\sum_{i=m+1}^{n}\int_0^t L_{i-1}d\tau + \sum_{i=m+1}^{n}\int_0^t A_{i-1}d\tau\right| \\
&= \max_{\forall t\in J}\left|\int_0^t\left(\sum_{i=m}^{n-1}L_i\right)d\tau + \int_0^t\left(\sum_{i=m}^{n-1}A_i\right)d\tau\right|.
\end{aligned} \tag{3.2}$$

From [12], we have

$$\begin{aligned}
\sum_{i=m}^{n-1}L_i &= D^2(s_{n-1}) - D^2(s_{m-1}), \\
\sum_{i=m}^{n-1}A_i &= F(s_{n-1}) - F(s_{m-1}).
\end{aligned} \tag{3.3}$$

So,

$$\begin{aligned}
\|s_n - s_m\| &= \max_{\forall t\in J}\left|\int_0^t\left[D^2(s_{n-1}) - D^2(s_{m-1})\right]d\tau + \int_0^t[F(s_{n-1}) - F(s_{m-1})]d\tau\right| \\
&\leq \int_0^t\left|D^2(s_{n-1}) - D^2(s_{m-1})\right|d\tau + \int_0^t|F(s_{n-1}) - F(s_{m-1})|d\tau \leq \alpha\|s_n - s_m\|.
\end{aligned} \tag{3.4}$$

Let $n = m+1$, then

$$\|s_n - s_m\| \leq \alpha\|s_m - s_{m-1}\| \leq \alpha^2\|s_{m-1} - s_{m-2}\| \leq \cdots \leq \alpha^m\|s_1 - s_0\|. \tag{3.5}$$

From the triangle inquality we have

$$\begin{aligned}
\|s_n - s_m\| &\leq \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \cdots + \|s_n - s_{n-1}\| \leq \left[\alpha^m + \alpha^{m1} + \cdots + \alpha^{n-m-1}\right]\|s_1 - s_0\| \\
&\leq \alpha^m\left[1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}\right]\|s_1 - s_0\| \leq \left[\frac{1-\alpha^{n-m}}{1-\alpha}\right]\|u_1(x,t)\|.
\end{aligned} \tag{3.6}$$

Since $0 < \alpha < 1$, we have $(1 - \alpha^{n-m}) < 1$, then

$$\|s_n - s_m\| \le \frac{\alpha^m}{1-\alpha} \max_{\forall t \in J} |u_1(x,t)|. \tag{3.7}$$

But $|u_1(x,t)| < \infty$, so, as $m \to \infty$, then $\|s_n - s_m\| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete. □

Theorem 3.3. *The series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem* (1.1) *using VIM converges when* $0 < \alpha < 1$, $0 < \beta < 1$.

Proof. One has the following:

$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}\left(\left[u_n(x,t) - f(x) - \int_0^t D^2(u_n(x,\tau))d\tau - \int_0^t F(u_n(x,\tau))d\tau\right]\right), \tag{3.8}$$

$$u(x,t) = u(x,t) - L_t^{-1}\left(\left[u(x,t) - f(x) - \int_0^t D^2(u(x,\tau))d\tau - \int_0^t F(u(x,\tau))d\tau\right]\right). \tag{3.9}$$

By subtracting relation (3.9) from (3.8),

$$\begin{aligned} u_{n+1}(x,t) - u(x,t) &= u_n(x,t) - u(x,t) \\ &\quad - L_t^{-1}\left(u_n(x,t) - u(x,t) - \int_0^t \left[D^2(u_n(x,\tau)) - D^2(u(x,\tau))\right]d\tau \right. \\ &\qquad \left. - \int_0^t [F(u_n(x,\tau)) - F(u(x,\tau))]d\tau\right), \end{aligned} \tag{3.10}$$

if we set, $e_{n+1}(r,t) = u_{n+1}(r,t) - u_n(r,t)$, $e_n(r,t) = u_n(r,t) - u(r,t)$, $|e_n(r,t^*)| = \max_t |e_n(r,t)|$ then since e_n is a decreasing function with respect to t from the mean value theorem we can write,

$$\begin{aligned} e_{n+1}(r,t) &= e_n(r,t) \\ &\quad + L_t^{-1}\left(-e_n(r,t) + \int_0^t \left[D^2(u_n(x,\tau)) - D^2(u(x,\tau))\right]d\tau \right. \\ &\qquad \left. - \int_0^t [F(u_n(x,\tau)) - F(u(x,\tau))]d\tau\right) \\ &\le e_n(r,t) + L_t^{-1}\left[-e_n(r,t) + L_t^{-1}|e_n(r,t)|(\nu(L_1 + TL_2))\right] \\ &\le e_n(r,t) - Te_n(r,\eta) + T(M'L_1 + M'L_2)L_t^{-1}L_t^{-1}|e_n(r,t)| \\ &\le (1 - T(1-\alpha))|e_n(r,t^*)|, \end{aligned} \tag{3.11}$$

where $0 \le \eta \le t$. Hence, $e_{n+1}(r,t) \le \beta|e_n(r,t^*)|$.

Therefore,

$$\|e_{n+1}\| = \max_{\forall t\in J}|e_{n+1}| \le \beta \max_{\forall t\in J}|e_n| \le \beta\|e_n\|. \tag{3.12}$$

Since $0 < \beta < 1$, then $\|e_n\| \to 0$. So, the series converges and the proof is complete. □

Theorem 3.4. *If the series solution* (2.38) *of problem* (1.1) *is convergent then it converges to the exact solution of the problem* (1.1) *by using HAM.*

Proof. We assume:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) \tag{3.13}$$

where

$$\lim_{m\to\infty} u_m(x,t) = 0. \tag{3.14}$$

We can write,

$$\sum_{m=1}^{n}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = u_1 + (u_2 - u_1) + \cdots + (u_n - u_{n-1}) = u_n(x,t). \tag{3.15}$$

We have,

$$\lim_{n\to\infty} u_n(x,t) = 0. \tag{3.16}$$

So, using (3.16) and the definition of the nonlinear operator L, we have

$$\sum_{m=1}^{\infty} L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = L\left[\sum_{m=1}^{\infty}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right]\right] = 0. \tag{3.17}$$

Therefore from (2.32), we can obtain that,

$$\sum_{m=1}^{\infty} L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = hH(x,t)\sum_{m=1}^{\infty}\mathfrak{R}_{m-1}(u_{m-1}(x,t)) = 0. \tag{3.18}$$

Since $h \ne 0$ and $H(x,t) \ne 0$, we have

$$\sum_{m=1}^{\infty}\mathfrak{R}_{m-1}(u_{m-1}(x,t)) = 0. \tag{3.19}$$

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation (3.19) and simplifying it, we have

$$\begin{aligned}
\sum_{m=1}^{\infty}&\Re_{m-1}(u_{m-1}(x,t))\\
&=\sum_{m=1}^{\infty}\left[u_{m-1}(x,t)-\int_0^t D^2(u_{m-1}(x,\tau))d\tau-\int_0^t F(u_{m-1}(x,\tau))d\tau-(1-\chi_m)f(x)\right]\\
&=u(x,t)-f(x)-\int_0^t D^2(u_{m-1}(x,\tau))d\tau\\
&\quad-\int_0^t[F(u_{m-1}(x,\tau))d\tau].
\end{aligned}\tag{3.20}$$

From (3.19) and (3.20), we have

$$u(x,t)=G(x,t)+\int_0^t(t-\tau)D^2(u(x,\tau))d\tau-\int_0^t(t-\tau)F(u(x,\tau))d\tau,\tag{3.21}$$

therefore, $u(x,t)$ must be the exact solution of (1.1). □

Theorem 3.5. *If $|u_m(x,t)|\leq 1$, then the series solution* (2.46) *of problem* (1.1) *converges to the exact solution.*

Proof. We can write the solution $u(x,t)$ as follows:

$$\begin{aligned}
u(x,t)&=\sum_{m=0}^{\infty}u_m(x,t)\\
&=\sum_{m=0}^{\infty}\left\{\int_0^t D^2(u_{m-1}(x,\tau))d\tau+\int_0^t F(u_{m-1}(x,\tau))d\tau\right.\\
&\qquad\left.+\ (1-\chi_m)f(x)+\int_0^t(t-\tau)g_1(u_{m-1}(x))d\tau+\int_0^t(t-\tau)(g_2(x))d\tau\right\}.
\end{aligned}\tag{3.22}$$

If $|u_m(x,t)|<1$, therefore, $u(x,t)=\sum_{m=0}^{\infty}u_m(x,t)$ must be the exact solution of (1.1). □

4. Numerical Example

In this section, we compute a numerical example which is solved by the MADM, VIM, HAM and MHPM. The program has been provided with Mathematica 6 according to the following algorithm. In this algorithm ε is a given positive value.

Table 1: Numerical results of Example 4.2.

t	Error (MADM, $n = 5$)	Error (VIM, $n = 3$)	Error (HAM, $n = 2$)	Error (MHPM, $n = 3$)
0.5	0.0588302	0.0587111	0.0348265	0.0588668
0.6	0.0670054	0.0677865	0.0437778	0.0674631
0.7	0.068559	0.0674742	0.0424178	0.0670032
0.8	0.0745342	0.0720383	0.0558752	0.0720428
0.9	0.0745342	0.0746331	0.0566234	0.07458943
1.0	0.0766331	0.0775012	0.0599735	0.0775367

Algorithm 4.1. One has the following.

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relation (2.10) for MADM, (2.20) for VIM, (2.38) for HAM and (2.46) for MHPM.

Step 3. If $|u_{n+1} - u_n| < \varepsilon$ then go to Step 4, else $n \leftarrow n + 1$ and go to Step 2.

Step 4. Print $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

Example 4.2 (see [3]). Consider the Fisher equation with $s = 3$.

$$u_t = u_{xx} + u\left(1 - u^3\right), \tag{4.1}$$

subject to initial conditions

$$u(x,0) = \left(\frac{1}{\left(1 + e^{(3/\sqrt{10})x}\right)^{1/3}}\right)^2, \tag{4.2}$$

with the exact solution is $\{(1/2)\tanh[-(3/2\sqrt{10})(x - (7/\sqrt{10})t)] + (1/2)\}^{2/3}$.

5. Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which convergent are rapidly to exact solutions. In this paper, the HAM has been successfully employed to obtain the approximate analytical solution of the Fisher equation. For this purpose, we showed that the HAM is more rapid convergence than the MADM, VIM and MHPM.

References

[1] T. A. Abassy, M. A. El-Tawil, and H. El Zoheiry, "Toward a modified variational iteration method," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 137–147, 2007.

[2] P. K. Brazhnik and J. J. Tyson, "On traveling wave solutions of Fisher's equation in two spatial dimensions," *SIAM Journal on Applied Mathematics*, vol. 60, no. 2, pp. 371–391, 2000.

[3] M. Matinfar and M. Ghanbari, "The application of the modified variational iteration method on the generalized Fisher's equation," *Journal of Applied Mathematics and Computing*, vol. 31, no. 1-2, pp. 165–175, 2009.

[4] S. T. Mohyud-Din and M. A. Noor, "Modified variational iteration method for solving Fisher's equations," *Journal of Applied Mathematics and Computing*, vol. 31, no. 1-2, pp. 295–308, 2009.

[5] M. Matinfar and M. Ghanbari, "Solving the Fisher's equation by means of variational iteration method," *International Journal of Contemporary Mathematical Sciences*, vol. 4, no. 5–8, pp. 343–348, 2009.

[6] A.-M. Wazwaz and A. Gorguis, "An analytic study of Fisher's equation by using Adomian decomposition method," *Applied Mathematics and Computation*, vol. 154, no. 3, pp. 609–620, 2004.

[7] A. Golbabai and M. Javidi, "A spectral domain decomposition approach for the generalized Burger's-Fisher equation," *Chaos, Solitons and Fractals*, vol. 39, no. 1, pp. 385–392, 2009.

[8] M. Javidi, "Modified pseudospectral method for generalized Burger's-Fisher equation," *International Mathematical Forum*, vol. 1, no. 29–32, pp. 1555–1564, 2006.

[9] H. N. A. Ismail and A. A. Abd Rabboh, "A restrictive Padé approximation for the solution of the generalized Fisher and Burger-Fisher equations," *Applied Mathematics and Computation*, vol. 154, no. 1, pp. 203–210, 2004.

[10] M. Moghimi and F. S. A. Hejazi, "Variational iteration method for solving generalized Burger-Fisher and Burger equations," *Chaos, Solitons and Fractals*, vol. 33, no. 5, pp. 1756–1761, 2007.

[11] T. Öziş and C. Köröğlu, "A novel approach for solving the Fisher equation using Exp-function method," *Physics Letters A*, vol. 372, no. 21, pp. 3836–3840, 2008.

[12] I. L. El-Kalla, "Convergence of the Adomian method applied to a class of nonlinear integral equations," *Applied Mathematics Letters*, vol. 21, no. 4, pp. 372–376, 2008.

[13] S. H. Behiry, H. Hashish, I. L. El-Kalla, and A. Elsaid, "A new algorithm for the decomposition solution of nonlinear differential equations," *Computers & Mathematics with Applications*, vol. 54, no. 4, pp. 459–466, 2007.

[14] A.-M. Wazwaz, "Construction of solitary wave solutions and rational solutions for the KdV equation by Adomian decomposition method," *Chaos, Solitons and Fractals*, vol. 12, no. 12, pp. 2283–2293, 2001.

[15] M. A. Fariborzi Araghi and Sh. S. Behzadi, "Solving nonlinear Volterra—Fredholm integro-differential equations using the modified Adomian decomposition method," *Computational Methods in Applied Mathematics*, vol. 9, no. 4, pp. 321–331, 2009.

[16] A. M. Wazwaz, *A First Course in Integral Equations*, WSPC, Jersey City, NJ, USA, 1997.

[17] J.-H. He, "Variational principles for some nonlinear partial differential equations with variable coefficients," *Chaos, Solitons and Fractals*, vol. 19, no. 4, pp. 847–851, 2004.

[18] J.-H. He, "Variational iteration method for autonomous ordinary differential systems," *Applied Mathematics and Computation*, vol. 114, no. 2-3, pp. 115–123, 2000.

[19] S.-Q. Wang and J.-H. He, "Variational iteration method for solving integro-differential equations," *Physics Letters A*, vol. 367, no. 3, pp. 188–191, 2007.

[20] J.-H. He, "Variational iteration method—some recent results and new interpretations," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 3–17, 2007.

[21] S. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, vol. 2 of *CRC Series: Modern Mechanics and Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.

[22] S. Liao, "Notes on the homotopy analysis method: some definitions and theorems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 983–997, 2009.

[23] A. Golbabai and B. Keramati, "Solution of non-linear Fredholm integral equations of the first kind using modified homotopy perturbation method," *Chaos, Solitons and Fractals*, vol. 39, no. 5, pp. 2316–2321, 2009.

[24] M. Javidi and A. Golbabai, "Modified homotopy perturbation method for solving non-linear Fredholm integral equations," *Chaos, Solitons and Fractals*, vol. 40, no. 3, pp. 1408–1412, 2009.